

REAL ENRIQUES SURFACES WITHOUT REAL POINTS AND ENRIQUES-EINSTEIN-HITCHIN 4-MANIFOLDS

ALEXANDER DEGTYAREV AND VIATCHESLAV KHARLAMOV

ABSTRACT. We prove that the moduli space of real Enriques surfaces which have no real points (and, thus, the moduli space of compact orientable 4-dimensional Einstein manifolds E whose universal covering is a $K3$ -surface and $\pi_1(E) = \mathbb{Z}/2 \times \mathbb{Z}/2$) is connected. The proof is based on a systematic study of real elliptic pencils and gives explicit models of all empty real Enriques surfaces.

1. INTRODUCTION

N. Hitchin [H] proved that the Euler characteristic $\chi(E)$ and signature $\sigma(E)$ of a compact orientable 4-dimensional Einstein manifold E satisfy the inequality $|\sigma(E)| \leq \frac{2}{3}\chi(E)$, the equality holding only if either E is flat or the universal covering X of E is a $K3$ -surface and $\pi_1(E) = 1, \mathbb{Z}/2$, or $\mathbb{Z}/2 \times \mathbb{Z}/2$. In the latter cases, E is a $K3$ -surface if $\pi_1 = 1$, an Enriques surface if $\pi_1 = \mathbb{Z}/2$, or the quotient of an Enriques surface by a free antiholomorphic involution if $\pi_1 = \mathbb{Z}/2 \times \mathbb{Z}/2$. It is the Einstein manifolds of the last type that we call *Enriques-Einstein-Hitchin varieties*. The varieties of the other three extremal types (flat, $\pi_1 = 1$, and $\pi_1 = \mathbb{Z}/2$) are known to form connected families: two varieties with the same fundamental group can be deformed continuously into each other. To our knowledge, the number of connected components of the moduli space of Enriques-Einstein-Hitchin varieties was not known. In this paper we give the answer: we prove that *their moduli space is connected*.

As is known (modulo Calabi-Yau theorem this statement is also contained in [H]), the universal covering X of an Enriques-Einstein-Hitchin manifold E carries a canonical complex structure, so that X is a $K3$ -surface, one nontrivial element of $\pi_1(E) = \mathbb{Z}/2 \times \mathbb{Z}/2$ acts on X holomorphically, and the two others, antiholomorphically. This correspondence establishes a homotopy equivalence between the moduli space of Enriques-Einstein-Hitchin varieties and that of Enriques surfaces with free anti-holomorphic involution (cf. [I]). An Enriques surface with a free anti-holomorphic involution is, by definition, an empty real Enriques surface, and the connectedness of the moduli space of Enriques-Einstein-Hitchin varieties follows from the main result of the present paper:

1991 *Mathematics Subject Classification*. 14J28, 14P25, and 53C25.

Key words and phrases. Enriques surface, elliptic pencil, real algebraic surface, Einstein manifold.

1.1. Main Theorem. *All empty real Enriques surfaces (or, equivalently, compact Einstein 4-manifolds with $|\sigma| = \frac{2}{3}\chi$ and $\pi_1 = \mathbb{Z}/2 \times \mathbb{Z}/2$) are of the same deformation type.*

As a corollary, all empty real structures on Enriques surfaces are equivariantly diffeomorphic (see also the remark at the end of the paper). Note that this is not the case for many other deformation classes of complex surfaces; the simplest example is $P^1 \times P^1$, which admits two nondiffeomorphic real structures with empty real part.

Our proof of the main theorem is based on a systematic study of real elliptic pencils and gives, in addition, explicit models of all empty real Enriques surfaces. It relies upon Horikawa's proof of connectedness of the moduli space of complex Enriques surfaces (see [Hor]) and Cosec's and Dolgachev's study of complex elliptic pencils on Enriques surfaces (see [CD] and [D]).

As it is often the case, on Enriques surfaces empty real structures differ essentially from those with nonempty real part. If the real part is nonempty, one can apply so called Donaldson-Hitchin trick and reduce the problem to a meticulous study of real rational surfaces with a nonsingular anti-bicanonical curve (see [DK], where this approach is applied to surfaces whose real part contains a component of nonpositive Euler characteristic). However, if the real part is empty, Donaldson's trick still applies but it yields another empty real Enriques surface. That is why we had to choose an alternative approach to cover this case. The remaining case of surfaces with all components homeomorphic to S^2 or $\mathbb{R}P^2$ is now close to completion and will appear in [DIK]; in addition to Donaldson's trick it requires an explicit description of the moduli space and arithmetical study of $(\mathbb{Z}/2 \times \mathbb{Z}/2)$ -actions in the intersection lattice of a $K3$ -surface, see [DK].

Acknowledgements. We thank: Fields institute, the organizers of the special semester dedicated to V. Arnold, and the organizers of this semester conference on real algebraic geometry for giving us an occasion to meet and to complete this project; M. Kreck and G. B. M. van der Geer for their encouragements.

2. PRELIMINARIES

In this section we fix main definitions, introduce principal notation and recall some known basic results (most of them are found, e.g., in [BPV] and [CD]).

2.1. $K3$ - and Enriques surfaces. A *$K3$ -surface* is a compact complex analytic surface X with $\pi_1(X) = 1$ and $c_1(X) = 0$. An *Enriques surface* is the quotient of a $K3$ -surface by a fixed point free holomorphic involution. A *real Enriques surface* is an Enriques surface equipped with a *real structure*, i.e., anti-holomorphic involution. The real structure lifts to the covering $K3$ -surface (for a fixed point free involution a proof is given in [H]; in others cases it is straightforward). Thus, real Enriques surfaces are in a one-to-one correspondence with $K3$ -surfaces equipped with an action of a group $G \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ generated by two commuting anti-holomorphic involutions with fixed point free composition. (Note that the isomorphism $G \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ is not fixed, cf. Remark in 5.2.)

For an Enriques surface E the map $\text{Pic } E \rightarrow H_2(E; \mathbb{Z}), D \mapsto [D]$, is an isomorphism. The nontrivial element of $\text{Tors } H_2(E; \mathbb{Z}) = \mathbb{Z}/2$ is equal to $[K_E] = w_2(E)$, where $K_E \in \text{Pic } E$ is the canonical class. Denote $L = H_2(E; \mathbb{Z}) / \text{Tors}$; it is a lattice isomorphic to $E_8 \oplus U$. Here, E_8 is the negative lattice generated by the root

system of the same name and U is a hyperbolic plane. In U there are two pairs of generators x_1, x_2 with $x_1^2 = x_2^2 = 0$, $x_1x_2 = 1$; we call such generators *standard*.

Throughout the paper we denote by E a fixed Enriques surface, by X , its universal covering, and by $\text{conj}: E \rightarrow E$, the real structure on E (if E is real). The eigenlattices of $\text{conj}_*: L \rightarrow L$ are denoted by L^\pm . For an element $y \in L$ we use \hat{y} to denote one of the two lifts of y to $H_2(E; \mathbb{Z})$. Given $y \in L^-$ or $x \in H_2(E; \mathbb{Z})$ with $(x \bmod \text{Tors}) \in L^-$, we let $\delta(y) = (1 + \text{conj}_*)\hat{y}$ and $\delta(x) = (1 + \text{conj}_*)x$. Clearly, δ can take only two values: 0 and $w_2(E)$.

As usual, we denote by $|D|$ the linear system generated by a divisor D . In the case of Enriques surface the Riemann-Roch theorem takes the form

$$\dim |D| - \dim H^1(E; \mathcal{O}_E(D)) + \dim |K_E - D| = \frac{1}{2}D^2.$$

Since, in addition, K_E is not effective (as $2K_E = 0$), this implies that for any element $x \in H_2(E; \mathbb{Z})$ with $x^2 \geq 0$ either x or $-x$ (but not both) is realized by an effective divisor.

A nonsingular rational curve R on an Enriques surface E (or $K3$ -surface X) is called a *nodal curve*. As follows from the adjunction formula, $R^2 = -2$. Hence, nodal curves are determined by their classes in $H_2(E; \mathbb{Z})$ (respectively, $H_2(X; \mathbb{Z})$): two nodal curves of the same class coincide. In the case of Enriques surface R is, in fact, determined by its class in L ; if E is real and $([R] \bmod \text{Tors}) \in L^-$, then $\delta(R) = 0$. An Enriques or $K3$ -surface is called *nodal* (respectively, *unnodal*) if it has (respectively, does not have) a nodal curve. Note that ‘most’ Enriques surfaces are unnodal, cf. the proof of 5.2.1.

2.2. Elliptic pencils. An *elliptic pencil* on an algebraic surface is a linear system of dimension one whose generic member is an irreducible smooth elliptic curve. Recall (see [CD] or [BPV], Lemma 17.1) that any elliptic pencil on an Enriques surface E is base point free and, thus, defines an elliptic fibration of $E \rightarrow \mathbb{C}p^1$ with exactly two multiple fibers P, P' , so that $\dim |P| = \dim |P'| = 0$ and $P - P' = K_E$. The pencil is then the linear system $|2P| = |2P'|$.

Let $|2P|$ be an elliptic pencil on E and P, P' its multiple fibers. Their pull-backs \tilde{P}, \tilde{P}' in X , which are irreducible smooth elliptic curves, are members of an elliptic pencil on X , which we denote by $|\tilde{P}|$. The corresponding fibration is induced from $|2P|$ by the double covering of its base branched at the points corresponding to P and P' . We will call $|\tilde{P}|$ the *pull-back of $|2P|$* and $|2P|$, the *projection of $|\tilde{P}|$* .

The pull-back $|\tilde{P}|$ is τ -equivariant. Conversely, if $|\tilde{P}|$ is a τ -invariant elliptic pencil on X , the projections of its members to E have self-intersection 0 and are homologous and, hence, linear equivalent to each other; thus, they form an elliptic pencil. (In particular, τ always acts nontrivially on the base of an equivariant pencil, as otherwise one would obtain an elliptic pencil in E without multiple fibers.)

An elliptic pencil on a real Enriques surface is called *real* if it is conj -invariant. Clearly, the pull-back of a real elliptic pencil is real.

2.2.1. Proposition. *An elliptic pencil $|2P|$ is real if and only if $([P] \bmod \text{Tors}) \in L^-$. If this is the case, the two multiple fibers, P and P' , of the pencil are real if $\delta([P]) = 0$ and conjugate to each other if $\delta([P]) = w_2(E)$.*

Proof. Since conj is antiholomorphic, it transforms holomorphic curves to holomorphic curves reversing their complex orientation (and their complex structure). Thus, the first statement follows from $\text{Pic}(E) = H_2(E; \mathbb{Z})$. The second statement follows from $P - P' = K_E$, where P and P' are the multiple fibers. \square

2.2.2. Proposition. *The involution induced by conj on the base of a real elliptic pencil on a real Enriques surfaces has nonempty real part.*

Proof. Let $|2P|$ be the pencil, B its base, and $|\tilde{P}|$ the pull-back of $|2P|$ in X . The base \tilde{B} of $|\tilde{P}|$ is the double covering of B branched at the two points corresponding to the two multiple fibers. Since $|\tilde{P}|$ is also a real elliptic pencil, the involution induced by conj on B must lift to an involution on \tilde{B} . On the other hand, any real structure on B with $B_{\mathbb{R}} = \emptyset$ lifts to an order 4 automorphism of \tilde{B} . \square

3. MODELS

3.1. Nonspecial Horikawa models. Let X be the $K3$ -surface obtained as the minimal resolution of the double covering of $Y = \mathbb{C}p^1 \times \mathbb{C}p^1$ branched over a reduced bi-degree $(4, 4)$ curve $C \subset Y$ with at worst simple singularities. Denote by $s: Y \rightarrow Y$ the Cartesian product of the nontrivial involutions $(u:v) \mapsto (-u:v)$ of the factors. Up to isomorphism s is the only holomorphic involution on Y with isolated fixed points. If C is s -symmetric, s lifts to two different involutions on X , commuting with the deck translation d of $X \rightarrow Y$. If, besides, C contains no fixed points of s , exactly one of these two involutions, which we denote by τ , is fixed point free (see, e.g., [Hor] or [BPV]), and, hence, the orbit space $E = X/\tau$ is an Enriques surface. The two rulings of Y define two τ -invariant elliptic pencils on X , which project to two elliptic pencils on E . We call these pencils *basic*; their multiple fibers correspond to the generatrices of Y through the fixed points of s .

Suppose that Y is equipped with a real structure c commuting with s and C is a real curve. Then $s \circ c$ is another real structure on Y and C . We denote the real point sets of these structures by $Y_{\mathbb{R}}^{(i)}$ and $C_{\mathbb{R}}^{(i)}$, $i = 1, 2$ ($i = 1$ corresponding to c) and call them the *halves* of Y and C . The involutions c and $s \circ c$ lift to four different commuting real structures $(t^{(1)}, t^{(2)} = \tau \circ t^{(1)}, d \circ t^{(1)}, \text{ and } d \circ t^{(2)})$ on X , which, in turn, descend to two real structures on E , called the *expositions* of E . A choice of an exposition is determined by a choice of one of the two liftings $t^{(1)}, d \circ t^{(1)}$ of c to X .

The involutions s and c generate a $\mathbb{Z}/2 \times \mathbb{Z}/2$ -action on Y , so that one nontrivial element acts holomorphically and has isolated fixed points and two other nontrivial elements act anti-holomorphically.

3.1.1. Proposition. *Up to isomorphism, there are five such actions. Four of them are decomposable, i.e., split into product of actions on the factors:*

- (1) $Y_{\mathbb{R}}^{(1)}, Y_{\mathbb{R}}^{(2)}$ are homeomorphic to $S^1 \times S^1$, each ruling has two invariant fibers;
- (2) $Y_{\mathbb{R}}^{(1)}$ is $S^1 \times S^1$, $Y_{\mathbb{R}}^{(2)} = \emptyset$, and one ruling has two invariant fibers;
- (3) $Y_{\mathbb{R}}^{(1)}$ is $S^1 \times S^1$, $Y_{\mathbb{R}}^{(2)} = \emptyset$, and there is no invariant fibers;
- (4) $Y_{\mathbb{R}}^{(1)} = Y_{\mathbb{R}}^{(2)} = \emptyset$ and there is no invariant fibers;

and one is indecomposable:

- (5) $Y_{\mathbb{R}}^{(1)}$ and $Y_{\mathbb{R}}^{(2)}$ are homeomorphic to S^2 , and s has two real fixed points.

With an abuse of the language we will say that the above representation of E is a *decomposable* or, respectively, *indecomposable Horikawa representation*.

Proof. Any (anti-)automorphism of Y is given by a linear expression in bihomogeneous coordinates. It is easy to see that an indecomposable action (whose anti-holomorphic involutions transpose the rulings) can be converted to a canonical

form with s as above and $c: [(u_1 : v_1), (u_2 : v_2)] \mapsto [(\bar{u}_2 : \bar{v}_2), (\bar{u}_1 : \bar{v}_1)]$. A decomposable action splits into product; up to isomorphism and interchanging c and $s \circ c$ there are two actions on $\mathbb{C}P^1$: the holomorphic involution is the map $s: (u : v) \mapsto (-u : v)$, and one of the anti-holomorphic ones is either $c_a: (u : v) \mapsto (\bar{u} : \bar{v})$ or $c_b: (u : v) \mapsto (\bar{v} : \bar{u})$. (Note that $s \circ c_a$ is isomorphic to c_a , and $s \circ c_b: (u : v) \mapsto (-\bar{u} : \bar{v})$ is fixed point free.) Combining c_a , c_b , and $s \circ c_b$, up to permutation of the factors and of c and $s \circ c$ one obtains the four actions listed in the statement:

- (1) $c = c_a \times c_a$;
- (2) $c = c_a \times c_b$;
- (3) $c = c_b \times c_b$;
- (4) $c = c_b \times (s \circ c_b)$. \square

Remark. Note that for 3.1.1(1) and (5) both the expositions have $E_{\mathbb{R}} \neq \emptyset$. Thus, in this paper we are only concerned with 3.1.1(2)–(4).

Homologically, decomposable and indecomposable actions differ by the induced action in $H_2(Y; \mathbb{Z}) \cong U$: the holomorphic involution always acts identically, and the anti-holomorphic ones induce multiplication by (-1) in the decomposable case and $y_1 \mapsto -y_2, y_2 \mapsto -y_1$ in the indecomposable one (where y_1, y_2 are the classes of the rulings on Y).

3.2. Special Horikawa models. In the construction of 3.1 one can as well take for Y a rational ruled surface Σ_2 with a (-2) -section S_0 and for C , a curve linearly equivalent to a 4-fold generic section and disjoint from S_0 . (Y can be thought of as the minimal resolution of the singular point of a quadric cone in $\mathbb{C}P^3$; then C is cut on Y by a quartic surface not through the vertex.) Up to isomorphism there is a unique holomorphic involution $s: Y \rightarrow Y$ with isolated fixed points. It preserves S_0 and has four fixed points: two in S_0 and two others in the generatrices through them. As in 3.1, if C is s -invariant and does not contain the fixed points of s , precisely one of the two lifts of s to X , denoted by τ , is fixed point free, and the quotient $E = X/\tau$ is an Enriques surface. The ruling of Y lifts to a τ -invariant pencil in X ; its projection to E is called the *basic* pencil; its multiple fibers correspond to the generatrices of Y through the fixed points of s . The exceptional section S_0 lifts to two disjoint nodal curves in X ; they are interchanged by τ and, thus, project to one nodal curve in E , called the *basic* nodal curve.

As before, a real structure $c: Y \rightarrow Y$ in respect to which both s and C are real defines two real structures on E , called *expositions* of E . In the $(\mathbb{Z}/2 \times \mathbb{Z}/2)$ -actions generated by s and c one element acts holomorphically and with isolated fixed points and the two other nontrivial elements act anti-holomorphically.

3.2.1. Proposition. *Up to isomorphism there are two such actions:*

- (1) *with $Y_{\mathbb{R}}^{(1)}$ and $Y_{\mathbb{R}}^{(2)}$ homeomorphic to $S^1 \times S^1$ and two invariant generatrices;*
- (2) *with $Y_{\mathbb{R}}^{(1)} = S^1 \times S^1$, $Y_{\mathbb{R}}^{(2)} = \emptyset$, and no invariant generatrices.*

Proof. We blow up the two fixed points of s not on S_0 and blow down the generatrices through them. This is an equivariant transformation whose result is $\mathbb{C}P^1 \times \mathbb{C}P^1$ with an action which has invariant generatrices (say, the image of S_0). The statement follows now from 3.1.1. \square

3.3. Basic elliptic pencils. Consider a (special or nonspecial) real Horikawa representation of a real Enriques surface E . Denote by $|2P_1|, |2P_2|$ the basic elliptic pencils on E in the nonspecial case, and by $|2P|$ and R , the basic elliptic pencil and basic nodal curve in the special case. Let $(P_1, P'_1), (P_2, P'_2)$, and (P, P') be the

multiple fibers of the pencils. The classes $([P_1], [P_2]) \subset L$ and $([P], [P + R]) \subset L$ are called the *basic U -pairs*; the sublattice in L generated by the basic pair is called the *basic lattice*. The following is a direct consequence of the construction:

3.3.1. Lemma. *The basic lattice of a Horikawa representation is a conj_* -invariant hyperbolic plane $U \subset L$. If the representation is special or decomposable nonspecial, conj_* acts on U by multiplication by (-1) . Otherwise (indecomposable nonspecial representation), conj_* transposes the standard generators $[P_1]$ and $[P_2]$ of U . \square*

Remark. Note that basic lattice determines basic U -pair. Indeed, U has a unique pair of standard generators represented by effective divisors. In the special case one has to distinguish between $[P]$ and $[P + R]$. Since $(P + R) \cdot R = -1 < 0$, this class cannot be represented by a fiber of an elliptic pencil: such a fiber would contain R as a component and, hence, intersect it trivially.

A conj_* -invariant hyperbolic plane $U \subset L$ is said to be of *type I*, or *decomposable*, if conj_* acts on U as multiplication by (-1) , and of *type II*, or *indecomposable*, if conj_* transposes a pair of generators of U . In the case of type I, i.e., $U \subset L^-$, the unordered pair of values $\delta(x_1), \delta(x_2)$ does not depend on a choice of a pair x_1, x_2 of standard generators of U . According to these values we will further subdivide type I into $I(0, 0)$, $I(0, w_2)$, and $I(w_2, w_2)$. The following is a consequence of Propositions 3.1.1 and 3.2.1 (and the fact that always $\delta([R]) = 0$):

3.3.2. Lemma. *The basic lattice is of type II if and only if the representation is nonspecial indecomposable. In the other cases the types are:*

- $I(0, 0)$ for nonspecial action 3.1.1(1) and special action 3.2.1(1);
- $I(0, w_2)$ for nonspecial action 3.1.1(2);
- $I(w_2, w_2)$ for nonspecial actions 3.1.1(3), (4) and special action 3.2.1(2). \square

3.4. Selected models of empty surfaces. Our proof of the main theorem appeals to nonspecial Horikawa representations with decomposable action 3.1.1(2). As follows from Lemma 3.3.2 and Remark in 3.1, this type is distinguished by the type of its basic lattice, which must be $I(0, w_2)$. In some affine coordinates (x, y) on Y the action is given by $s: (x, y) \mapsto (-x, -y)$, $c: (x, y) \mapsto (\bar{x}^{-1}, \bar{y})$. Since $Y_{\mathbb{R}}^{(1)} = S^1 \times S^1$ and $Y_{\mathbb{R}}^{(2)} = \emptyset$, one has:

3.4.1. Proposition. *If the real part of E is empty, so is $C_{\mathbb{R}}^{(1)}$. Conversely, if $C_{\mathbb{R}}^{(1)} = \emptyset$, the real part of E is empty for one and only one of the expositions.*

Let us represent branch curves C by polynomials in x, y . Consider the vector space of s -invariant polynomials of bidegree $(4, 4)$, its real part

$$\mathcal{C} = \{p = \sum a_{i,j} x^i y^j \mid i = j \pmod{2}, 0 \leq i, j \leq 4, \bar{a}_{4-i,j} = a_{i,j}\},$$

the corresponding projective space PC , and subsets $\mathcal{M}_0 \subset \mathcal{M} \subset PC$

$$\mathcal{M} = \{p \in PC \mid C_{\mathbb{R}}^{(1)} = \emptyset\},$$

$$\mathcal{M}_0 = \left\{ p \in \mathcal{M} \mid \begin{array}{l} \text{the curve } C = \{p = 0\} \text{ has at worst simple singularities,} \\ a_{0,0} \neq 0, a_{0,4} \neq 0, a_{4,0} \neq 0, a_{4,4} \neq 0 \end{array} \right\}.$$

3.4.2. Theorem. *For each $p \in \mathcal{M}_0$ the Horikawa construction gives a unique empty real Enriques surfaces. The space \mathcal{M}_0 is connected; in particular, all empty real Enriques surfaces obtained by the Horikawa construction from points of \mathcal{M}_0 are of the same deformation type.*

Proof. The first statement is contained in 3.4.1. The connectedness follows from two observations. First, \mathcal{M} is convex in PC and, hence, connected. (Indeed, $C_{\mathbb{R}}^{(1)} = \emptyset$ if and only if, up to multiplication by (-1) , $p > 0$ on $Y_{\mathbb{R}}^{(1)}$.) Second, the conditions $a_{0,0} \neq 0$, $a_{0,4} \neq 0$, $a_{4,0} \neq 0$, $a_{4,4} \neq 0$, as well as the restriction on the singularities, are of codimension at least 2. The deformation type of the resulting surface is preserved since the (unique) exposition with empty real part, corresponding to $p > 0$ on $Y_{\mathbb{R}}^{(1)}$, varies continuously with p . \square

4. EXISTENCE OF PENCILS AND MODELS

4.1. Real elliptic pencils on E . Recall that an effective divisor $D = \sum m_i R_i$ on E with all R_i distinct and irreducible is called a *divisor of canonical type* if $D \cdot R_i = K_E \cdot R_i = 0$ for all i . It is called *indecomposable* if it is connected and $\text{g.c.d.}(m_i) = 1$.

4.1.1. Lemma. *If D is a divisor of canonical type and the class of D is primitive in L , then $|2D|$ is an elliptic pencil on E .*

Proof. As shown in [CD], Proposition 3.1.2, $|D|$ or $|2D|$ is an elliptic pencil whenever D is an indecomposable divisor of canonical type. Let us prove that D is indecomposable. Assume that $D = \sum n_i D_i$, $n_i \geq 1$, with all D_i indecomposable and $D_i \cdot D_j = 0$. Then $|D_1|$ or $|2D_1|$ is an elliptic pencil, and the other components are its fibers. Hence, the classes of all D_i are some multiples of the class of a multiple fiber of the pencil. Since $([D] \bmod \text{Tors})$ is primitive, this implies that $D = D_1$ and that $|D|$ is not an elliptic pencil. \square

4.1.2. Theorem. *A real Enriques surface E admits a real elliptic pencil if and only if there exists an element $x \in L^-$ with $x^2 = 0$. If E is unnodal, for any primitive element $x \in L^-$ with $x^2 = 0$ either x or $-x$ can be realized as the class of a multiple fiber of a real elliptic pencil.*

Proof. The ‘only if’ part is obvious (cf. 2.2.1). For the ‘if’ part, pick a primitive element $x \in L^-$ with $x^2 = 0$ and an effective divisor D with $[D] = \pm x$. According to [CD], Theorem 3.2.1, $D \sim D' + \sum m_i R_i$, $m_i \geq 1$, where R_i are nodal curves and D' is a divisor of canonical type. Moreover, the class $[D'] \in L$ is uniquely determined by x and is obtained from x by a series of reflections. Since x is primitive, so is $[D']$ and, by Lemma 4.1.1, $|2D'|$ is an elliptic pencil. Due to the uniqueness of $[D']$ it is conj-invariant. \square

4.2. Existence of real Horikawa models. The following two pure complex results are known. Their proofs are found, e.g., in [BPV], Theorems 18.1 and 18.2.

4.2.1. Lemma. *If an Enriques surface E has a pair $|2P_1|, |2P_2|$ of elliptic pencils with $P_1 \cdot P_2 = 1$, then E admits a nonspecial Horikawa representation with $|2P_1|, |2P_2|$ as basic pencils. \square*

4.2.2. Lemma. *If an Enriques surface E has an elliptic pencil $|2P|$ and a nodal curve R with $P \cdot R = 1$, then E admits a special Horikawa representation with $|2P|$ and R as the basic pencil and nodal curve. \square*

4.2.3. Theorem. *A real Enriques surface E admits a real Horikawa representation if and only if L contains a conj_* -invariant hyperbolic plane U of type I or II. If E is unnodal, U can be taken for the basic lattice of a representation, whose type is determined by Lemma 3.3.2. In general, if U is of type I (respectively, type II), E admits either a special Horikawa representation or a nonspecial decomposable (respectively, indecomposable) Horikawa representation.*

Proof. The necessity of the condition follows from 3.3.1. To prove the ‘if’ part, consider the standard generators $x_1, x_2 \in U$ realized by effective divisors. According to [CD], Lemma 3.3.1, by a sequence of reflections against classes of nodal curves x_1, x_2 can be taken to some classes $y_1, y_2 \in L$, unique up to reordering, so that either

- (1) $y_1 = [D_1]$ and $y_2 = [D_2]$, or
- (2) $y_1 = [D_1]$ and $y_2 = [D_1 + R]$,

where D_1, D_2 are indecomposable divisors of canonical type and R is a nodal curve. In case (1) the order of y_1, y_2 is also determined by x_1, x_2 ; hence, the pair (y_1, y_2) behaves in respect to conj_* in the same way as (x_1, x_2) and a real Horikawa representation is given by Lemma 4.2.1. In case (2) the uniqueness implies that y_1, y_2 are conj_* -skew-invariant and a representation is given by Lemma 4.2.2. Finally, if E is unnodal, no reflection is necessary and x_1, x_2 are themselves realized by indecomposable divisors of canonical type. \square

5. PROOF OF THE MAIN THEOREM

5.1. Eigenlattices of conj_* . Let D_4 denote the negative lattice generated by the root system of the same name.

5.1.1. Lemma. *For an empty real Enriques surface $L^+ = D_4$ and $L^- = D_4 \oplus U$.*

Proof. It suffices to prove that $L^+ = D_4$, since D_4 admits a unique, up to isomorphism, primitive embedding into $E_8 \oplus U$, see [N], Theorem 1.14.2 and Remark 1.14.6.

Since $E_{\mathbb{R}}$ is empty, $\text{rk } L^+ = 4$, $\sigma(L^+) = -4$, and the discriminant form $\text{discr } L^+$ of L^+ is even. The dimension of $\text{discr } L^+$ is either 0, or 2, or 4. In the first case L^+ is unimodular and must have signature $0 \pmod 8$ (see [S]). In the last case $L^+(\frac{1}{2})$ is unimodular and even, which also contradicts to the signature congruence. Thus, L^+ is an even integral negative lattice of rank 4 with $\text{discr } L^+$ even and of dimension 2. The only such lattice is D_4 , see [N], Remark 1.14.6. \square

Remark. Our case is very special and much simpler than the general situation treated by Nikulin. For example, the fact that D_4 is determined up to isomorphism by its genus is proved in a few lines. Indeed, since the discriminant group contains a vector of square 1, any such lattice is a sublattice of index 2 of a negative unimodular lattice of rank 4, i.e., $4(-1)$. Since the latter is odd, the lattice in question is uniquely determined as its maximal even sublattice. For a similar reason the orthogonal complement of D_4 in $E_8 \oplus U$ is also unique, and the uniqueness of an embedding $D_4 \subset E_8 \oplus U$ is then straightforward.

5.1.2. Lemma. *If E is an empty real Enriques surface, L contains a hyperbolic plane of type $I(0, w_2)$.*

Proof. Due to 5.1.1 $L^- \cong D_4 \oplus U$ does contain a hyperbolic plane U . Since U is unimodular and D_4 is determined by its genus, the embedding $U \rightarrow L^-$ is unique

up to isomorphism. Let x_1, x_2 be a standard pair of generators of U , and e_1, \dots, e_4 some standard generators of D_4 (so that $e_i^2 = -2$, $e_i e_j = 0$ for $1 \leq i < j \leq 3$, and $e_i e_4 = 1$ for $1 \leq i \leq 3$). Consider the following three possibilities:

Case 1: $\delta(x_1) = 0$ and $\delta(x_2) = w_2$. Then U is of type $I(0, w_2)$.

Case 2: $\delta(x_1) = \delta(x_2) = w_2$. If there is an index i with $\delta(e_i) = 0$, the sublattice generated by $x_1 + x_2 + e_i$ and x_1 is of type $I(0, w_2)$. Otherwise, $\delta(e_1 + e_4) = 0$ and the sublattice generated by $x_1 + x_2 + e_1 + e_4$ and x_1 is of type $I(0, w_2)$.

Case 3: $\delta(x_1) = \delta(x_2) = 0$ for *any* hyperbolic plane $U \subset L^-$. Take for U the basic lattice of a real Horikawa representation, which exists due to Theorem 4.2.3. If the representation is nonspecial and decomposable, its basic pencils $|2P_1|, |2P_2|$ have real multiple fibers and, hence, the only intersection point of P_1 and P_2 is real. If the representation is special, the basic pencil $|2P|$ has real multiple fibers and $x_2 - x_1$ is realized by a real nodal curve R , and the intersection point of P and R is real. Existence of a real point contradicts to the assumption $E_{\mathbb{R}} = \emptyset$. \square

5.2. Proof of the theorem.

5.2.1. Lemma. *Any real Enriques surface can be made unnodal by a small real deformation.*

Proof. Surfaces with nodal curves form a countable union of hypersurfaces in the period space of complex Enriques surfaces: these hypersurfaces are the hyperplane sections defined by (-2) -vectors in the $K3$ -lattice (see, e.g., the construction of the period space in [BPV]). For our purpose, it is sufficient to consider the period space of local deformations. Due to the local Torelli theorem, the real part of the local period space is nonsingular and represents small real deformations of a given real Enriques surface (cf. [Kh]). Thus, in any neighborhood of a real point there are real points representing unnodal surfaces. \square

5.2.2. Lemma. *Any unnodal empty real Enriques surface is obtained by the Horikawa construction from a point of \mathcal{M}_0 .*

Proof. Due to 5.1.2 L contains a hyperbolic plane of type $I(0, w_2)$, which, due to 4.2.3 (and since E is unnodal) is the basic lattice of a real Horikawa representation based on a decomposable action. Due to 3.3.2, this is the action selected in 3.4. \square

Proof of Theorem 1.1. To deform one empty real Enriques surface to another we deform them both to unnodal surfaces (see 5.2.1), which, due to 5.2.2, are represented by points of \mathcal{M}_0 , and connect the results by a path in \mathcal{M}_0 (see 3.4.2). \square

Remark. The choice of one of the two lifts of the real structure on an empty real Enriques surface to the covering $K3$ -surface defines a nonramified double covering of the moduli space of empty real Enriques surfaces. This covering is nontrivial: the two covering real structures can be exchanged by a diffeomorphism or by the monodromy along a loop in the moduli space. This is easily seen on Horikawa models with decomposable action 3.1.1(2) or 3.1.1(4). In particular, similar to empty real Enriques surfaces, the $K3$ -surfaces equipped with an ordered pair of commuting anti-holomorphic fixed point free involutions with fixed point free composition are all of the same deformation type.

REFERENCES

[BPV] W. Barth, C. A. M. Peters, A. Van de Ven, *Compact Complex Surfaces*, Springer, 1984.

- [CD] F. R. Cosec, I. V. Dolgachev, *Enriques surfaces I*, Birkhäuser, 1989.
- [DIK] A. Degtyarev, I. Itenberg, V. Kharlamov, *Real Enriques surfaces* (to appear).
- [DK] A. Degtyarev, V. Kharlamov, *On the moduli space of real Enriques surfaces*, C.R. Acad. Sci. **324** (1997), no. 3, 317–322.
- [D] I.V. Dolgachev, *On automorphisms of Enriques surfaces*, Invent Math. **76** (1984), 163–177.
- [I] M. Itoh, *Conformal geometry of Ricci flat 4-manifolds*, Kodai Math. J. **17** (1994), 179–200.
- [H] N. Hitchin, *Compact four-dimensional Einstein manifolds*, J. Differential Geometry **9** (1974), 435–441.
- [Hor] E. Horikawa, *On the periods of Enriques surfaces. I*, Math. Ann. **234** (1978), 73–108.
- [Kh] V. Kharlamov, *The topological type of nonsingular surfaces in $\mathbb{R}p^3$ of degree 4*, Funktsional. Anal. i Prilozhen. **10** (1976), no. 4, 55–68 (Russian); English transl. in Functional Anal. Appl. **10** (1976), no. 4, 295–305.
- [N] V. Nikulin, *Integral symmetric bilinear forms and some of their applications*, Izv. Akad. Nauk SSSR **43** (1979), no. 1, 117–177 (Russian); English transl. in Math. USSR-Izv. **14** (1980), 103–167.
- [S] J.-P. Serre, *Cours d'arithmétique*, Presses Universitaires de France, 1970.

BILKENT UNIVERSITY, TURKEY
AND STEKLOV MATHEMATICAL INSTITUTE, ST. PETERSBURG BRANCH, RUSSIA
E-mail address: `degt@fen.bilkent.edu.tr`, `degt@pdmi.ras.ru`

INSTITUT DE RECHERCHE MATHÉMATIQUE AVANCÉE,
UNIVERSITÉ LOUIS PASTEUR ET CNRS, FRANCE
E-mail address: `kharlam@math.u-strasbg.fr`