REAL PLANE SEXTICS WITHOUT REAL POINTS

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Abstract. We prove that the equisingular deformation type of a simple real plane sextic curve with smooth real part is determined by its real homological type, *i.e.*, the polarization, exceptional divisors, and real structure recorded in the homology of the covering $K3$-surface. As an illustration, we obtain an equisingular deformation classification of real plane sextics with empty real part (for completeness, we consider the few non-simple ones as well).

1. Introduction

Recall that a *real algebraic variety* is a complex algebraic variety X equipped with an anti-holomorphic involution $\sigma\colon X \to X$, referred to as the *real structure* and typically omitted from the notation. The fixed point set $X_{\mathbb{R}}$ of σ is called the *real part* of X. If X is smooth, $\dim_{\mathbb{C}} X = n$, then $X_{\mathbb{R}}$ is either empty or a smooth manifold of real dimension n. Similar terminology applies to pairs (X, C), and in this paper we deal with real (with respect to the only, up to automorphism, real structure on $\mathbb{P}^2 := \mathbb{P}^2_{\mathbb{C}}$, *viz.* $[z_0 : z_1 : z_2] \mapsto [\bar{z}_0 : \bar{z}_1 : \bar{z}_2]$) algebraic curves $C \subset \mathbb{P}^2$.

From the topological point of view, the ultimate question is a deformation classification of such curves. There is a large amount of literature on either singular *complex* curves or *smooth* real ones. The combination of the two, *viz.* equisingular equivariant deformations of singular real curves, is considerably less studied: worth mentioning are D. A. Gudkov *et al.* [15] on irreducible quartic curves with arbitrary singularities and V. Kharlamov [20], where one-nodal quintics are used as a means of a deformation classification of the smooth ones. The deformation classification of all nodal rational quintics was obtained by A. Jaramillo Puentes in [17].

When it comes to sextics, of great help is the advanced theory of $K3$-surfaces which works well both in singular complex and smooth real cases. However, as was discovered in [16], the naked notion of real homological type (see §2.4 below) is no longer sufficient: in [16], in order to obtain the deformation classification of one-nodal real sextics, one had to compute the fundamental polyhedra, often infinite, of certain groups generated by reflections. This was reconfirmed by J. Josi [19, 18], who studied nodal rational real sextics; however, he observed that the problem does not arise when none of the nodes is real. A generalisation of this fact to arbitrary real sextics with smooth real part is one of the principal results of our paper, see Theorem 1.3 below.

2010 *Mathematics Subject Classification.* Primary: 14J28, 14P25; Secondary: 14H50, 14H10, 14J10.

Key words and phrases. Plane sextic, simple sextic, integral lattice, $K3$-surface, equisingular deformation, equivariant deformation, trigonal curve.

The first author was partially supported by the TÜBİTAK grant 123F111.

The second author was supported in part by the ANR grants ANR-18-CE40-0009 ENUMGEOM and ANR-22-CE40-0014 SINTROP.

From now on, to shorten the terminology, we refer to real curves $C \subset \mathbb{P}^2$ with empty real part $C_{\mathbb{R}} = \varnothing$ as *empty curves*. The degree of such a curve (as well as that of each irreducible component thereof) is obviously even, and there is but one real projective equivalence class of empty conics. The deformation class of an empty quartic is determined by its set of singularities, which can be $\varnothing$, $2\mathbf{A}_1$, $2\mathbf{A}_2$, $2\mathbf{A}_3$, or $4\mathbf{A}_1$, and, in the last two cases, by whether the two conic components are real or complex conjugate. It is also well known (by a simple convexity and codimension argument) that in each even degree there a unique equivariant deformation class of smooth empty curves. Thus, sextics constitute the first nontrivial case. This problem was brought to our attention by A. Libgober who was interested in the existence of empty sextics with the set of singularities $8\mathbf{A}_2$ (see [22]). We answer this question in the affirmative; moreover, we obtain a complete equisingular equivariant deformation classification of empty sextics.

1.1. **Principal results.** In this paper, we mainly deal with *simple* (*i.e.*, ones with simple, *aka* **A**–**D**–**E**, singularities) sextics $C \subset \mathbb{P}^2$ (though, see Addendum 1.2 for the non-simple case); it is these sextics that are closely related to $K3$-surfaces.

One of our principal results is the equisingular equivariant deformation classification of empty sextics. The following theorem is proved in §3.2.

Theorem 1.1. *There are*

- 169 *equisingular equivariant deformation families, contained in*
- 159 *real forms* (aka *real lattice types, see §2.2*) *of*
- 139 *complex lattice types* (*see §2.1*) *of*
- 104 *sets of singularities*

of empty sextics; *they are listed in Tables 1–4* (*see Convention 2.13*).

Theorem 1.1 is proved lattice theoretically; however, in Appendix A we provide an explicite geometric description of most empty sextics with large total Milnor number.

It is worth mentioning that, according to [1, 2], with the only exception of the set of singularities $2\mathbf{A}_9$, $d = 1$ (see §B.2), each complex lattice type in Theorem 1.1 constitutes a single connected equisingular deformation family.

For completeness, in the next statement we discuss empty sextics with a non-simple singular point; the proof is found in §3.3.

Addendum 1.2. *There are two deformation families of reduced non-simple empty sextics. Any such sextic splits into three conics tangent to each other at a common pair of complex conjugate points* (*the set of singularities* $2\mathbf{J}_{10}$ *in the notation of* [3]). *One of the conics is always real, whereas the two others are either real or complex conjugate* (cf. *item* (1) *in Convention 2.13*).

Theorem 1.1 is derived from the following statement, proved in §3.1, which is of an independent interest. In line with the general framework of $K3$-surfaces, we reduce the deformation classification to the purely arithmetic study of the so-called *real homological types* (see §2.4), which capture the immediate homological information about the polarization, exceptional divisors, and real structure.

Theorem 1.3. *Two real simple sextics without real singular points are in the same equisingular equivariant deformation class if and only if their real homological types are isomorphic.*

TABLE 1. The case $\mu = 18$ (see Convention 2.13)

S	d	n	Remarks
$6\mathbf{A}_3$	4^3	1	(3,0)(0,0,3); see (A.17), (A.19)
$2\mathbf{A}_5 \oplus 4\mathbf{A}_2$	3^1	1	(1,1,1); see (A.14), (A.15)
$2\mathbf{A}_5 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_1$	2	1	(1,1)($\mathbf{A}_3^{\mathrm{ii}}$)
$2\mathbf{A}_5 \oplus 2\mathbf{A}_4$	1	1^{2*}	
$2\mathbf{A}_6 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$	1	1	
$2\mathbf{A}_7 \oplus 4\mathbf{A}_1$	4^3	1	(1,1)(1,1,0); see (A.18), (A.20)
$2\mathbf{A}_7 \oplus 2\mathbf{A}_2$	2	1	
$2\mathbf{A}_8 \oplus 2\mathbf{A}_1$	1	1	
—	3^1	2	[2]; see (A.22), (A.23)
$2\mathbf{A}_9$	1	1^{2*}	2 complex families; see §B.2
$2\mathbf{D}_5 \oplus 2\mathbf{A}_4$	1	1	
$2\mathbf{D}_6 \oplus 2\mathbf{A}_3$	2^1	1	(1,1)($\mathbf{D}_6^{\mathrm{ii}}$, $\mathbf{A}_3^{\mathrm{i}}$); see (A.26), (A.27)
$2\mathbf{D}_8 \oplus 2\mathbf{A}_1$	2^1	1	(1,1); see (A.28)
$2\mathbf{D}_9$	1	1	
$2\mathbf{E}_6 \oplus 2\mathbf{A}_3$	1	1	
$2\mathbf{E}_7 \oplus 2\mathbf{A}_2$	1	1	
$2\mathbf{E}_8 \oplus 2\mathbf{A}_1$	1^1	1	see §A.15

1.2. **Contents of the paper.** In §2 we build the necessary algebraic/arithmetic framework for the deformation classification of real sextics. Upon introducing both complex and real versions of the so-called lattice and homological types, in §2.5 and §2.6 we discuss their easily comprehensible and computable geometric invariants.

In §3 we prove the principal results of the paper: Theorem 1.3 is proved first, and Theorem 1.1 is derived therefrom by a computer-aided computation.

In Appendix A, we describe an explicite geometric construction for the majority of special (see §2.5.1) empty sextics by means of the double covering $p\colon \mathbb{P}^2 \dashrightarrow \Sigma_2$ and trigonal curves in the Hirzebruch surface Σ_2.

In Appendix B, we work out a couple of examples illustrating the computation leading to the proof of Theorem 1.1.

1.3. **Acknowledgement.** This paper was conceived and most of its results were obtained during our joint research stay at the *Max-Planck-Institut für Mathematik*, Bonn. We are grateful to this institution and its friendly staff for the hospitality and excellent working conditions.

2. Lattice types and homological types

In this section, we describe the complex and real versions of the so-called lattice and homological types of a (real) simple plane sextic curve. Intuitively, at least in the complex setting, the lattice type captures algebro-geometric properties of the sextics, whereas the homological type also takes into account the topology of the ground field $\mathbb{C}$.

In the last two subsections, we introduce the invariants of lattice/homological types used in Tables 1–4 (see Convention 2.13).

Table 2. The case $\mu = 16$ (see Convention 2.13)

S	d	n	Remarks
$8\mathbf{A}_2$	3^1	1	(1,1,1); see (A.14)
$2\mathbf{A}_3 \oplus 4\mathbf{A}_2 \oplus 2\mathbf{A}_1$	1	1	
$4\mathbf{A}_3 \oplus 4\mathbf{A}_1$	2	1	(1,1); from $2\mathbf{D}_6 \oplus 2\mathbf{A}_3$, see §A.11.1
—	4^1	4	[4]; see (A.16)–(A.18)
$4\mathbf{A}_3 \oplus 2\mathbf{A}_2$	2	1	from $6\mathbf{A}_3$; see §A.4.2
$2\mathbf{A}_4 \oplus 2\mathbf{A}_2 \oplus 4\mathbf{A}_1$	1	1	
$2\mathbf{A}_4 \oplus 4\mathbf{A}_2$	1	1	
$2\mathbf{A}_4 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_1$	1	1	
$4\mathbf{A}_4$	1	1^{2*}	
—	5^1	2	[3]; see §A.5
$2\mathbf{A}_5 \oplus 6\mathbf{A}_1$	2	1	(1,1); from $2\mathbf{A}_5 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_1$
$2\mathbf{A}_5 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$	1	1	
—	3	2	[2]; from $2\mathbf{A}_5 \oplus 4\mathbf{A}_2$, see §A.3.1
—	2	1	from $2\mathbf{A}_5 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_1$
—	6^1	2	[2]; see §A.6
$2\mathbf{A}_5 \oplus 2\mathbf{A}_3$	1	1	
$2\mathbf{A}_6 \oplus 4\mathbf{A}_1$	1	1	
$2\mathbf{A}_6 \oplus 2\mathbf{A}_2$	1	1	
$2\mathbf{A}_7 \oplus 2\mathbf{A}_1$	1	1^{2*}	
—	2	1	from $2\mathbf{D}_8 \oplus 2\mathbf{A}_1$
—	4^1	2^2	[2]; see §A.7
$2\mathbf{A}_8$	1	1	
—	3^1	2	[2]; see (A.22)
$2\mathbf{D}_4 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_1$	2	1	(1,1)($\mathbf{A}_3^{\mathrm{i}}$); from $2\mathbf{D}_6 \oplus 2\mathbf{A}_3$, see §A.11.1
$2\mathbf{D}_4 \oplus 2\mathbf{A}_4$	1	1	
$4\mathbf{D}_4$	2^3	2	[1]; see §A.9
$2\mathbf{D}_5 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$	1	1	
$2\mathbf{D}_5 \oplus 2\mathbf{A}_3$	1	1	
—	2	1	from $2\mathbf{D}_6 \oplus 2\mathbf{A}_3$
—	4^1	2	[2]; see §A.10
$2\mathbf{D}_6 \oplus 4\mathbf{A}_1$	2^1	2	[1]($\mathbf{D}_6^{\mathrm{ii}}$); see (A.25)–(A.26)
—	2	1	(1,1)($\mathbf{D}_6^{\mathrm{i}}$); from $2\mathbf{D}_8 \oplus 2\mathbf{A}_1$, see §A.12.1
$2\mathbf{D}_6 \oplus 2\mathbf{A}_2$	1	1	
$2\mathbf{D}_7 \oplus 2\mathbf{A}_1$	1	1	
$2\mathbf{D}_8$	1	1	
—	2^1	1^2	see (A.28)–(A.29)
$2\mathbf{E}_6 \oplus 4\mathbf{A}_1$	1	1	
$2\mathbf{E}_6 \oplus 2\mathbf{A}_2$	1	1	
—	3^1	2	[2]; see §A.13
$2\mathbf{E}_7 \oplus 2\mathbf{A}_1$	1	1	
—	2^1	1	see §A.14
$2\mathbf{E}_8$	1^1	1	see §A.15

Table 3. The case $\mu = 14$ (see Convention 2.13)

S	d	n	Remarks
$4\mathbf{A}_2 \oplus 6\mathbf{A}_1$	1	1	
$6\mathbf{A}_2 \oplus 2\mathbf{A}_1$	1	1	
—	3	2	[2]; from $8\mathbf{A}_2$, see §A.3.1
$2\mathbf{A}_3 \oplus 8\mathbf{A}_1$	2	2	[1]($\mathbf{A}_3^{\mathrm{ii}}$); from $2\mathbf{D}_6 \oplus 4\mathbf{A}_1$, see §A.11.2
—	2	1	(1,1)($\mathbf{A}_3^{\mathrm{i}}$); from $2\mathbf{D}_6 \oplus 2\mathbf{A}_3$, see §A.11.1
$2\mathbf{A}_3 \oplus 2\mathbf{A}_2 \oplus 4\mathbf{A}_1$	1	1	
—	2	1	from $6\mathbf{A}_3$; see §A.4.2
$2\mathbf{A}_3 \oplus 4\mathbf{A}_2$	1	1	
$4\mathbf{A}_3 \oplus 2\mathbf{A}_1$	1	1	
—	2	1	from $4\mathbf{A}_3 \oplus 4\mathbf{A}_1$ (A.17); see §A.4.1
—	4	2	[2]; from $4\mathbf{A}_3 \oplus 4\mathbf{A}_1$ (A.16), see §A.4.1
$2\mathbf{A}_4 \oplus 6\mathbf{A}_1$	1	1	
$2\mathbf{A}_4 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$	1	1	
$2\mathbf{A}_4 \oplus 2\mathbf{A}_3$	1	1	
$2\mathbf{A}_5 \oplus 4\mathbf{A}_1$	1	1	
—	2	1	from $2\mathbf{A}_5 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$; see §A.6.1
$2\mathbf{A}_5 \oplus 2\mathbf{A}_2$	1	1	
—	3	2	[2]; from $2\mathbf{A}_5 \oplus 4\mathbf{A}_2$, see §A.3.1
$2\mathbf{A}_6 \oplus 2\mathbf{A}_1$	1	1	
$2\mathbf{A}_7$	1	1	
—	2	1^2	from $2\mathbf{A}_7 \oplus 4\mathbf{A}_1$; see §A.4.2
$2\mathbf{D}_4 \oplus 6\mathbf{A}_1$	2	2	[1]; from $2\mathbf{D}_6 \oplus 4\mathbf{A}_1$, see §A.11.2
$2\mathbf{D}_4 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$	1	1	
$2\mathbf{D}_4 \oplus 2\mathbf{A}_3$	1	1	
—	2	1^2	from $2\mathbf{D}_6 \oplus 2\mathbf{A}_3$; see §A.11.1
$2\mathbf{D}_5 \oplus 4\mathbf{A}_1$	1	1	
—	2	1	from $2\mathbf{D}_6 \oplus 4\mathbf{A}_1$; see §A.11.3
$2\mathbf{D}_5 \oplus 2\mathbf{A}_2$	1	1	
$2\mathbf{D}_6 \oplus 2\mathbf{A}_1$	1	1	
—	2	1	from $2\mathbf{D}_6 \oplus 4\mathbf{A}_1$; see §A.11.3
$2\mathbf{D}_7$	1	1	
$2\mathbf{E}_6 \oplus 2\mathbf{A}_1$	1	1	
$2\mathbf{E}_7$	1	1	

2.1. **Complex lattice types.** Recall that, given a simple sextic C in $\mathbb{P}^2$, the minimal resolution $X := X_C$ of singularities of the double covering of $\mathbb{P}^2$ ramified at C is a $K3$-surface; the covering projection is denoted by $\pi\colon X \to \mathbb{P}^2$. *Via* the Poincaré duality isomorphism, we always identify $H^2(X) = H_2(X)$ (unless stated otherwise, all coefficients are in $\mathbb{Z}$) and regard the latter as a unimodular lattice:

$$H_2(X) \simeq \mathbf{L} := 2\mathbf{E}_8 \oplus 3\mathbf{U}.$$

TABLE 4. The case $\mu \leqslant 12$ (see Convention 2.13)

S	d	n	Remarks
$12\mathbf{A}_1$	2	2	[1]; see §A.11.2
$2\mathbf{A}_2 \oplus 8\mathbf{A}_1$	1	1	
—	2	1	see §A.11.3
$4\mathbf{A}_2 \oplus 4\mathbf{A}_1$	1	1	
$6\mathbf{A}_2$	1	1	
—	3	2	[2]; see §A.3.1
$2\mathbf{A}_3 \oplus 6\mathbf{A}_1$	1	1	
—	2	1	see §A.11.1
$2\mathbf{A}_3 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$	1	1	
$4\mathbf{A}_3$	1	1	
—	2	1^2	see §A.4.2
$2\mathbf{A}_4 \oplus 4\mathbf{A}_1$	1	1	
$2\mathbf{A}_4 \oplus 2\mathbf{A}_2$	1	1	
$2\mathbf{A}_5 \oplus 2\mathbf{A}_1$	1	1	
—	2	1	see §A.6.1
$2\mathbf{A}_6$	1	1	
$2\mathbf{D}_4 \oplus 4\mathbf{A}_1$	1	1	
—	2	1	see §A.11.3
$2\mathbf{D}_4 \oplus 2\mathbf{A}_2$	1	1	
$2\mathbf{D}_5 \oplus 2\mathbf{A}_1$	1	1	
$2\mathbf{D}_6$	1	1	
$2\mathbf{E}_6$	1	1	
$10\mathbf{A}_1$	1	1	
—	2	1	see §A.11.2
$2\mathbf{A}_2 \oplus 6\mathbf{A}_1$	1	1	
$4\mathbf{A}_2 \oplus 2\mathbf{A}_1$	1	1	
$2\mathbf{A}_3 \oplus 4\mathbf{A}_1$	1	1	
—	2	1	see §A.11.1
$2\mathbf{A}_3 \oplus 2\mathbf{A}_2$	1	1	
$2\mathbf{A}_4 \oplus 2\mathbf{A}_1$	1	1	
$2\mathbf{A}_5$	1	1	
$2\mathbf{D}_4 \oplus 2\mathbf{A}_1$	1	1	
$2\mathbf{D}_5$	1	1	
$8\mathbf{A}_1$	1	1	
—	2	1	see §A.11.2
$2\mathbf{A}_2 \oplus 4\mathbf{A}_1$	1	1	
$4\mathbf{A}_2$	1	1	
$2\mathbf{A}_3 \oplus 2\mathbf{A}_1$	1	1	
$2\mathbf{A}_4$	1	1	
$2\mathbf{D}_4$	1	1	
$6\mathbf{A}_1$	1	1	
$2\mathbf{A}_2 \oplus 2\mathbf{A}_1$	1	1	
$2\mathbf{A}_3$	1	1	
$4\mathbf{A}_1$	1	1	
$2\mathbf{A}_2$	1	1	
$2\mathbf{A}_1$	1	1	

Throughout the paper, $\mathbf{A}_n$, $\mathbf{D}_n$, $\mathbf{E}_n$ are *negative definite* root lattices corresponding to the Dynkin diagrams of the same name, and $\mathbf{U}$ is the *hyperbolic plane*:

$$\mathbf{U} = \mathbb{Z}u_1 + \mathbb{Z}u_2, \qquad u_1^2 = u_2^2 = 0, \quad u_1 \cdot u_2 = 1.$$

The classes of the exceptional divisors over the singularities of C span a root lattice $S \subset H_2(X)$. In view of the uniqueness of the decomposition of S into irreducible **A**–**D**–**E** components, this lattice can be identified with the (abstract) set of singularities of C. Let $h := h_C \in H_2(X)$ be the *polarization*, *i.e.*, the class of the pull-back of a line in $\mathbb{P}^2$. One has $h^2 = 2$. We consider the lattices

$$S_h := S \oplus \mathbb{Z}h, \quad \tilde{S}_h := \text{the primitive hull of } S_h \text{ in } H_2(X), \quad \tilde{S} := h^\perp \subset \tilde{S}_h.$$

The 2-polarized hyperbolic lattice $\tilde{S}_h \ni h$ (regarded up to polarized isometry) is called the (*complex*) *lattice type* of the original sextic C, *cf.* [28]. (Here and below, a *polarized isometry* φ is one preserving h, *i.e.*, $\varphi(h) = h$. If $\varphi(h) = -h$, we refer to φ as a *skew-polarized isometry*.) If X is very general in its equisingular family, $\tilde{S}_h$ is the Néron-Severi lattice $NS(X)$ of X. The Riemann-Roch theorem implies that, in any case, S is recovered as the maximal root lattice in $h^\perp \subset NS(X)$.

The next few statements are well known (see, *e.g.*, [7, 28]) and follow from the global Torelli theorem [26] and surjectivity of the period map [21] for $K3$-surfaces, as well as results of B. Saint-Donat [27] and V. Nikulin [25]; see also [5, 14].

Theorem 2.1. *A 2-polarized lattice $\tilde{S}_h \ni h$ represents a complex lattice type if and only if its isomorphism class is an* abstract (complex) lattice type*, i.e.,*

1. *$\tilde{S}_h$ is hyperbolic and admits a primitive isometry into $\mathbf{L}$;*
2. *$\tilde{S}_h$ is generated over $\mathbb{Q}$ by h and the roots in $h^\perp \subset \tilde{S}_h$;*
3. *there is no element $e \in \tilde{S}_h$ such that $e^2 = 0$ and $e \cdot h = 1$.*

2.2. **Real lattice types.** Let $C \subset \mathbb{P}^2$ be a simple real sextic and $\pi\colon X \to \mathbb{P}^2$ the covering $K3$-surface. The real structure σ on $\mathbb{P}^2$ lifts to two commuting real structures $\sigma^\pm\colon X \to X$, so that the composition $\sigma^+ \circ \sigma^-$ is the deck translation τ of the covering. The two lifts $\sigma^\pm$ are distinguished by the projections $\pi(\operatorname{Fix}\sigma^\pm) \subset \mathbb{P}^2_{\mathbb{R}}$ of their real parts; these projections, called the *halves* of $\mathbb{P}^2_{\mathbb{R}}$, are disjoint except for the common boundary $C_{\mathbb{R}}$.

In this paper, we consider real sextics with smooth (possibly empty) real part $C_{\mathbb{R}}$. In this case, exactly one of the two halves of $\mathbb{P}^2_{\mathbb{R}}$ is orientable, and the corresponding lift is denoted by σ^+. The isomorphism class of $(\tilde{S}_h \ni h, \sigma_S)$, where $\sigma_S\colon \tilde{S}_h \to \tilde{S}_h$ is the restriction of $\sigma^+_*\colon H_2(X) \to H_2(X)$, is called the *real lattice type* of C. Alternatively, the conjugacy classes, in the group of (skew-)polarized isometries of $\tilde{S}_h \ni h$, of involutive skew-polarized isometries σ_S as above are called the *real forms* of the complex lattice type $\tilde{S}_h \ni h$.

An obvious condition necessary for geometric realizability of a real form by a real sextic with smooth real part is that σ_S should not fix, as a set, any of the $\mathbf{A}$–$\mathbf{D}$–$\mathbf{E}$ components of S.

2.3. **Complex homological types.** To a simple sextic $C \subset \mathbb{P}^2$ we can associate its (*complex*) *homological type*, *i.e.*, the triple

$$H_2(X) \supset \tilde{S}_h \ni h$$

considered up to isometry. An *abstract* (*complex*) *homological type* is an isometry class of triples $\mathbf{L} \supset \tilde{S}_h \ni h$, where $\tilde{S}_h \ni h$ is an abstract complex lattice type (see Theorem 2.1) and $\tilde{S}_h$ is primitive in $\mathbf{L}$.

The generic *transcendental lattice* $T := \tilde{S}_h^\perp \subset \mathbf{L}$ has two positive squares; hence, all positive definite 2-subspaces in $T \otimes \mathbb{R}$ can be oriented in a coherent way. A choice of one of these two coherent orientations is called an *orientation* of the abstract homological type. The homological type of a sextic has a canonical orientation, *viz.* the one given by $\mathbb{R}\operatorname{Re}\omega \oplus \mathbb{R}\operatorname{Im}\omega$, where $\omega \neq 0$ is a holomorphic 2-form on X.

Theorem 2.2 (see [7])**.** *The equisingular deformation families of simple sextics are in a canonical bijection with the oriented abstract homological types.*

2.4. **Real homological types.** The *real homological type* of a real sextic $C \subset \mathbb{P}^2$ with smooth real part is the quadruple

$$\bigl(H_2(X) \supset \tilde{S}_h \ni h, \sigma^+_*\bigr) \tag{2.3}$$

considered up to isometry commuting with σ^+_*. Here, the orientation of the complex homological type is redundant, as $-\sigma^+_*$ is an orientation reversing isometry.

In this paper, we are particularly interested in empty sextics.

Theorem 2.4. *A quadruple $(\mathbf{L} \supset \tilde{S}_h \ni h, \theta)$ represents the real homological type of an empty sextic if and only if*

1. $\mathbf{L} \supset \tilde{S}_h \ni h$ *is an abstract complex homological type,*
2. θ *is an involutive skew-polarized isometry of* $\mathbf{L} \supset \tilde{S}_h \ni h$*, and*
3. $\operatorname{Ker}(1-\theta) \simeq \mathbf{E}_8(2) \oplus \mathbf{U}(2)$.

Proof. The first two requirements are obvious, whereas the necessity of item (3) follows from the deformation classification of smooth real sextics found in [24]. Conversely, in view of Theorem 2.2, the sufficiency is given by the construction of anti-holomorphic maps (see, *e.g.*, [4] or [12, §13.4.3] for an explanation in the anti-holomorphic setup); since the real structure obtained preserves the polarisation, it automatically commutes with the deck translation. □

Definition 2.5. If the real sextic $C \subset \mathbb{P}^2$ is empty, there is a distinguished sphere $\operatorname{Fix}\sigma^- \subset X$ disjoint from all exceptional divisors. Choosing an orientation, we obtain a class $s := [\operatorname{Fix}\sigma^-] \in H_2(X)$, called the *empty sphere*. It is immediate that s is σ_*^+-skew-invariant, and

$$s^2 = -2, \quad s \cdot \tilde{S}_h = 0, \quad s = h \bmod 2H_2(X);$$

since $h^2 = 2$, the primitive hull $U_{h,s}$ of $\mathbb{Z}h \oplus \mathbb{Z}s$ in $H_2(X)$ is isomorphic to $\mathbf{U}$ and, thus, splits as an orthogonal direct summand of $H_2(X)$, orthogonal to $\tilde{S} = h^\perp \subset \tilde{S}_h$.

Corollary 2.6. *Let $\tilde{S}_h \ni h$ be a complex lattice type and $\theta_S \colon \tilde{S}_h \to \tilde{S}_h$ an involutive skew-polarized isometry, and let θ_h be the restriction of θ_S to $\tilde{S}$. If $(\tilde{S}_h \ni h, \theta_S)$ represents the real lattice type of an empty sextic, then*

$$\text{both } \tilde{S}_\pm := \operatorname{Ker}(1 \mp \theta_h) \text{ admit primitive isometries into } \mathbf{E}_8(2) \oplus \mathbf{U}(2), \tag{2.7}$$

$$\mathbb{Z}h \text{ is an orthogonal direct summand in } \tilde{S}_h; \tag{2.8}$$

here and below, given a lattice $M := (M, b)$, where b is the bilinear form on M, the notation $M(2)$ stands for the lattice $(M, 2b)$.

Proof. If θ_S extends to an empty real structure $\theta = \sigma_*^+$, by Theorem 2.4 we have

$$\begin{aligned} L_+ &:= \operatorname{Ker}(1-\sigma_*^+) \simeq \mathbf{E}_8(2) \oplus \mathbf{U}(2), \\ L_- &:= \operatorname{Ker}(1+\sigma_*^+) \simeq \mathbf{E}_8(2) \oplus \mathbf{U}(2) \oplus \mathbf{U}, \end{aligned} \tag{2.9}$$

and statement (2.7) is immediate for the primitive sublattice $\tilde{S}_+ \subset L_+$. For $\tilde{S}_-$, we observe in addition that it is orthogonal to $U_{h,s} \subset L_-$, and $U_{h,s}^\perp \subset L_-$ equals $\mathbf{E}_8(2) \oplus \mathbf{U}(2)$ in view of the uniqueness of the latter lattice in its genus. □

Remark 2.10. As an immediate consequence of Corollary 2.6, we have $\tilde{S}_\pm = \tilde{S}_\pm^\circ(2)$ for some even lattices $\tilde{S}_\pm^\circ$, and (2.7) reduces to the existence of primitive isometries $\tilde{S}_\pm^\circ \hookrightarrow \mathbf{E}_8 \oplus \mathbf{U}$, which is easily checked using Nikulin's theory [24]. *A posteriori*, our Theorem 1.1 implies that the necessary condition (2.7) is also sufficient, provided that the existence of the complex family is known. In the case of non-special sextics (see §2.5.1), this fact has a simple direct proof, see Theorem B.1 below.

2.5. **Invariants of complex lattice types.** Fix a complex lattice type $\tilde{S}_h \ni h$ and pick a representative sextic $C \subset \mathbb{P}^2$. The following objects are invariants of $\tilde{S}_h \ni h$.

2.5.1. The *kernel* of the extension, *i.e.*, the (finite) isotropic subgroup

$$\tilde{S}_h/S_h \subset \operatorname{discr} S_h := S_h^\vee/S_h.$$

In a sense, given S, this kernel *is* the lattice type; however, we confine ourselves to the exponent (maximal order of elements)

$$d := \exp(\tilde{S}_h/S_h)$$

and use a number of more geometric/numeric derivatives described below. The sextic is reducible if and only if $2 \mid d$, see [7]. Sextics with $3 \mid d$ are often said to be of *torus type*. In general, a sextic is called *special* if $d > 1$.

2.5.2. The *combinatorial type*, *i.e.*, the homeomorphism class of the pair (tub_C, C), where tub_C is a regular tubular neighbourhood of C. Roughly, this consists of (the degrees of) the irreducible components of C, its set of singularities S, and the position of (the branches of) the singular points of C on its components.

If $C_\mathbb{R}$ is smooth, all components of C must be of even degree; hence, these degrees (6), $(4, 2)$, or $(2, 2, 2)$ are determined by the number $c_\mathbb{C}$ of conic components.

2.5.3. The *alignment*, which is a partial description of the "position of the singular points on the components" mentioned in the previous section. Assume that $c_\mathbb{C} = 3$ and consider a singular point P of type $\mathbf{A}_{2p+1}$ or $\mathbf{D}_{2p+4}$, $p \geqslant 1$. Then, two of the three conic components of C are tangent to each other at P; they are called *tight* at P, whereas the third conic is called *loose*. Now, assume that C has exactly two points P_1, P_2 of the same type $\mathbf{A}_{2p+1}$ or $\mathbf{D}_{2p+4}$, $p \geqslant 1$. If $p = 1$, there are two possibilities, *viz.*

(i) the points are *misaligned*, *i.e.*, the loose conics at P_1, P_2 are distinct, or
(ii) the points are *aligned*, *i.e.*, P_1, P_2 have a common loose conic,

which are usually indicated *via*, *e.g.*, $\mathbf{A}_3^{\mathrm{i}}$ or $\mathbf{D}_6^{\mathrm{ii}}$. If $p > 1$, the pair is automatically misaligned, as in (i), and this property is obviously preserved under perturbations (provided that the new curve still splits into three conics).

If $c_\mathbb{C} = 1$, one can also consider the set of singularities of the quartic component, but we use this extra piece of data only once, in Remark 2.15 below.

2.5.4. Splitting conics and lines. A conic or line $B \subset \mathbb{P}^2$ is *splitting* (*Z-splitting* in [28]) for C if

- the pull-back $\pi^{-1}(B)$ splits into two smooth rational curves B', B'', and
- the classes $[B']$, $[B'']$ are distinct and lie in $\tilde{S}_h \subset H_2(X)$.

The latter condition ensures that the splitting conics and lines are stable in the sense that they follow all equisingular deformations of C.

Theorem 2.11 (Shimada [28]). *A complex lattice type is determined by its combinatorial type and the numbers of splitting lines and conics.*

In view of (2.8), an empty sextic cannot have splitting lines; therefore, it suffices to consider the number $s_\mathbb{C}$ of its splitting conics.

Remark 2.12. Whenever (2.8) holds, both conic components and splitting conics are found using vectors $v \in \tilde{S}/S$ whose shortest representative in $\tilde{S}$ has square -4. The conic components are in a bijection with such vectors of order 2 (equivalently, invariant under the deck translation), whereas splitting conics correspond to pairs of opposite vectors of any larger order (equivalently, 2-element orbits of the deck translation); in the latter case, the individual vectors are in a bijection with the pull-backs of the splitting conics in X. Splitting conics of order 3 are often referred to as *torus structures*; if $\{f=0\}$ is such a conic, the equation of C is $f^3+g^2=0$ for some cubic polynomial g.

As a consequence, the presence of a splitting conic implies that $d>2$.

2.5.5. The group $\operatorname{Sym}(\tilde{S}_h \ni h)$ of *stable symmetries*, *i.e.*, the subgroup of $O_h(\tilde{S}_h)$ preserving the exceptional divisors (as a set) and acting identically on $\operatorname{discr}\tilde{S}_h$. If C is generic, so that $\tilde{S}_h$ is the Néron-Severi lattice of X, this is indeed the group of symplectic automorphisms of X commuting with the deck translation. These automorphisms descend to $\mathbb{P}^2$ and are stable in the sense that they follow all equisingular deformations of C; we refer to [8] for further details.

We are mainly interested in nontrivial *stable involutions*; the number of such involutions is denoted by $m_{\mathbb{C}}$.

2.6. **Invariants of real forms.** Given a real form σ_S of a complex lattice type $\tilde{S}_h \ni h$, the invariants introduced in §2.5 have the following real refinements.

The conic count $c_{\mathbb{C}}$ splits into $c_{\mathbb{R}} := (r_{\mathrm{c}}, c_{\mathrm{c}})$, where

- r_{c} is the number of real conic components and
- c_{c} is the number of pairs of complex conjugate ones,

so that $c_{\mathbb{C}} = r_{\mathrm{c}} + 2c_{\mathrm{c}}$. Similarly, we let $s_{\mathbb{R}} := (r_{\mathrm{s}}, c_{\mathrm{s}}, q_{\mathrm{s}})$, where

- r_{s} is the number of real splitting conics whose pull-backs in X are also real,
- c_{s} is the number of real splitting conics whose pull-backs in X are complex conjugate to each other, and
- q_{s} is the number of pairs of complex conjugate splitting conics,

so that $s_{\mathbb{C}} = r_{\mathrm{s}} + c_{\mathrm{s}} + 2q_{\mathrm{s}}$.

Besides, we have the group $\operatorname{Sym}_{\mathbb{R}}(\tilde{S} \ni h, \sigma_S)$ of equivariant stable symmetries and the number $m_{\mathbb{R}}$ of equivariant nontrivial stable involutions, *cf.* §2.5.5.

Now, we are ready to describe the data presented in Tables 1–4.

Convention 2.13. The rows of Tables 1–4 list all complex lattice types $\tilde{S}_h \ni h$, $S \neq 0$, admitting at least one real form represented by an empty sextic.

The S-column refers to the set of singularities S.

The d-column shows the exponent $d = \exp(\tilde{S}_h/S_h)$ (see §2.5.1) and the number $m_{\mathbb{C}} = m_{\mathbb{R}}$ (if greater than 0) of stable involutions (see §2.5.5) as a superscript.

The n-column shows the number n of real forms of $\tilde{S}_h \ni h$ and the number of deformation families (if greater than 1) *per real form* as a superscript; the latter is followed by a $*$ if all real homological types (within one real form) share the same equivariant transcendental lattice (T, σ_T).

Finally, in the remark column, we describe the invariants of the real forms:

- $c_{\mathbb{R}}$ (if $n=1$ and $c_{\mathbb{C}}=3$), as a two element list $(*,*)$,
- $s_{\mathbb{R}}$ (if $n=1$ and $s_{\mathbb{C}}>0$), as a three element list $(*,*,*)$, and
- the alignment (see §2.5.3), whenever applicable.

(To save space, we omit the labels $c_{\mathbb{R}}$, $s_{\mathbb{R}}$.) If $n>1$, we encounter but the four cases below, and the multiple values of $c_{\mathbb{R}}$ and $s_{\mathbb{R}}$ are replaced with a reference [1]–[4]:

(1) one has $s_{\mathbb{C}}=0$ and the two real forms differ by $c_{\mathbb{R}}=(1,1)$ or $(3,0)$;
(2) one has $c_{\mathbb{C}}\leqslant 1$ and the two real forms differ by $s_{\mathbb{R}}=(0,1,0)$ or $(1,0,0)$;
(3) one has $c_{\mathbb{C}}=0$ and the two real forms differ by $s_{\mathbb{R}}=(0,2,0)$ or $(2,0,0)$;

(4) there are four real forms, one with $c_{\mathbb{R}} = (1, 1)$ and $s_{\mathbb{R}} = (1, 1, 0)$ and three with $c_{\mathbb{R}} = (3, 0)$ and $s_{\mathbb{R}} = (0, 0, 1)$, $(0, 2, 0)$, or $(2, 0, 0)$.

In the same column, we explain the construction of real curves, by a perturbation (see Remark 2.14 below) from a larger set of singularities S' (*via* "from S'") and/or by a reference to Appendix A or Appendix B. The common reference for most non-special ($d = 1$) curves is §B.1.

Remark 2.14. Another well-known consequence of the standard theory of $K3$-surfaces [26, 21, 27] is the fact that perturbations of simple sextics are unobstructed. Literally the same argument shows that the statement holds in the equivariant setting as well. More precisely, let (2.3) be the real homological type of a real sextic with smooth real part. (The last condition is not essential, but then one would have to make a choice between $\sigma^{\pm}$.) Pick a primitive σ_*^+-invariant root sublattice $S' \subset S$, and denote by $\tilde{S}'_h$ the primitive hull of $S' \oplus \mathbb{Z}h \subset \tilde{S}_h$. Then, there is a family of real sextics C_t, $t \in [0, 1]$, such that $C_0 = C$ and the real homological type of each C_t, $t \in (0, 1]$, is $\bigl(H_2(X) \supset \tilde{S}'_h \ni h, \sigma_*^+\bigr)$.

Remark 2.15. *A posteriori*, we conclude that the chosen invariants do completely describe our results.

First, whenever a complex lattice type has several real forms, these forms are distinguished by the pairs $(c_{\mathbb{R}}, s_{\mathbb{R}})$, see (1)–(4) above.

Second, for the sets of singularities *listed in the tables* and complex lattice types without components of odd degree, the invariants $(d, c_{\mathbb{C}}, s_{\mathbb{C}})$ and the alignment (i) *vs.* (ii) in §2.5.3 (whenever applicable) almost single out those admitting at least one empty real structure. The two exceptions are the sets of singularities $4\mathbf{A}_3 \oplus 2\mathbf{A}_1$ and $2\mathbf{A}_3 \oplus 6\mathbf{A}_1$, both with $c_{\mathbb{C}} = 1$: in addition, we obviously need to exclude the complex lattice types where the quartic component has a single type $\mathbf{A}_3$ point.

Finally, we observe that, for all empty sextics, one has $\mathrm{Sym}_{\mathbb{R}} = \mathrm{Sym}$.

3. Deformation classification

In general, it is not true (*cf.* [16]) that the equisingular equivariant deformation class of a real sextic $C \subset \mathbb{P}^2$ is determined by its real homological type. It is, however, true in the special case where C has no real singular points (*cf.* [18] for nodal sextics); this fact and its implications for empty real sextics are the principal results of this section.

3.1. **Proof of Theorem 1.3.** Fix a quadruple

$$(\mathbf{L} \supset \tilde{S}_h \ni h, \theta) \tag{3.1}$$

and assume that it represents the real homological type of a real sextic without real singular points. Fix a $(-\theta)$-invariant Weyl chamber Δ of $h^\perp \subset \tilde{S}_h$, which we regard as a distinguished set of vectors in $h^\perp$. Denote by $S := \mathbb{Z}\Delta$ the sublattice generated by the roots in $h^\perp$.

Put $L_{\pm} = \mathrm{Ker}(1 \mp \theta)$ and consider the

- hyperbolic lattices $P_{\pm} := (\tilde{S}_h \cap L_{\pm})^\perp \subset L_{\pm}$,
- projectivized positive cones $\mathcal{C}_{\pm} := \bigl\{x \in P_{\pm} \otimes \mathbb{R} \bigm| x^2 = 1\bigr\}$, and
- walls $H_v := \bigl\{(x_+, x_-) \in \mathcal{C}_+ \times \mathcal{C}_- \bigm| x_+ \cdot v = x_- \cdot v = 0\bigr\}$, where $v \in \mathbf{L}$ is a vector such that $v^2 = -2$ and $v \cdot h = 0$.

A *marking* of a real sextic curve $C \subset \mathbb{P}^2$ as above is an isometry $\psi\colon H_2(X_C) \to \mathbf{L}$ such that

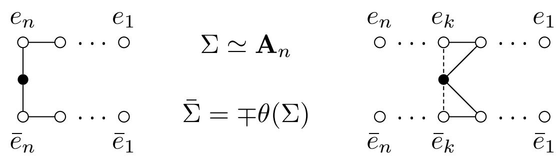


FIGURE 1. Extended graphs D_v

- ψ is real, *i.e.*, $\psi \circ \sigma_*^+ = \theta \circ \psi$,
- $\psi(h_C) = h$, and
- ψ establishes a bijection between the exceptional divisors of X_C and Δ.

The *period space* of marked real sextics is the space

$$\Omega := \bigl(\mathcal{C}_+ \times \mathcal{C}_- \smallsetminus \bigcup H_v\bigr)/\pm 1,$$

and the *period map* sends a sextic C to the class of $\psi(\omega_+, \omega_-)$, where $\omega_+ = \operatorname{Re}\omega$, $\omega_- = \operatorname{Im}\omega$, and ω is a non-zero real (in the sense that $\sigma_*^+(\omega) = \bar\omega$) holomorphic 2-form on X_C. The period map makes Ω a fine moduli space of marked real simple sextics with the given real homological type (3.1): in view of [5] and [27], this fact follows from the construction of anti-holomorphic maps, as explained in the proof of Theorem 2.4. Certainly, this space is disconnected, and the rest of the proof deals with showing that the connected components of Ω constitute a single orbit of the automorphism group of (3.1); in other words, any component can be sent to any other by a change of the marking.

First, observe that each of $\mathcal{C}_\pm$ has two connected components. Hence, so does $(\mathcal{C}_+ \times \mathcal{C}_-)/\pm 1$, and these two components are interchanged by $-\theta$.

Now, consider a wall H_v for some $v \in \mathbf{L} \smallsetminus S$, $v^2 = -2$, $v \cdot h = 0$. It is obvious that $\operatorname{codim} H_v \geqslant 2$ unless $v \in L_\pm$, and $H_v = \varnothing$ unless

$$\text{the lattice } S + \mathbb{Z}v \text{ is negative definite.} \tag{3.2}$$

Therefore, from now on we assume that $v \in L_\pm$ and (3.2) holds. In particular, the latter implies that $|v \cdot e| \leqslant 1$ for each $e \in \Delta$. To complete the proof, we need to find, for any such vector v, an automorphism of (3.1) that would interchange pairs of components adjacent to H_v (more precisely, any pair of components whose closures share a common very general point of H_v).

If $v \cdot e = 0$ for each vector $e \in \Delta$, the reflection

$$r_v \colon x \mapsto x + (x \cdot v)v \tag{3.3}$$

is the desired automorphism.

In general, we depict the union $\Delta \cup v$ by an analog D_v of Dynkin diagram in which dotted edges $[e_1, e_2]$ (corresponding to the value $e_1 \cdot e_2 = -1$) are allowed. To minimize the number of such edges, we occasionally change the signs of the vectors. In particular, assuming that $v \in L_\pm$, we let $\bar v := v$ and $\bar e := \mp\theta(e)$ for $e \in S$, so that $[\bar e_1, \bar e_2]$ is always an edge of the same type (solid, empty, or dotted) as $[e_1, e_2]$. Due to the assumption that C has no real singular points, the Dynkin diagram of Δ itself is a disjoint union of trees (of types $\mathbf{A}$–$\mathbf{D}$–$\mathbf{E}$) split into pairs exchanged by the complex conjugation; thus, changing, if necessary (*i.e.*, if $v \in L_+$), the signs in half of the components, we can assume that these pairs are of the form $\Sigma, \bar\Sigma$.

Observe that, in any induced *subtree* $T \subset D_v$, the signs can be changed so that T has no dotted edges; then, by (3.2), T must be an ordinary simply laced Dynkin diagram. We conclude that v is adjacent to at most one pair of components $\Sigma, \bar{\Sigma}$, as otherwise D_v would contain $\tilde{\mathbf{D}}_4$. Likewise, if v is adjacent to a pair of vertices $e_1, e_2 \in \Sigma$ (hence, also to $\bar{e}_1, \bar{e}_2 \in \bar{\Sigma}$), then (*cf.* Figure 1, right)

- e_1 and e_2 are adjacent in Σ, as otherwise $D_v \supset \tilde{\mathbf{D}}_4$,
- $[v, e_1]$ and $[v, e_2]$ are edges of the opposite types, as otherwise $D_v \supset \tilde{\mathbf{A}}_2$,
- as a consequence, v is not adjacent to any other vertex $e \in \Sigma$.

Finally, we have $\Sigma \simeq \mathbf{A}_n$, $n \geqslant 1$, as otherwise D_v would contain a subgraph $\tilde{\mathbf{D}}_m$. Summarizing, we arrive at the two configurations shown in Figure 1. In the left figure, the union $\Sigma \cup v \cup \bar{\Sigma}$ is $\mathbf{A}_{2n+1}$, with the standard basis

$$e_1, \ldots, e_n, e_{n+1} := v, e_{n+1} := \bar{e}_n, \ldots, e_{2n+1} := \bar{e}_1.$$

In the right figure, the *lattice* $\mathbb{Z}\Sigma + \mathbb{Z}v + \mathbb{Z}\bar{\Sigma}$ is still $\mathbf{A}_{2n+1}$: we merely change the definition of

$$e_{n+1} := v - (e_k + \ldots + e_n + \bar{e}_n + \ldots + \bar{e}_k).$$

In both cases, we need an element r of the Weyl group of $\mathbf{A}_{2n+1}$ commuting with θ, preserving (as a set) the subset $\{e_1, \ldots, e_n, \mp e_{n+1}, \ldots, \mp e_{2n+1}\}$, and acting *via* -1 on its orthogonal complement. If $v \in \mathbf{L}_+$ (the "$-$" sign above), then $-r = \theta$ is the automorphism induced by the only nontrivial symmetry of the Dynkin graph $\mathbf{A}_{2n+1}$. If $v \in \mathbf{L}_-$ (the "$+$" sign), then r is

$$e_i \leftrightarrow e_{i+n+1},\ i = 1, \ldots, n, \qquad e_{n+1} \mapsto -(e_1 + \ldots + e_{2n+1}).$$

Instead of decomposing r into a product of reflections, we observe that it preserves $\sum ie_i \bmod (2n+2)\mathbf{A}_{2n+1}$. Thus, $r = \mathrm{id}$ on $\operatorname{discr} \mathbf{A}_{2n+1}$ and, therefore, r extends to $\mathbf{L}$ identically on the orthogonal complement $\mathbf{A}_{2n+1}^{\perp}$. □

3.2. **Proof of Theorem 1.1.** Theorem 1.3 reduces the proof to the enumeration of appropriate real homological types, which is done in several steps:

(1) we list all triples $(\tilde{S}_h \ni h, \theta_S)$ satisfying (2.7) and (2.8);
(2) for each triple, we list all eigenlattices $T_\pm$ of the prospective equivariant transcendental lattices (T, θ_T);
(3) for each pair (T_+, T_-), the prospective transcendental lattices (T, θ_T) are obtained as appropriate finite index extensions of $T_+ \oplus T_-$;
(4) we list the isomorphism classes of unimodular equivariant finite index extensions $(\mathbf{L}, \theta)$ of $(\tilde{S}_h \oplus T, \theta_S \oplus \theta_T)$;
(5) from the real homological types $(\mathbf{L} \supset \tilde{S}_h \ni h, \theta)$ thus obtained we select those representing empty sextics, see Theorem 2.4(3).

Step (1). We start with the known classification of complex lattice types of simple sextics [30] and select those satisfying obvious restriction, *viz.* the fact that

(3.4) the number of singular points of each type is even.

For each complex lattice type $\tilde{S}_h$ thus obtained, we list the conjugacy classes of skew-polarized involutions $\theta_S \colon \tilde{S}_h \to \tilde{S}_h$ such that

(3.5) θ_S does not fix as a set any irreducible summand of S.

Then, we select those pairs $(\tilde{S}_h, \theta_S)$ that satisfy (2.7), see Remark 2.10.

Remark 3.6. *A posteriori*, we confirm that any complex lattice type $\tilde{S}_h \ni h$ that admits an involution as in (3.5) has $\mathbb{Z}h$ as an orthogonal direct summand, *cf.* (2.8); hence, there are neither linear/cubic components nor splitting lines.

Furthermore, within each set of singularities, any two complex lattice types that admit an involution as in (3.5) are distinguished by the invariants $(d, c_\mathbb{C}, s_\mathbb{C})$ and alignment (see §2.5), whereas within each complex lattice type, the involutions satisfying (3.5) are distinguished by the invariants $(c_\mathbb{R}, s_\mathbb{R})$ introduced in §2.6. In this sense, our tables are complete.

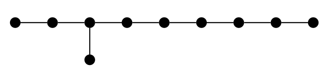
Moreover, with two exceptions, the complex lattice types admitting an involution as in (3.5) are distinguished by $(d, c_\mathbb{C}, s_\mathbb{C})$ and alignment from those not admitting one. The exceptions are the sets of singularities $4\mathbf{A}_3 \oplus 2\mathbf{A}_1$ and $2\mathbf{A}_3 \oplus 6\mathbf{A}_1$ with a single conic component passing through an *odd* number of $\mathbf{A}_3$-type points (or, in other words, a quartic component with a single $\mathbf{A}_3$-type singularity).

Remark 3.7. Another experimental observation is the fact that, for any involution satisfying (3.5), the eigenlattices $\tilde{S}_\pm$ are of the form $\tilde{S}^\circ_\pm(2)$ (*cf.* Remark 2.10), where $\tilde{S}^\circ_\pm$ are *even root lattices.*

Step (2). Let $\tilde{S}_\pm = \tilde{S}^\circ_\pm(2)$, see Remark 2.10 or Remark 3.7, pick a primitive isometry $\tilde{S}^\circ_\pm \hookrightarrow \mathbf{E}_8 \oplus \mathbf{U}$, and denote by $T^\circ_\pm$ the orthogonal complement. Then, we have $T_+ = T^\circ_+(2)$ and $T_- = T^\circ_-(2) \oplus \mathbb{Z}s$, $s^2 = -2$, see Definition 2.5. This construction determines the genera of $T_\pm$ (see, *e.g.*, [24]), and we check, on a case by case basis, that each lattice obtained is unique in its genus.

Since it is easier to work with lattices rather than their discriminant forms, we construct an isometry $\tilde{S}^\circ_\pm \hookrightarrow \mathbf{E}_8 \oplus \mathbf{U}$ by representing the Dynkin diagram of the root system of $\tilde{S}^\circ_\pm$, see Remark 3.7, as an induced subgraph of the graph

(3.8)



which is the Coxeter scheme of a fundamental polyhedron of the group generated by reflections of $\mathbf{E}_8 \oplus \mathbf{U}$, see, *e.g.*, [29].

For the uniqueness in the genus, we encounter three cases:

- definite lattices of rank 1: the uniqueness is obvious;
- hyperbolic lattices of rank 2: we are not aware of any general theory, but our needs are completely covered by the reduced form found, *e.g.*, in [13];
- hyperbolic lattices of higher rank: we use Miranda–Morrison's theory [23].

Step (3). Here and at step (4), we need to solve the following problem: given two non-degenerate even lattices M_1 and M_2, find all (up to $O(M_1) \times O(M_2)$ or a certain prescribed subgroup thereof) classes of even finite index extensions

$$\tilde{M} \supset M_1 \oplus M_2 \text{ such that both } M_1 \text{ and } M_2 \text{ are primitive in } \tilde{M}. \tag{3.9}$$

To this end, recall that a non-degenerate lattice M is naturally a subgroup of its dual lattice $M^\vee \subset M \otimes \mathbb{Q}$, and the *discriminant form* of a non-degenerate even lattice M is the finite abelian group $\operatorname{discr} M := M^\vee/M$ equipped with the induced $\mathbb{Q}/2\mathbb{Z}$-valued quadratic form (see [24], where the notation is q_M). We denote by $\mathfrak{I}_*(M)$ the image of the natural homomorphism $O_*(M) \to \operatorname{Aut}_*(\operatorname{discr} M)$, where $*$ is a placeholder for a number of extra symbols used below to restrict the groups, *e.g.*, preserving h, commuting with an involution, *etc.*

According to [24, Proposition 1.4.1], the isomorphism classes of even finite index extensions of a non-degenerate even lattice M are in a canonical bijection with the isotropic subgroups $\mathcal{G} \subset \operatorname{discr} M$. It follows (*cf.* [24, Proposition 1.5.1]) that the classes of extensions as in (3.9) are in a bijection with the double cosets

$$\mathfrak{I}(M_1)\backslash\{\psi\colon \mathcal{K} \hookrightarrow \operatorname{discr} M_2\}/\mathfrak{I}(M_2), \tag{3.10}$$

where $\mathcal{K} \subset \operatorname{discr} M_1$ is a subgroup and ψ is an injective anti-isometry, so that the isotropic subgroup $\mathcal{G} \subset \operatorname{discr} M_1 \oplus \operatorname{discr} M_2$ as above is the graph of ψ.

In the special case where M is a lattice with an involution and M_1, M_2 are the eigenlattices, $\mathcal{K}$ is a group of exponent 2. Conversely, if $\mathcal{K}$ is of exponent 2, then $\mathrm{id}_1 \oplus -\mathrm{id}_2$ extends to an involution of M, see [24, Corollary 1.5.2].

Back to the proof of Theorem 1.1, the construction of the prospective equivariant transcendental lattices T from a given pair (T_+, T_-) reduces to the computation of the subgroups $\mathfrak{I}(T_\pm) \subset \operatorname{Aut}(\operatorname{discr} T_\pm)$. At the same time, for step (4), we find the image $\mathfrak{I}_\theta(T)$ of the subgroup $O_\theta(T) \subset O(T)$ centralizing θ_T: it is the subgroup of

$$\mathfrak{I}(T_+) \times \mathfrak{I}(T_-) \subset \operatorname{Aut}(\operatorname{discr} T_+) \times \operatorname{Aut}(\operatorname{discr} T_-)$$

preserving $\mathcal{K}$ as a set and commuting with ψ.

Remark 3.11. In view of (3.12) below, the exponent 2 subgroup $\mathcal{K} \subset \operatorname{discr} T_+$ in (3.10) is subject to an extra restriction

$$|\mathcal{K}|^2 \cdot |\operatorname{discr} \tilde{S}_h| = |\operatorname{discr} T_+| \cdot |\operatorname{discr} T_-|.$$

The computation of $\mathfrak{I}(T_\pm)$ is straightforward if the eigenlattice is definite of rank 1, and is easily done using Miranda-Morrison theory [23] if the eigenlattice is hyperbolic of rank at least 3. If it is hyperbolic of rank 2, we compute $O(T_\pm)$ explicitly. Up to rescaling, we encounter but the following three classes of lattices (abbreviating $[a, b, c] := \mathbb{Z}u + \mathbb{Z}v$, where $u^2 = a, u \cdot v = b, v^2 = c$):

- $T_\pm \simeq [a, b, 0]$ represents 0 (equivalently, $-\det T_\pm$ is a perfect square): the group $O(T_\pm)$ equals $(\mathbb{Z}/2)^2$ or $\mathbb{Z}/2$ depending on whether the two isotropic directions can or cannot be interchanged;
- $T_\pm \simeq [1, 0, c]$: the group $O(T_\pm)$ is given by the solutions to Pell's equation;
- $T_\pm \simeq [-2, b, -2]$, $b \geq 3$ odd: since the generators u, v constitute the two walls of a fundamental polyhedron, the group $O(T_\pm)$ is generated by -1, the symmetry $u \leftrightarrow v$ of the polyhedron, and reflection r_v, see (3.3).

Step (4). Similar to (3.10) in step (3), the extensions are in a bijection with the double cosets

$$\mathfrak{I}_{h,\theta}(\tilde{S}_h)\backslash\{\psi\colon \operatorname{discr} \tilde{S}_h \to \operatorname{discr} T\}/\mathfrak{I}_\theta(T), \tag{3.12}$$

where

- ψ is a *bijective* anti-isometry (since the resulting lattice $\mathbf{L}$ is unimodular; any extension (3.12) is isomorphic to $\mathbf{L}$ as it is unique in its genus),
- $\mathfrak{I}_\theta(T)$ is the subgroup computed in step (3) together with T, and
- $\mathfrak{I}_{h,\theta}(\tilde{S}_h)$ is the image of the subgroup $O_{h,\theta}(\tilde{S}_h) \subset O(\tilde{S}_h)$ preserving h and centralizing θ_S: it is easily found as a subgroup of the finite group $O(S)$; in fact, it suffices to use the group of symmetries of the Dynkin diagram.

Step (5). Once a quadruple $(\mathbf{L} \supset \tilde{S}_h \ni h, \theta)$ has been constructed, the verification of condition (3) in Theorem 2.4 is straightforward. □

3.3. **Proof of Addendum 1.2.** Since a sextic curve C in question must have at least two (conjugate) non-simple singular points, these points are triple, hence adjacent to $\mathbf{J}_{10}$, see [3]. Then, the δ-invariant $\delta(C) \ge 12$, and C splits into at least three components. Thus, C is a union of three conics, all passing through a fixed pair of conjugate points O, O' with fixed tangent lines OP, $O'P$, where $P \in \mathbb{P}^2_{\mathbb{R}}$. It follows that the three conics are members of a pencil $\mathcal{P}$ with two double base points. Up to real projective transformation, such a pencil is unique: it is determined by the triple (O, O', P). The two degenerate fibers of $\mathcal{P}$, *viz.* the double line OO' and the union of OP, $O'P$, divides $\mathcal{P}_{\mathbb{R}}$ into two intervals; in one of them the conics are empty, whereas in the other their real parts form a family of nested ovals. This description makes the statement of Addendum 1.2 immediate. □

Appendix A. Symmetric sextics

In this appendix, we describe an explicite geometric construction for the majority of special ($d > 1$) empty sextics by means of the double covering $p\colon \mathbb{P}^2 \dashrightarrow \Sigma_2$ and trigonal curves in the Hirzebruch surface Σ_2. In the extremal cases, where the deck translation of p is a stable involution (see §2.5.5) of the sextic in question, this construction is canonical, thus providing also a deformation classification. As a by-product, we visualize the distinctions between multiple deformation families within the same real lattice type.

A.1. **Preliminaries.** Assume that an empty sextic $C \subset \mathbb{P}^2$ is preserved by an equivariant involution $s\colon \mathbb{P}^2 \to \mathbb{P}^2$. As is known, s has an isolated real fixed point $O \notin C$ (since $C_{\mathbb{R}} = \varnothing$) and a real fixed line L, and the quotient by s of the blow-up $\mathbb{P}^2(O)$ is the Hirzebruch surface Σ_2 (see [8, 10, 11] for details). Thus, we have a commutative diagram

$$\begin{array}{ccc} X & \overset{\tilde{p}}{\dashrightarrow} & Y \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\bar{\pi}} \\ \mathbb{P}^2 & \overset{p}{\dashrightarrow} & \Sigma_2, \end{array} \tag{A.1}$$

where each surface is real and each arrow is a (birational) real double covering. The ramification loci are as follows:

- for π, the original sextic $C = p^{-1}(\bar{C})$,
- for p, the exceptional section E and the section $\bar{L} := p(L)$ disjoint from E,
- for $\bar{\pi}$, the exceptional section E and the image $\bar{C} := p(C)$; this image is a *proper* (*i.e.*, disjoint from E) trigonal curve in Σ_2,
- for $\tilde{p}$, the pull-back $\bar{\pi}^{-1}(\bar{L})$.

Since the line L has real points, so does $\bar{L}$ and the real structure on Σ_2 is standard.

To avoid excessive notation, below we systematically use $^-$ to denote the image under p of a curve in $\mathbb{P}^2$. Conversely, for a curve $\bar{B} \subset \Sigma_2$, we silently denote by B its pullback in $\mathbb{P}^2$.

A.1.1. *Trigonal curves and real forms.* Below, we always start with an extremal trigonal curve $\bar{C} \subset \Sigma_2$, *i.e.*, $\mu(\bar{C}) = 8$; such a curve appears from a stable involution of a sextic C. (For irreducible sextics this fact is proved in [8]; in general, the assertion would follow from comparing the dimensions of the moduli spaces, but we do not engage into this discussion and merely state the fact.) Up to automorphism of Σ_2, there are but finitely many such curves: complex curves are classified by means of *dessins d'enfants* $\Gamma \subset S^2$, and the real forms of each curve are the reflections of S^2 preserving Γ.

Remark A.2. In §A.13–§A.15, the curve $\bar{C} = \bar{C}_r$ depends on a parameter $r \in \mathbb{R}$, so that $\bar{C}_0$ is isotrivial whereas all curves with $r \neq 0$ are non-isotrivial and pairwise isomorphic over $\mathbb{C}$. In all other cases, the *complex* curve with the prescribed set of singularities is unique.

A.1.2. *Coordinates.* Till the rest of this appendix, we use real affine coordinates (x, y) in Σ_2 so that the exceptional section E is given by $y = \infty$. In these coordinates, a proper real section is given by

$$y = f(x) := ax^2 + bx + c, \quad a, b, c \in \mathbb{R}, \tag{A.3}$$

and a proper real trigonal curve is given by

$$y^3 + y^2 p_2(x) + y p_4(x) + p_6(x) = 0, \quad p_d \in \mathbb{R}[x],\ \deg p_d \leqslant d.$$

The coordinates about the fiber $x = \infty$ are $u := 1/x$ and $v := y/x^2$, in which (A.3) takes the form

$$v = a + bu + cu^2. \tag{A.4}$$

A.1.3. *The construction.* Conversely, given a proper trigonal curve $\bar{C} \subset \Sigma_2$ and a proper section $\bar{L} \subset \Sigma_2$, the double covering of Σ_2 ramified at $\bar{L} \cup E$ blows down to $\mathbb{P}^2$ and the pull-back of $\bar{C}$ is a sextic curve. If both $\bar{C}$ and $\bar{L}$ are real, so is C.

Remark A.5. The set of singularities of C depends on that of $\bar{C}$ and on the position of $\bar{L}$ with respect to $\bar{C}$, see [8, 11]. If

(1) $\bar{L}$ is *generic*, *i.e.*, transverse to $\bar{C}$,

then each singular point of $\bar{C}$ doubles in C; thus, if $\bar{C}$ is also extremal, we have $\mu(C) = 16$. Besides, we consider but the following types of degenerate sections:

(2) $\bar{L}$ is simple tangent to $\bar{C}$ at a pair of conjugate points, or
(3) $\bar{L}$ passes through a pair of conjugate type $\mathbf{A}$ singular points of $\bar{C}$.

Both degenerations are of codimension 2 and result in sextics C with $\mu(C) = 18$.

In the real case, each of the two complements below admits a chessboard colorings, thus splitting into two (open) regions:

$$(\Sigma_2)_{\mathbb{R}} \smallsetminus (\bar{C}_{\mathbb{R}} \cup E_{\mathbb{R}}) = \Sigma_2^+ \cup \Sigma_2^-, \qquad (\Sigma_2)_{\mathbb{R}} \smallsetminus (\bar{L}_{\mathbb{R}} \cup E_{\mathbb{R}}) = \Sigma_2^{+L} \cup \Sigma_2^{-L},$$

and $C_{\mathbb{R}}$ is empty if and only if

$$\bar{C}_{\mathbb{R}} \text{ lies in one of the halves } \Sigma_2^{\pm L}, \text{ henceforth denoted by } \Sigma_2^{-L}, \tag{A.6}$$

and the lift of the real structure is chosen so that $\mathbb{P}^2_{\mathbb{R}}$ projects to the closure of Σ_2^{+L}. A section satisfying (A.6) is called *non-separating* (with respect to $\bar{C}$).

Note that (A.6) implies (but, in general, is not equivalent to) that

$$\bar{L}_{\mathbb{R}} \text{ lies in one of the halves } \Sigma_2^{\pm}, \text{ henceforth denoted by } \Sigma_2^{-}; \tag{A.7}$$

furthermore, the lift of the real structure on Σ_2 to Y in (A.1) is to be chosen so that $Y_{\mathbb{R}}$ project to the closure of Σ_2^+.

Lemma A.8. *For a proper real trigonal curve $\bar{C} \subset \Sigma_2$, the space of non-separating real sections has exactly two connected components $\mathcal{S}_{\pm}(\bar{C})$, which are both convex in the affine space* (A.3) *of proper sections.*

Proof. The set of non-separating sections is the union of two open convex sets, *viz.*

$$\mathcal{S}_\epsilon := \bigl\{f \text{ as in (A.3)} \bigm| \epsilon f(x) > \epsilon y \text{ for each point } (x,y) \in \bar{C}_{\mathbb{R}}\bigr\}, \quad \epsilon = \pm.$$

(At $x = \infty$ this condition is to be modified according to (A.4).) To prove that, say, $\mathcal{S}_+ \neq \varnothing$, let $b = 0$. By the compactness of $\bar{C}_R \smallsetminus E_{\mathbb{R}}$, the condition above holds

- for all $|x| \leqslant 1$ whenever $a \geqslant 0$ and $c \gg 0$, and
- for all $|x| \geqslant 1$ whenever $a \gg 0$ and $c \geqslant 0$, *cf.* (A.4).

Thus, $\mathcal{S}_+$ contains a section $y = f(x)$ as in (A.3) with $b = 0$ and $a, c \gg 0$. □

A.1.4. *The deformation classification.* A real trigonal curve $\bar{C} \subset \Sigma_2$ is said to be *symmetric* if it is equisingular deformation equivalent, over $\mathbb{R}$, to its image under at least one of the automorphisms

$$r\colon (x,y) \mapsto (-x,y) \quad \text{or} \quad (x,y) \mapsto (-x,-y)$$

reversing the orientation of the real fibers of the ruling. If this is the case, then r, followed by the deformation, interchanges $\Sigma_2^\pm$, as well as $\mathcal{S}_\pm(\bar{C})$ in Lemma A.8.

Corollary A.9 (of Lemma A.8)**.** *Let $\bar{C} \subset \Sigma_2$ be an extremal real trigonal curve, and let $\mathcal{M}(\bar{C})$ be the space of pairs (C,s), where $C \subset \mathbb{P}^2$ is an empty sextic, $\mu(C) = 16$, and $s\colon \mathbb{P}^2 \to \mathbb{P}^2$ a real stable involution such that $C/s \cong \bar{C}$. Then $\mathcal{M}(\bar{C})$ has one* (*if $\bar{C}$ is symmetric*) *or two* (*otherwise*) *connected components.*

Proof. As explained in §A.1.3, the condition $\mu(C) = 16$ is equivalent to the assertion that the section $\bar{L}$ in the construction of C is generic. By Lemma A.8, the space of all sections satisfying (A.6) has two convex components, which are interchanged by an automorphism if Σ_2 if and only if $\bar{C}$ is symmetric, and it remains to observe that non-generic sections satisfying (A.6) constitute the intersection with $\mathcal{S}(\bar{C})_\pm$ of a real algebraic variety of codimension at least 2, *cf.* Remark A.5(2) and (3). □

A.1.5. *Splitting conics and sections.* Fix a proper trigonal curve $\bar{C} \subset \Sigma_2$. A *splitting section* is a proper section $\bar{B} \subset \Sigma_2$ such that

(1) the local intersection index of $\bar{B}$ and $\bar{C}$ is even at each point, and
(2) the proper transforms of $\bar{B}$ and $\bar{C}$ in the minimal resolution of singularities of $\bar{C}$ are disjoint.

A proper section $\bar{B}$ is splitting if and only if the proper preimage $\bar{\pi}^{-1}(\bar{B})$ splits into a pair of disjoint sections of the rational Jacobian elliptic surface Y, see (A.1). If $\bar{L}$ is another proper section, so that we have diagram (A.1), the section $\bar{B}$ is splitting if and only if $B := p^{-1}(\bar{B})$ is an s-invariant splitting conic for C, see §2.5.4.

Remark A.10. Strictly speaking, the existence of an involutive stable symmetry s and the presence of splitting conics are two independent properties of the complex lattice type; however, often (although not always), the former implies the latter. Typically, an s-invariant splitting conic B projects to a splitting section $\bar{B}$. We refrain from a general statement and merely indicate splitting sections in the models, referring implicitly to the classification of complex lattice types to identify the value of d.

Observation A.11. Consider an s-invariant real splitting conic B and its image $\bar{B} := p(B)$. Due to condition (1) above, $\bar{B}_{\mathbb{R}}$ lies entirely in (the closure of) one of the two regions $\Sigma_2^\pm$. It is immediate from the construction that

- if $\bar{B}$ is in the closure of Σ_2^-, see (A.7), the two components of $\bar{\pi}^{-1}(\bar{B})$ are complex conjugate and, hence, B contributes to c_{s} in $s_{\mathbb{R}}$ (see §2.6);
- if $\bar{B}$ is in the closure of Σ_2^+, both components of $\bar{\pi}^{-1}(\bar{B})$ are real and, hence, B contributes to r_{s} in $s_{\mathbb{R}}$.

This observation is used to identify the real lattice types obtained in §A.1.3.

A.1.6. *Perturbations of trigonal curves.* The description of non-isotrivial trigonal curves by means of *dessins d'enfants* implies that, just like in the case of plane sextics, *all* **A***-type singular points of a non-isotrivial trigonal curve* $\bar{C} \subset \Sigma_2$ *can be perturbed arbitrarily and independently.* We use this observation to realize some real lattice types not admitting an involutive stable symmetry.

A.2. **The computation.** In the rest of this appendix, we consider, one-by-one, the extremal trigonal curves $\bar{C} \subset \Sigma_2$ appearing from the stable involutions s of the complex lattice types listed in Tables 1 and 2 and, for each such curve (designated by its sets of singularities), list its real forms, *cf.* §A.1.1.

For each real form, we compute the connected components of the equisingular equivariant moduli spaces of pairs (C, s), where C is an empty sextic obtained by the construction of §A.1.3 from $\bar{C}$ and a section $\bar{L}$ as in Remark A.5. If $\bar{L}$ is generic, we refer to Corollary A.9; otherwise, we describe the 1-parameter family of sections explicitly, still arriving at two connected components (interchanged by an automorphism and deformation if $\bar{C}$ is symmetric, see §A.1.4).

Remark A.12. Usually, a real lattice type admits at most one stable involution and, hence, the moduli space of pairs (C, s) is that of sextics C. In the two exceptional cases (see Remarks A.21, A.24 below), we explain that one of the three involutions is distinguished and it is this involution that is used in the construction.

By computing the invariants (most notably, $c_{\mathbb{R}}$ and $s_{\mathbb{R}}$, see §2.6), we establish that each real form of the given complex lattice type has indeed been realized; in most cases, we reestablish the deformation classification by showing that each connected component belongs to its own real form. (Usually, it suffices to compare the counts.) The two exceptional cases are emphasized in §A.7 and §A.12 below.

Occasionally, we consider a few perturbations of the original extremal trigonal curve $\bar{C} \subset \Sigma_2$ (see §A.1.6) and use them to construct representatives of some other real lattice types, not admitting a stable involution.

Remark A.13. It is worth pointing out that, since, upon the perturbation, the trigonal curve is no longer extremal, the involution is not stable and the construction of §A.1.3 does *not* give us the complete stratum of sextics.

A.3. **The trigonal curve $\bar{C}(4\mathbf{A}_2)$.** The curve $\bar{C}$ is given by

$$4y^3 - (24x^3 + 3)y + (8x^6 + 20x^3 - 1) = 0, \tag{A.14}$$

see [10]; its four cusps are

$$P_1 = \left(0, -\tfrac{1}{2}\right), \quad P_{2,3,4} = \left(\epsilon, \tfrac{3}{2}\right), \quad \epsilon^3 = 1.$$

The non-separating real sections passing through P_3, P_4 are

$$a(x^2 + x + 1) + \frac{3}{2}, \quad |a + 1| > 1; \tag{A.15}$$

they give rise to the set of singularities $2\mathbf{A}_5 \oplus 4\mathbf{A}_2$. A generic section gives rise to $8\mathbf{A}_2$. All sextics are of torus type ($d = 3$); this fact follows from the presence of the splitting sections $\bar{B}_i$ passing through P_j, $j \ne i$, see [10] and §A.1.5.

In spite of its appearance, the curve $\bar{C}$ is symmetric: in appropriate coordinates, with the cusps at $\pm(1,1)$ and $\pm(2i - i\sqrt{3}, 0)$, it can be given by a polynomial that is skew-invariant under $(x, y) \mapsto (-x, -y)$. Hence, in each of the two cases, we obtain a single real lattice type and a single deformation family, see §A.1.4.

A.3.1. *Perturbations.* Perturbing the cusp P_1 to $\mathbf{A}_1$ or $\varnothing$ (see §A.1.6), we realize the sets of singularities

$$2\mathbf{A}_5 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1, \quad 2\mathbf{A}_5 \oplus 2\mathbf{A}_2, \quad 6\mathbf{A}_2 \oplus 2\mathbf{A}_1, \quad 6\mathbf{A}_2,$$

all with $d = 3$, as the real splitting section $\bar{B}_1$ remains intact. The resulting curve is no longer symmetric; hence, in each of the four cases, we obtain two real forms, which differ by $s_{\mathbb{R}}$, see §2.6(2) and Observation A.11.

A.4. **The trigonal curve $\bar{C}(2\mathbf{A}_3 \oplus 2\mathbf{A}_1)$.** The curve $\bar{C}$ splits into the zero section $\bar{D}_0$: $y = 0$ and two other sections $\bar{D}_\pm$ tangent to L_0 at points $P_\pm$, respectively, and intersecting at two other points Q_1, Q_2. The three real forms are (listing $\bar{D}_\pm$ only)

$$y = (x \pm 1)^2: \quad P_\pm = (\pm 1, 0), \quad Q_1 = (0, 1), \quad Q_2 = (\infty, 1), \tag{A.16}$$

$$y = \pm(x \pm 1)^2: \quad P_\pm = (\pm 1, 0), \quad Q_i = (\pm i, \pm 2i), \tag{A.17}$$

$$y = (x \pm i)^2: \quad P_\pm = (\pm i, 0), \quad Q_1 = (0, -1), \quad Q_2 = (\infty, 1). \tag{A.18}$$

The forms (A.17) and (A.18) are symmetric, whereas (A.16) is not. Non-separating real sections passing through Q_1, Q_2 exist in (A.17) only; they are

$$\bar{L}\colon\ y = a(x^2 + 1) + 2x, \quad |a| > 1, \tag{A.19}$$

giving rise to the set of singularities $6\mathbf{A}_3$. Non-separating real sections through both $P_\pm$ are

$$\bar{L}\colon\ y = a(x^2 + 1), \quad |a| > 1, \tag{A.20}$$

in (A.18); they give rise to the set of singularities $2\mathbf{A}_7 \oplus 4\mathbf{A}_1$. Finally, if $\bar{L}$ is a generic non-separating section, we obtain $4\mathbf{A}_3 \oplus 4\mathbf{A}_1$ (see §A.4.1 below).

In all cases, we have $d = 4$ due to the presence of a pair of splitting sections (not necessarily real) $\bar{B}_i$ passing through both $P_\pm$ and Q_i, $i = 1, 2$.

Remark A.21. The sextic C with the set of singularities $2\mathbf{A}_7 \oplus 4\mathbf{A}_1$ has three stable involutions, and we choose the only one fixing the two type $\mathbf{A}_7$ points.

A.4.1. *The sextic $C(4\mathbf{A}_3 \oplus 4\mathbf{A}_1)$ with $d = 4$ and perturbations thereof.* The four real forms of this complex lattice type can be distinguished as follows, *cf.* §2.6(4).

If $\bar{C}$ is as in (A.16), then $c_{\mathbb{R}} = (3, 0)$. Both splitting sections

$$\bar{B}_i\colon\ y = \pm(x^2 - 1), \quad i = 1, 2,$$

are real (hence so are the splitting conics B_i), and their real parts are in the same half $\Sigma_2^\pm$. Thus, $s_{\mathbb{R}} = (0, 2, 0)$ or $(2, 0, 0)$ (see Observation A.11).

If $\bar{C}$ is as in (A.17), then $\bar{B}_1$, $\bar{B}_2$ are conjugate and $c_{\mathbb{R}} = (3, 0)$, $s_{\mathbb{R}} = (0, 0, 1)$.

If $\bar{C}$ is as in (A.18), then $c_{\mathbb{R}} = (1, 1)$. Both splitting sections

$$\bar{B}_i\colon\ y = \pm(x^2 + 1), \quad i = 1, 2,$$

are real and their real parts are in the opposite halves $\Sigma_2^\pm$; hence, $s_{\mathbb{R}} = (1, 1, 0)$.

In the first case, (A.16), we can perturb Q_2 (see §A.1.6), leaving a single splitting section $\bar{B}_1$, hence still $d = 4$. This gives rise to two real forms, distinguished by $s_{\mathbb{R}}$ (see §2.6(2) and Observation A.11), of the set of singularities $4\mathbf{A}_3 \oplus 2\mathbf{A}_1$. The same pair of real forms is obtained by perturbing Q_2 in the last case (A.18).

In the second case, (A.17), we can perturb a conjugate pair of nodes of the *original sextic* C. (This perturbation is no longer symmetric.) This operation destroys both splitting conics, resulting in the set of singularities $4\mathbf{A}_3 \oplus 2\mathbf{A}_1$ with $d = 2$. This real lattice type is also obtained from $2\mathbf{D}_6 \oplus 2\mathbf{A}_3$ (see the end of §A.11.1 below), where $d = 2$ in the first place.

A.4.2. *Other perturbations.* Perturbing P_2 to $\mathbf{A}_2$ or $\varnothing$ (see §A.1.6) in (A.17) and taking for $\bar{L}$ either (an appropriate perturbation of) (A.19) or a generic section, we obtain reducible ($d = 2$) curves with the sets of singularities

$$4\mathbf{A}_3 \oplus 2\mathbf{A}_2, \quad 2\mathbf{A}_3 \oplus 2\mathbf{A}_2 \oplus 4\mathbf{A}_1, \quad 4\mathbf{A}_3.$$

Perturbing Q_1, Q_2 in (A.18) and taking a perturbation of (A.20) for $\bar{L}$, we obtain a reducible ($d = 2$) curve with the set of singularities $2\mathbf{A}_7$.

A.5. **The trigonal curve $\bar{C}(2\mathbf{A}_4)$.** The curve $\bar{C}$ is given by

$$\begin{gathered} 4y^3 - 3yp(x) + q(x) = 0, \text{ where} \\ p(x) = x^4 - 12x^3 + 14x^2 + 12x + 1, \\ q(x) = (x^2 + 1)(x^4 - 18x^3 + 74x^2 + 18x + 1). \end{gathered}$$

see [11]; it has $\mathbf{A}_4$ singular points at $(0, 1/2)$ and $(\infty, 1/2)$. A generic non-separating section $\bar{L}$ gives rise to the two real forms for the set of singularities $4\mathbf{A}_4$. The curve is special ($d = 5$) due to the presence of the splitting sections

$$\bar{B}_\pm\colon\ y = \frac{1}{2}(x^2 + 1) \pm 3x.$$

Both $\bar{B}_\pm$ are real and their real parts are in the same half $\Sigma_2^\pm$; hence, $s_{\mathbb{R}} = (0, 2, 0)$ or $(2, 0, 0)$, see §2.6(3) and Observation A.11.

A.6. **The trigonal curve $\bar{C}(\mathbf{A}_5 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1)$.** The curve $\bar{C}$ splits into a "parabola" $\bar{D}_2$ and a section $\bar{D}_1$ inflection tangent to $\bar{D}_2$ at $(1/4, 1/2)$:

$$\bar{D}_2 :\ y^2 = x, \qquad \bar{D}_1 :\ y = -x^2 + \frac{3}{2}x + \frac{3}{16},$$

see [9]. Taking for $\bar{L}$ a generic non-separating section, we arrive at the two real forms for the set of singularities $2\mathbf{A}_5 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$ with $d = 6$. Indeed, the curve is clearly reducible, hence $2 \mid d$, and it is of torus type due to the splitting section

$$\bar{B}\colon\ y = x + \frac{1}{4},$$

resulting also in $s_{\mathbb{R}} = (0, 1, 0)$ or $(1, 0, 0)$, see §2.6(2) and Observation A.11.

A.6.1. *Perturbations.* Perturbing the cusp at infinity to $\mathbf{A}_1$ or $\varnothing$ (see §A.1.6) and thus destroying the torus structure, we arrive at reducible ($d = 2$) curves with the sets of singularities $2\mathbf{A}_5 \oplus 4\mathbf{A}_1$ and $2\mathbf{A}_5 \oplus 2\mathbf{A}_1$.

A.7. **The trigonal curve $\bar{C}(\mathbf{A}_7 \oplus \mathbf{A}_1)$.** The curve $\bar{C}$ splits into a "conic" $\bar{D}_2$ with a node at $(\infty, 0)$ and a section $\bar{D}_1$ quadruple tangent to $\bar{D}_2$ at $(0, 1)$:

$$\bar{D}_2\colon\ \epsilon x^2 + y^2 = 1, \qquad \bar{D}_1\colon\ y = -\frac{1}{2}\epsilon x^2 + 1, \quad \epsilon = \pm 1.$$

Since both real forms are asymmetric, a generic non-separating section $\bar{L}$ gives rise to *two real families* for each of the two real forms of $2\mathbf{A}_7 \oplus 2\mathbf{A}_1$. One has $d = 4$ and $s_{\mathbb{R}} = (0, 1, 0)$ or $(1, 0, 0)$, see §2.6(2), due to the splitting section $\bar{B}$: $y = 1$.

A.8. **The trigonal curve $\bar{C}(\mathbf{A}_8)$.** The curve $\bar{C}$ is given by

$$-y^3 + y^2 - x^3(2y - x^3) = 0 \tag{A.22}$$

or, parametrically, by

$$x = \frac{t}{t^3 + 1}, \qquad y = \frac{1}{(t^3 + 1)^2},$$

see [9]. According to [9, Lemma 2.3.3], a real section $\bar{L}$ tangent to $\bar{C}$ at two complex conjugate points is uniquely determined by the tangency points, which must be at

$$t = -2^{-4/3} \pm si, \quad s > 0, \quad 8s^2 \neq 3 \cdot 2^{1/3}, \tag{A.23}$$

and the explicit equation found in [9] shows that $\bar{L}$ is non-separating, thus giving rise to the set of singularities $2\mathbf{A}_8 \oplus 2\mathbf{A}_1$. A generic section gives rise to $2\mathbf{A}_8$. In both cases, the curve is of torus type ($d = 3$) due to the splitting section $\bar{B}$: $y = 0$. As $\bar{C}$ is asymmetric, each complex lattice type has two real forms: they differ by $s_{\mathbb{R}} = (0, 1, 0)$ or $(1, 0, 0)$, see §2.6(2) and Observation A.11.

A.9. **The trigonal curve $\bar{C}(2\mathbf{D}_4)$.** The curve $\bar{C}$ is isotrivial, splitting into three sections $\bar{D}_0$: $y = 0$ and $\bar{D}_\pm$. We consider the two real forms

$$\bar{D}_\pm\colon\ y = \pm x \quad \text{or} \quad y = \pm ix.$$

A generic section $\bar{L}$ gives rise to the set of singularities $4\mathbf{D}_4$, with the two real forms distinguished by $c_{\mathbb{R}} = (1, 1)$ or $(3, 0)$, see §2.6(1).

Remark A.24. The curve C has three stable involutions, and we chose the only one whose action on $\operatorname{Sing} C$ coincides with that of σ. Then, we can assume that all singular points of $\bar{C}$ are real and thus avoid considering its other real forms, *viz.* those with a pair of complex conjugate singular points.

A.10. **The trigonal curve $\bar{C}(\mathbf{D}_5 \oplus \mathbf{A}_3)$.** The curve $\bar{C}$ splits into a "parabola" $\bar{D}_2$ and section $\bar{D}_1$ tangent to $\bar{D}_2$ at $(1, 1)$ and passing through its cusp $(\infty, 0)$:

$$\bar{D}_2 :\ y^2 = x, \qquad \bar{D}_1 :\ y = \frac{1}{2}(x + 1);$$

it has a $\mathbf{D}_5$ singularity at $(\infty, 0)$. Taking for $\bar{L}$ a generic non-separating section, we arrive at the two real forms for the set of singularities $2\mathbf{D}_5 \oplus 2\mathbf{A}_3$. One has $d = 4$ and $s_{\mathbb{R}} = (0, 1, 0)$ or $(1, 0, 0)$, see §2.6(2), due to the splitting section $\bar{B}$: $y = 1$.

A.11. **The trigonal curve $\bar{C}(\mathbf{D}_6 \oplus 2\mathbf{A}_1)$.** The curve $\bar{C}$ splits into the section $\bar{D}_0$: $y = x$ and two other sections $\bar{D}_\pm$ intersecting $\bar{D}_0$ at the point $Q = (\infty, 0)$ of their tangency and at two other points $P_\pm$. Both real forms are symmetric:

$$\bar{D}_\pm\colon\ y = \pm 1: \qquad P_\pm = (\pm 1, \pm 1), \tag{A.25}$$

$$\bar{D}_\pm\colon\ y = \pm i: \qquad P_\pm = (\pm i, \pm i). \tag{A.26}$$

Only (A.26) admits non-separating sections passing through $P_\pm$; they are

$$\bar{L}\colon\ y = a(x^2 + 1) + x, \quad 2|a| > 1, \tag{A.27}$$

and the resulting sextics have the set of singularities $2\mathbf{D}_6 \oplus 2\mathbf{A}_3$. A generic section $\bar{L}$ gives rise to the set of singularities $2\mathbf{D}_6 \oplus 4\mathbf{A}_1$ ($\mathbf{D}_6^{\mathrm{ii}}$). In both cases, $d = 2$. Hence, $d \mid 2$ (and thus $d = 2$, as we keep the curves reducible) for all perturbations below.

A.11.1. *Perturbations of $\bar{L}$ as in* (A.27) *and $\bar{C}$ as in* (A.26). We can perturb Q to

$$\mathbf{D}_4 \oplus \mathbf{A}_1 \to \mathbf{D}_4 \quad \text{or} \quad \mathbf{A}_3 \oplus 2\mathbf{A}_1 \to 4\mathbf{A}_1 \xrightarrow{*} 3\mathbf{A}_1 \xrightarrow{*} 2\mathbf{A}_1,$$

arriving at the sets of singularities

$2\mathbf{D}_4 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_1$ ($\mathbf{A}_3^{\mathrm{i}}$), $\quad 2\mathbf{D}_4 \oplus 2\mathbf{A}_3$, $\quad$ or

$4\mathbf{A}_3 \oplus 4\mathbf{A}_1$, $\quad 2\mathbf{A}_3 \oplus 8\mathbf{A}_1$ ($\mathbf{A}_3^{\mathrm{i}}$, *cf.* §A.11.2), $\quad 2\mathbf{A}_3 \oplus 6\mathbf{A}_1$, $\quad 2\mathbf{A}_3 \oplus 4\mathbf{A}_1$,

respectively, all with $d = 2$. Here, the two codimension one perturbations are easily described explicitly,

$$\bar{D}_\pm\colon\ y = \pm(1 - \varepsilon) + \varepsilon x \quad \text{or} \quad y = \pm i(1 - \varepsilon) + \varepsilon x, \quad \text{or}$$
$$\bar{D}_0\colon\ y = x + \varepsilon(x^2 - 1) \quad \text{or} \quad y = x + \varepsilon(x^2 + 1),$$

respectively (we describe both real forms of $\bar{C}$), upon which we can use §A.1.6 to perturb type $\mathbf{A}$ singular points. In the last two cases, marked with a *, we perturb one or both points of intersection of $\bar{D}_+$ and $\bar{D}_-$, thus keeping D_0 a separate conic component of C.

Alternatively, starting from $\mathbf{A}_3 \oplus 2\mathbf{A}_1$ and dissolving one of the nodes, we arrive at the set of singularities $4\mathbf{A}_3 \oplus 2\mathbf{A}_1$ with $d = 2$, *cf.* the end of §A.4.1.

A.11.2. *Perturbations of $\bar{L}$ generic and $\bar{C}$ as in* (A.25) *or* (A.26). The perturbations

$$\mathbf{D}_4 \oplus \mathbf{A}_1, \quad \mathbf{A}_3 \oplus 2\mathbf{A}_1, \quad 4\mathbf{A}_1$$

in §A.11.1 produce two real forms, with $c_\mathbb{R} = (1, 1)$ or $(3, 0)$, see §2.6(1), for each of

$$2\mathbf{D}_4 \oplus 6\mathbf{A}_1, \quad 2\mathbf{A}_3 \oplus 8\mathbf{A}_1\ (\mathbf{A}_3^{\mathrm{ii}},\ \textit{cf.}\ \text{§A.11.1}), \quad 12\mathbf{A}_1.$$

The last two perturbations result in the sets of singularities $10\mathbf{A}_1$ and $8\mathbf{A}_1$, both sextics splitting into a quartic and a conic.

A.11.3. *Perturbations of $\bar{L}$ generic and $\bar{C}$ as in* (A.25). In addition to §A.11.2, we can also perturb Q to $\mathbf{D}_5$ (perturbing $\bar{D}_+ \cup \bar{D}_-$ to a "parabola" $\bar{D}$) or $\mathbf{A}_2 \oplus 2\mathbf{A}_1$:

$$\bar{D}\colon\ (y^2 - 1) + \varepsilon x = 0, \qquad \bar{D}_0\colon\ y = x + \delta x^2,$$

arriving at the sets of singularities

$$2\mathbf{D}_5 \oplus 4\mathbf{A}_1, \quad 2\mathbf{A}_2 \oplus 8\mathbf{A}_1.$$

Alternatively, the perturbation of the node $\bar{D}_+ \cap \bar{D}_0$, possibly preceded by the perturbation of Q to $\mathbf{D}_4 \oplus \mathbf{A}_1$ as in §A.11.1, gives rise to

$$2\mathbf{D}_6 \oplus 2\mathbf{A}_1, \quad 2\mathbf{D}_4 \oplus 4\mathbf{A}_1.$$

All four sextics obtained split into a quartic and a conic.

A.12. **The trigonal curve $\bar{C}(\mathbf{D}_8)$.** The trigonal curve $\bar{C}$ splits into the section $\bar{D}_1 = \{y = x\}$ and a "conic" $\bar{D}_2$ with a node at $(\infty, 0)$; the two real forms are

$$\bar{D}_2\colon\ y^2 - x^2 = 1, \tag{A.28}$$

$$\bar{D}_2\colon\ x^2 - y^2 = 1. \tag{A.29}$$

Arguing as in [10, 11], we conclude that, if a section $\bar{L}$ is bitangent to $\bar{D}_2$, the two tangency points must be of the form $(\pm x_0, y_0)$. These points can be complex conjugate only in case (A.28), and a double tangent section $\bar{L}$ does exist if and only if $0 < |y_0| < 1$. Then $\bar{L}$ is automatically non-separating and gives rise to the set of singularities $2\mathbf{D}_8 \oplus 2\mathbf{A}_1$. A generic section in (A.28) or (A.29) gives rise to the set of singularities $2\mathbf{D}_8$. As $\bar{C}$ is symmetric, each real form of $\bar{C}$ results in a single deformation family; thus, we obtain *two deformation families* for $2\mathbf{D}_8$. All sextics obtained are reducible, *i.e.*, we have $d = 2$.

A.12.1. *Perturbations.* Perturbing the type $\mathbf{D}_8$ point at infinity to $\mathbf{D}_6 \oplus \mathbf{A}_1$ (*e.g.*, replacing $\bar{D}_1$ with the line $y = x + \varepsilon$) and keeping $\bar{L}$ bitangent to D, we obtain the set of singularities $2\mathbf{D}_6 \oplus 4\mathbf{A}_1$ ($\mathbf{D}_6^{\mathrm{i}}$, *cf.* §A.11).

A.13. **The trigonal curve $\bar{C}(\mathbf{E}_6 \oplus \mathbf{A}_2)$.** The curve $\bar{C}$ is given by

$$y^3 + (ry + x)^2 = 0, \quad r \in \mathbb{R},$$

see [6]; it has an $\mathbf{E}_6$ singularity at $(\infty, 0)$, and a generic non-separating section $\bar{L}$ gives rise to the set of singularities $2\mathbf{E}_6 \oplus 2\mathbf{A}_2$. The sextics are of torus type ($d = 3$) due to the splitting section $\bar{B}$: $y = 0$. Since $\bar{C}$ is asymmetric, there are two real forms, with $s_{\mathbb{R}} = (0, 1, 0)$ or $(1, 0, 0)$, see §2.6(2) and Observation A.11).

A.14. **The trigonal curve $\bar{C}(\mathbf{E}_7 \oplus \mathbf{A}_1)$.** The curve $\bar{C}$ splits into a "parabola" $\bar{D}_2$ and section $\bar{D}_1$:

$$\bar{D}_2 :\ y^2 = x, \qquad \bar{D}_1 :\ y = r, \quad r \in \mathbb{R};$$

it has an $\mathbf{E}_7$ singularity at $(\infty, 0)$. Taking for $\bar{L}$ a generic non-separating section, we arrive at the set of singularities $2\mathbf{E}_7 \oplus 2\mathbf{A}_1$ with $d = 2$ (reducible curve). All curves $\bar{C}_r$ are symmetric; hence, we obtain a single connected deformation family.

A.15. **The trigonal curve $\bar{C}(\mathbf{E}_8)$.** The curve $\bar{C}$ is parametrized by

$$x = t^3 + 3rt, \quad y = t,$$

where $r \in \mathbb{R}$. The sign of $r \neq 0$ selects one of the two distinct real forms of non-isotrivial curves, see Remark A.2. Arguing as in [10, 11], we conclude that a section double tangent to $\bar{C}$ at two values $t_1 \neq t_2$ of the parameter exists if and only if

$$r = t_1^2 + 3t_1t_2 + t_2^2.$$

The values $t_{1,2}$ can be complex conjugate, $t_{1,2} = \alpha \pm \beta i$, if and only if

$$r = 5\alpha^2 + \beta^2 > 0,$$

and a double tangent section does exist if and only if $\alpha \neq 0$. In this case, the section is automatically non-separating, giving rise to the set of singularities $2\mathbf{E}_8 \oplus 2\mathbf{A}_1$.

A generic section gives rise to $2\mathbf{E}_8$. Since all curves $\bar{C}_r$ are symmetric, we obtain a single connected deformation family in each of the two cases.

Appendix B. Further examples

In this appendix, we consider a couple of examples illustrating the computation leading to the proof of Theorem 1.1. In §B.1 we give a simple proof of the fact that the necessary condition (2.7) is also sufficient in the case of non-special sextics. In §B.2 we show that, as one would expect, the two real deformation families with the set of singularities $2\mathbf{A}_9$ correspond to the two distinct complex ones.

B.1. **The non-special curves.** In the case of non-special ($d = 1$) empty sextics, the sufficiency of condition (2.7) (*cf.* Remark 2.10) can easily be proved directly, without a reference to the classification.

Theorem B.1. *A non-special empty sextic with an even set of singularities $2\mathcal{S}$ exists if and only if the root system $\mathcal{S}$ admits a primitive embedding into $\mathbf{E}_8 \oplus \mathbf{U}$, or, equivalently, the Dynkin diagram of $\mathcal{S}$ is an induced subgraph of* (3.8).

Proof. As explained in Theorem 2.4 and (2.9), the empty involution on $\mathbf{L}$ can be described as follows (see [24]): the eigenlattices are

$$L_+ = \mathbf{E}_8(2) \oplus \mathbf{U}(2), \qquad \text{with a standard basis } e'_1, \ldots, e'_{10},\ cf.\ (3.8),$$
$$L_- = \mathbf{E}_8(2) \oplus \mathbf{U}(2) \oplus \mathbf{U}, \quad \text{with a standard basis } e''_1, \ldots, e''_{10},\ u_1, u_2,$$

and $\mathbf{L}$ is the extension of $L_+ \oplus L_-$ *via* the vectors $e^\pm_i := \frac{1}{2}(e'_i \pm e''_i)$, $i = 1, \ldots, 10$. Now, we assume that $\mathcal{S} \subset L_+$ is generated by a subset e'_i, $i \in I$, of the basis, and it is immediate that the vectors $e^\pm_i$, $i \in I$, generate the set of singularities $2\mathcal{S}$. The polarisation is $h := u_1 + u_2$, and the existence of a sextic is given by Theorem 2.4.

The necessity of the condition follows from Corollary 2.6 and obvious observation that, if $S := 2\mathcal{S}$ is primitive in $\mathbf{L}$, then so are $S_\pm \subset L_\pm$. ☐

B.2. **The set of singularities $2\mathbf{A}_9$.** This is the only complex lattice type whose corresponding equisingular family has two real components of positive dimension, see [1]. We assert that each of the two components has an empty real representative. We have $\tilde{S}_h = 2\mathbf{A}_9 \oplus \mathbb{Z}h$, and the transcendental lattice is

$$T := \mathbb{Z}u \oplus \mathbb{Z}v_1 \oplus \mathbb{Z}v_2, \quad u^2 = -2, \quad v^2_{1,2} = 10;$$

the two lattices can be glued, in the sense of (3.12), by the anti-isometry ψ

$$\begin{aligned} &\tfrac{1}{2}u \mapsto [0,0,1], \\ &\tfrac{1}{2}v_1 \mapsto [5,0,0], \qquad \tfrac{2}{5}v_1 \mapsto [2,0,0], \\ &\tfrac{1}{2}v_2 \mapsto [0,5,0], \qquad \tfrac{2}{5}v_2 \mapsto [0,2,0] \end{aligned}$$

of the discriminants; we use the shorthand notation $[a, b, c]$ for the elements of

$$\operatorname{discr}(\mathbf{A}_9 \oplus \mathbf{A}_9 \oplus \mathbb{Z}h) = (\mathbb{Z}/10) \oplus (\mathbb{Z}/10) \oplus (\mathbb{Z}/2).$$

The "obvious" automorphisms of T, *viz.* inverting one of the generators and the transposition $v_1 \leftrightarrow v_2$, generate a subgroup $G \subset \operatorname{Aut} \operatorname{discr} T$ of index 2. On the other hand, a computation using [23] shows that the full image of $O(T)$ in $\operatorname{Aut} \operatorname{discr} T$ is of index 2; hence, the latter image equals G. Furthermore, $\psi_*(G)$ equals the image of $O_h(\tilde{S}_h)$ in $\operatorname{Aut} \operatorname{discr} \tilde{S}_h$, and it is this fact that gives rise to two homological types/connected components: $\tilde{S}_h$ and T can be glued *via* ψ or *via* $\psi' := \varphi \circ \psi$, where φ is, say, the transposition $[2,0,0] \leftrightarrow [0,2,0]$ of the two generators of $\operatorname{discr}_5 \tilde{S}_h$. A straightforward computation shows that, in both cases, the involution

$$-\big((\mathbf{A}_9 \leftrightarrow \mathbf{A}_9) \oplus (v_1 \leftrightarrow v_2)\big)$$

extends to one induced by an empty real structure.

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