

PLANE SEXTICS WITH A TYPE \mathbf{E}_8 SINGULAR POINT

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ABSTRACT. We construct explicit geometric models for and compute the fundamental groups of all plane sextics with simple singularities only and with at least one type \mathbf{E}_8 singular point. In particular, we discover four new sextics with nonabelian fundamental groups; two of them are irreducible. The groups of the two irreducible sextics found are finite. The principal tool used is the reduction to trigonal curves and Grothendieck's *dessins d'enfants*.

1. INTRODUCTION

1.1. Principal results. This paper is a continuation of my paper [6], where we started the study of the equisingular deformation families and the fundamental groups of plane *sextics* (i.e., curves $B \subset \mathbb{P}^2$ of degree six) with a distinguished triple singular point, using the representation of such sextics via trigonal curves in Hirzebruch surfaces and Grothendieck's *dessins d'enfants*. (All varieties in the paper are over \mathbb{C} and are considered in their Hausdorff topology.) Recall that, in spite of the fact that the deformation classification of sextics can be reduced to a relatively simple, although tedious, arithmetical problem (see [2]), the geometry of the pairs (\mathbb{P}^2, B) remains a *terra incognita*, as the construction relies upon the global Torelli theorem for $K3$ -surfaces and is quite implicit. On the contrary, the approach suggested in [6], although not resulting in a defining equation for B , gives one a fairly good understanding of the topology of (\mathbb{P}^2, B) . In particular, it is sufficient for the computation of the fundamental group $\pi_1(\mathbb{P}^2 \setminus B)$. A few other applications of this approach and more motivation can be found in [6]. For a brief overview of the latest achievements on the subject, see C. Eyral, M. Oka [8].

In the present paper, we deal with the case when the distinguished triple point in question is of type \mathbf{E}_8 . The case of a type \mathbf{E}_7 singular point was considered in [6], and the case of \mathbf{E}_6 is the subject of a forthcoming paper. As in [6], a simple trick with the skeletons reduces most sextics $B \subset \mathbb{P}^2$ to certain trigonal curves \bar{B}' in Σ_2 (instead of the original surface Σ_3). This trick simplifies dramatically the classification of the sextics and the computation of their fundamental groups. It is still unclear if there is a simple geometric relation between B and \bar{B}' .

Throughout the paper, we consider a plane sextic $B \subset \mathbb{P}^2$ satisfying the following conditions:

- (*) B has simple singularities only, and
 B has a distinguished singular point P of type \mathbf{E}_8 .

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It is worth mentioning that all sextics with a non-simple singular point, as well as their fundamental groups, are well known. If such a sextic B also has a type \mathbf{E}_8 singular point, then the equisingular deformation class of B is determined by its set of singularities, which is either $\mathbf{E}_8 \oplus \mathbf{E}_{12}$ or $\mathbf{E}_8 \oplus \mathbf{J}_{2,i}$, $i = 0, 1$, in Arnol'd's notation, and the fundamental group $\pi_1(\mathbb{P}^2 \setminus B)$ is abelian.

Recall that the total Milnor number $\mu(B)$ of a plane sextic $B \subset \mathbb{P}^2$ with simple singularities only is subject to the inequality $\mu(B) \leq 19$. (The double covering X of the plane \mathbb{P}^2 ramified at B is a $K3$ -surface, and since the exceptional divisors arising from the resolution of the singular points are all linearly independent and span a negative definite lattice, one has $\mu(B) \leq h^{1,1}(X) - 1 = 19$.) A sextic B is called *maximal* (sometimes *maximizing*) if $\mu(B) = 19$. Recall also that maximal sextics are *rigid*, i.e., two such sextics belong to a connected equisingular deformation family if and only if they are related by a projective transformation. The principal results of the present paper are Theorems 1.1.1 and 1.1.2 (an explicit classification of the maximal sextics with a type \mathbf{E}_8 singular point) and Theorems 1.1.3 and 1.1.5 (the computation of the fundamental groups).

1.1.1. Theorem. *Up to projective transformation, or, equivalently, up to equisingular deformation, there are 39 maximal irreducible sextics B satisfying (*). They realize 26 sets of singularities (see Table 1 on Page 9).*

1.1.2. Theorem. *Up to projective transformation, or, equivalently, up to equisingular deformation, there are 18 maximal reducible sextics B satisfying (*). They realize 17 sets of singularities (see Table 2 on Page 11).*

Theorems 1.1.1 and 1.1.2 are proved in §2 (see 2.5 and 2.6, respectively), where all sextics are constructed explicitly using trigonal curves. This construction is further used in §3 in the computation of the fundamental groups of the sextics. Alternatively, the statements could as well be derived from combining the results of J.-G. Yang [13] (the existence) and I. Shimada [12], where the maximal sets of singularities realized by more than one deformation family are enumerated.

1.1.3. Theorem. *With two exceptions, the fundamental group $\pi_1(\mathbb{P}^2 \setminus B)$ of a plane sextic $B \subset \mathbb{P}^2$ satisfying (*) is abelian. The exceptions are:*

- (1) *the sextic with the set of singularities $\mathbf{E}_8 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$; the group is*

$$G_6 := \langle \alpha_1, \alpha_2 \mid (\alpha_1 \alpha_2^{-1})^5 \alpha_2^6 = 1, [\alpha_1, \alpha_2^3] = 1, (\alpha_1 \alpha_2)^2 \alpha_1 = (\alpha_2 \alpha_1)^2 \alpha_2 \rangle;$$

- (2) *the sextic with the set of singularities $\mathbf{E}_8 \oplus \mathbf{D}_6 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$; the group is*

$$G_\infty := \langle \alpha_1, \alpha_2 \mid (\alpha_1 \alpha_2^{-1})^5 \alpha_2^6 = 1, [\alpha_1, \alpha_2^3] = 1, (\alpha_1 \alpha_2)^2 = (\alpha_2 \alpha_1)^2 \rangle.$$

Each group G_i above, $i = 6$ or ∞ , can be represented as a semi-direct product of its abelianization, which is a cyclic group of order i , and its commutant $[G_i, G_i] \cong SL(2, \mathbb{F}_5)$, which is the only perfect group of order 120.

Theorem 1.1.3 is proved in 4.5. It is worth mentioning that the group G_6 in 1.1.3(1) is finite, substantiating my conjecture that the group of *any* irreducible sextic with simple singularities that is not of torus type is finite. Recall that a plane sextic B is said to be of *torus type* if its equation can be represented in the form $f_2^3 + f_3^2 = 0$ for some homogeneous polynomials f_2, f_3 of degree 2 and 3, respectively. The groups of sextics of torus type are all infinite: they factor to the reduced braid group $\mathbb{B}_3/(\sigma_1 \sigma_2)^3 \cong \mathbb{Z}_2 * \mathbb{Z}_3$.

1.1.4. Remark. More precisely, one has

$$G_\infty = \mathbb{Z} \times SL(2, \mathbb{F}_5) \quad \text{and} \quad G_6 = \mathbb{Z}_{12} \odot SL(2, \mathbb{F}_5),$$

where the latter central product is the quotient of $\mathbb{Z}_{12} \times SL(2, \mathbb{F}_5)$ by the diagonal $\mathbb{Z}_2 \subset \mathbb{Z}_2 \times \text{Center } SL(2, \mathbb{F}_5)$; for details, see 3.6 and 3.5, respectively.

Recall that, due to O. Zariski [14] (see also [7]), any perturbation $B \rightarrow B'$ of reduced plane curves induces an epimorphism $\pi_1(\mathbb{P}^2 \setminus B) \twoheadrightarrow \pi_1(\mathbb{P}^2 \setminus B')$ of their fundamental groups. In particular, if $\pi_1(\mathbb{P}^2 \setminus B)$ is abelian, so is $\pi_1(\mathbb{P}^2 \setminus B')$. Next theorem describes the few perturbations of plane sextics as in Theorem 1.1.3 that have nonabelian fundamental groups.

1.1.5. Theorem. *With two exceptions, the fundamental group of a sextic B' that is a proper perturbation of a plane sextic B satisfying (*) is abelian. The exceptions are:*

- (1) *the perturbation $\mathbf{E}_8 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2 \rightarrow 2\mathbf{A}_4 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_2$, and*
- (2) *the perturbation $\mathbf{E}_8 \oplus \mathbf{D}_6 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2 \rightarrow \mathbf{D}_6 \oplus \mathbf{D}_5 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$.*

For both curves, the perturbation epimorphism $\pi_1(\mathbb{P}^2 \setminus B) \twoheadrightarrow \pi_1(\mathbb{P}^2 \setminus B')$ is an isomorphism. In particular, the fundamental groups of the curves are G_6 and G_∞ , respectively, see Theorem 1.1.3.

This theorem is proved in 4.6. It covers over two hundred new (compared to [6]) sets of simple singularities realized by sextics with abelian fundamental groups. Recall that, according to [4], any induced subgraph of the combined Dynkin graph of a plane sextic B with simple singularities only can be realized by a perturbation of B ; in other words, the singular points of B can be perturbed independently. The total number of such sets of singularities currently known is over 1400.

1.2. Classical Zariski pairs. Among the new sets of singularities realized by sextics with abelian fundamental groups is $\mathbf{E}_6 \oplus \mathbf{A}_8 \oplus \mathbf{A}_2 \oplus 2\mathbf{A}_1$ (a perturbation of $\mathbf{E}_8 \oplus \mathbf{A}_8 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$, Nos. 9 and 17 in Table 1). The corresponding sextic is included into a so called *classical Zariski pair*, i.e., a pair of irreducible sextics that share the same set of singularities but have different Alexander polynomials (see, e.g., [2] for details). In each pair, one of the curves is *abundant*, or of *torus type*, and its Alexander polynomial is $t^2 - t + 1$; the other curve is *non-abundant*, or not of torus type, and its Alexander polynomial is 1. Here, the term ‘abundant’ is due to the fact that the Alexander polynomial of the curve is larger than the minimal polynomial imposed by its singularities. Conjecturally, in each pair the fundamental group of the abundant curve is the reduced braid group $\mathbb{B}_3/(\sigma_1\sigma_2)^3 \cong \mathbb{Z}_2 * \mathbb{Z}_3$, whereas the group of the non-abundant curve is abelian (hence, equal to \mathbb{Z}_6). At present, the conjecture is known for all sets of singularities except

$$\mathbf{A}_{17} \oplus \mathbf{A}_1, \quad \mathbf{A}_{14} \oplus \mathbf{A}_2 \oplus 2\mathbf{A}_1, \quad 2\mathbf{A}_8 \oplus 2\mathbf{A}_1, \quad 2\mathbf{A}_8 \oplus \mathbf{A}_1.$$

The group of the abundant curve is known for all sets of singularities except $\mathbf{A}_{14} \oplus \mathbf{A}_2 \oplus 2\mathbf{A}_1$. In E. Artal et al. [1], it is stated that the fundamental group of a sextic with a single type \mathbf{A}_{19} singular point is abelian; by perturbation, this assertion implies that the sets of singularities $\mathbf{A}_{17} \oplus \mathbf{A}_1$ and $2\mathbf{A}_8 \oplus \mathbf{A}_1$ can also be realized by irreducible sextics with abelian groups, leaving the conjecture unsettled for two sets of singularities only.

1.3. Contents of the paper. The paper depends on a preliminary computation found in [6]; it is based on the theory of trigonal curves, Grothendieck’s *dessins d’enfants*, braid monodromy, and Zariski–van Kampen’s method. We refer to [6] for a brief exposition of these subjects.

In §2, we prove Theorems 1.1.1 and 1.1.2 by providing an explicit geometric construction, in terms of the skeletons of the trigonal models, for all maximal sextics satisfying (*). This construction is used in §3 to compute the fundamental groups of the maximal sextics. In §4, we analyze the perturbations of maximal sextics with a type \mathbf{E}_8 singular point and prove Theorems 1.1.3 and 1.1.5. An important technical result here is Proposition 4.1.1, describing the local fundamental groups of all perturbations of a type \mathbf{E}_8 singularity.

2. THE CLASSIFICATION

In this section, we prove Theorems 1.1.1 and 1.1.2.

2.1. Maximal trigonal curves. Recall that the *Hirzebruch surface* Σ_k , $k \geq 0$, is a geometrically ruled rational surface with an *exceptional section* E of square $-k$. Sometimes, the fibers of the ruling are referred to as *vertical lines* in Σ_k . A *trigonal curve* is a curve $\bar{B} \subset \Sigma_k$ disjoint from E and intersecting each generic fiber at three points.

In this paper, we consider trigonal curves with simple singularities only.

A *singular fiber*, sometimes also called a *vertical tangent*, of a trigonal curve $\bar{B} \subset \Sigma_k$ is a fiber of the ruling of Σ_k intersecting \bar{B} geometrically at less than three points. Locally, \bar{B} is the ramification locus of the Weierstraß model of a Jacobian elliptic surface, and to describe the (topological) type of a singular fiber we use the standard notation for the singular elliptic fibers, referring to the extended Dynkin graph of the corresponding configuration of the exceptional divisors. The types are as follows:

- $\tilde{\mathbf{A}}_0^*$: a simple vertical tangent;
- $\tilde{\mathbf{A}}_0^{**}$: a vertical inflection tangent;
- $\tilde{\mathbf{A}}_1^*$: a node of \bar{B} with one of the branches vertical;
- $\tilde{\mathbf{A}}_2^*$: a cusp of \bar{B} with vertical tangent;
- $\tilde{\mathbf{A}}_p, \tilde{\mathbf{D}}_q, \tilde{\mathbf{E}}_6, \tilde{\mathbf{E}}_7, \tilde{\mathbf{E}}_8$: a simple singular point of \bar{B} of the same type with minimal possible local intersection index with the fiber.

For more details, including the relation to Kodaira’s classification of singular elliptic fibers, we refer to [6] and [3].

The type $\tilde{\mathbf{A}}_0^{**}$, $\tilde{\mathbf{A}}_1^*$, and $\tilde{\mathbf{A}}_2^*$ singular fibers of a trigonal curve are called *unstable*, and all other singular fibers are called *stable*. Informally, a fiber is unstable if its type does not need to be preserved under equisingular, but not necessarily fiberwise, deformations of the curve.

A trigonal curve is called *stable* if all its singular fibers are stable.

The (*functional*) *j*-invariant $j = j_{\bar{B}}: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ of a trigonal curve $\bar{B} \subset \Sigma_k$ is defined as the analytic continuation of the function sending a point b in the base \mathbb{P}^1 of Σ_k to the *j*-invariant (divided by 12^3) of the elliptic curve covering the fiber F over b and ramified at $F \cap (\bar{B} + E)$. The curve \bar{B} is called *isotrivial* if $j_{\bar{B}} = \text{const.}$ Such curves can easily be enumerated, see, e.g., [3].

2.1.1. Definition. A non-isotrivial trigonal curve \bar{B} is called *maximal* if it has

the following properties:

- (1) \bar{B} has no singular fibers of type $\tilde{\mathbf{D}}_4$;
- (2) $j = j_{\bar{B}}$ has no critical values other than 0, 1, and ∞ ;
- (3) each point in the pull-back $j^{-1}(0)$ has ramification index at most 3;
- (4) each point in the pull-back $j^{-1}(1)$ has ramification index at most 2.

It is shown in [6] that the total Milnor number $\mu(\bar{B})$ of a non-isotrivial trigonal curve $\bar{B} \subset \Sigma_k$ with simple singularities only is subject to the inequality

$$\mu(\bar{B}) \leq 5k - 2 - \#\{\text{unstable fibers of } \bar{B}\},$$

which turns into an equality if and only if \bar{B} is maximal.

2.2. The trigonal model. The following statement, proved in [6], reduces the study of sextics with a type \mathbf{E}_8 singular point to the study of trigonal curves in the Hirzebruch surface Σ_3 .

2.2.1. Proposition. *There is a natural bijection ϕ , invariant under equisingular deformations, between the following two sets:*

- (1) plane sextics B with a distinguished type \mathbf{E}_8 singular point P , and
- (2) trigonal curves $\bar{B} \subset \Sigma_3$ with a distinguished type $\tilde{\mathbf{A}}_1^*$ singular fiber F .

A sextic B is irreducible if and only if so is $\bar{B} = \phi(B)$ and, with one exception, B is maximal if and only if \bar{B} is maximal and has no unstable fibers other than F . The exception is the reducible sextic B with the set of singularities $\mathbf{E}_8 \oplus \mathbf{E}_7 \oplus \mathbf{D}_4$; in this case, $\phi(B)$ is isotrivial.

The trigonal curve \bar{B} corresponding to a sextic B (more precisely, pair (B, P)) under ϕ above is called the *trigonal model* of B .

From now on, we disregard the exceptional case $\mathbf{E}_8 \oplus \mathbf{E}_7 \oplus \mathbf{D}_4$, which is treated separately in Subsection 3.7, and assume that \bar{B} is not isotrivial, hence maximal. Let $j_{\bar{B}}: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be the j -invariant of \bar{B} , and let $\text{Sk} = j_{\bar{B}}^{-1}([0, 1]) \subset S^2$ be its skeleton (see [6] or [3]). Recall that Sk is a connected planar map with at most 3-valent \bullet -vertices (the pull-backs of 0) and monovalent \circ -vertices (some of the pull-backs of 1) connected to \bullet -vertices. In fact, each edge connecting two \bullet -vertices contains a bivalent \circ -vertex in the middle, making Sk a bipartite graph, but these bivalent \circ -vertices are ignored.

Denote by u the monovalent \circ -vertex of Sk corresponding to the distinguished type $\tilde{\mathbf{A}}_1^*$ fiber F given by Proposition 2.2.1, and let v be the \bullet -vertex adjacent to u . The induced subgraph of Sk spanned by u and v , i.e., u , v , and the edge $[u, v]$, is called the *insertion*. In the drawings below, the insertion is shown in gray.

2.2.2. Lemma. *The mono- (respectively, bi-) valent \bullet -vertices of Sk correspond to the type $\tilde{\mathbf{E}}_6$ (respectively, $\tilde{\mathbf{E}}_8$) singular fibers of \bar{B} , and the monovalent \circ -vertices of Sk other than u correspond to the type $\tilde{\mathbf{E}}_7$ singular fibers of \bar{B} . Furthermore, one has*

$$(2.2.3) \quad 3\#\bullet(1) + 4\#\bullet(2) + \#\bullet(3) + 3\#\circ(1) = 8 - 2d,$$

where $\#\bullet(i)$ is the number of \bullet -vertices of valency i , $\bullet = \bullet$ or \circ , and d is the number of the $\tilde{\mathbf{D}}$ -type fibers of \bar{B} .

Proof. The first statement follows from the fact that all singular fibers of \bar{B} other than F are stable and the relation between the vertices of Sk and the fibers of \bar{B} (see [6] or [3]). Then the number t of the triple points of \bar{B} is given by

$$t = d + \#\bullet(1) + \#\bullet(2) + \#\circ(1) - 1,$$

and (2.2.3) follows from the vertex count given by Corollary 2.5.5 in [6]. \square

2.2.4. Remark. Recall that, in addition to the $\tilde{\mathbf{E}}$ -type singular fibers described in Lemma 2.2.2, each m -gonal region of Sk contains a unique type $\tilde{\mathbf{A}}_{m-1}$ ($\tilde{\mathbf{A}}_0^*$ if $m = 0$) or type $\tilde{\mathbf{D}}_{m+4}$ fiber of \bar{B} . The total number d of the $\tilde{\mathbf{D}}$ -type fibers is given by (2.2.3), and the d regions containing such fibers can be chosen arbitrary, resulting in general in distinct deformation classes of curves and even in distinct sets of singularities.

2.3. The reduction. We still assume that the trigonal model $\bar{B} \subset \Sigma_3$ is non-isotrivial and maximal and keep the notation u and v for, respectively, the \circ - and \bullet -vertices spanning the insertion in Sk .

2.3.1. Lemma. *If v is a monovalent vertex, then Sk is as shown in Figure 2(k) below and the set of singularities of B is $\mathbf{E}_8 \oplus \mathbf{E}_6 \oplus \mathbf{D}_5$.*

Proof. Under the assumptions, the insertion is an edge bounded by two monovalent vertices. On the other hand, it is a subgraph of a connected graph Sk . Hence, Sk is exhausted by the insertion, i.e., it is the graph shown in Figure 2(k). Then, (2.2.3) implies that $d = 1$; hence, the set of singularities of B is $\mathbf{E}_8 \oplus \mathbf{E}_6 \oplus \mathbf{D}_5$. \square

2.3.2. Lemma. *If v is a bivalent vertex, then Sk is the graph shown in Figure 2(j) below and the set of singularities of B is $2\mathbf{E}_8 \oplus \mathbf{A}_3$.*

Proof. If v is bivalent (corresponding to a type $\tilde{\mathbf{E}}_8$ singular fiber of \bar{B} , i.e., to a type \mathbf{E}_8 singular point of B other than P), then the vertex count (2.2.3) implies that Sk has exactly one more \bullet -vertex, which is trivalent, and one can easily see that the only skeleton with these properties is the one shown in Figure 2(j). Besides, (2.2.3) also implies $d = 0$. Hence, the set of singularities of B is $2\mathbf{E}_8 \oplus \mathbf{A}_3$. \square

Finally, consider the ‘general’ case, when v is trivalent. In this case, removing the insertion and patching the two other edges incident to v to a single edge, one obtains a new connected graph Sk' .

2.3.3. Lemma. *With one exception (see below), the graph Sk' constructed above is the skeleton of a certain stable maximal trigonal curve $\bar{B}' \subset \Sigma_2$. Conversely, attaching an insertion at the middle of any edge of the skeleton Sk' of a stable maximal trigonal curve $\bar{B}' \subset \Sigma_2$, one obtains a skeleton Sk defining a maximal trigonal curve \bar{B} as in Proposition 2.2.1.*

The exception is the skeleton Sk shown in Figure 3(g), when Sk' has no \bullet -vertices; the set of singularities of the corresponding curve B is $\mathbf{E}_8 \oplus \mathbf{D}_6 \oplus \mathbf{D}_5$.

Proof. First, notice that v cannot be adjacent to three monovalent \circ -vertices, as that would contradict to (2.2.3). Then, it is easy to see that, provided that Sk' has at least one \bullet -vertex (as any skeleton does), it is indeed a valid skeleton, and (2.2.3) transforms to the following vertex count for Sk' :

$$3\#\bullet(1) + 4\#\bullet(2) + \#\bullet(3) + 3\#\circ(1) = 4 - 2d.$$

Using [6, Corollary 2.5.5], one concludes that Sk' is the skeleton of a stable maximal trigonal curve $\bar{B}' \subset \Sigma_2$. The converse statement is obvious.

In the exceptional case, when Sk' has no \bullet -vertices, since Sk' is still connected, it must consist of a single circle. Then Sk is the graph shown in Figure 3(g), and the vertex count (2.2.3) implies that $d = 2$, i.e., each of the two regions of Sk contains a $\tilde{\mathbf{D}}$ -type fiber of \bar{B} and the set of singularities of B is $\mathbf{E}_8 \oplus \mathbf{D}_6 \oplus \mathbf{D}_5$. \square

2.4. Reducible vs. irreducible curves. Recall that a *marking* at a trivalent \bullet -vertex w of a skeleton Sk is a counterclockwise ordering $\{e_1, e_2, e_3\}$ of the three edges attached to w . A marking is uniquely defined by assigning index 1 to one of the three edges. Given a marking, the indices of the edges are considered defined modulo 3, so that $e_4 = e_1$, $e_5 = e_2$, etc.

2.4.1. Definition. A *marking* of a skeleton Sk is a collection of markings at all trivalent \bullet -vertices of Sk . Given a marking, one can assign a type $[i, j]$, $i, j \in \mathbb{Z}_3$, to each edge e of Sk connecting two trivalent \bullet -vertices, according to the indices of the two ends of e . A marking of a skeleton without mono- or bivalent \bullet -vertices is called *splitting* if it satisfies the following two conditions:

- (1) the types of all edges are $[1, 1]$, $[2, 3]$, or $[3, 2]$;
- (2) an edge connecting a \bullet -vertex w and a monovalent \circ -vertex is e_1 at w .

According to [3], the splitting markings of the skeleton of a maximal trigonal curve $\bar{B} \subset \Sigma_k$ are in a one-to-one correspondence with the *linear* components of \bar{B} (i.e., components of \bar{B} that are sections of Σ_k).

2.4.2. Lemma. Let Sk, Sk' be a pair of skeletons as in Lemma 2.3.3, so that Sk' is obtained from Sk by removing the insertion. Then the curve \bar{B} defined by Sk is reducible if and only if Sk' admits a splitting marking such that the insertion is attached to an edge of type $[2, 3]$ and is oriented as shown in Figure 1(b).

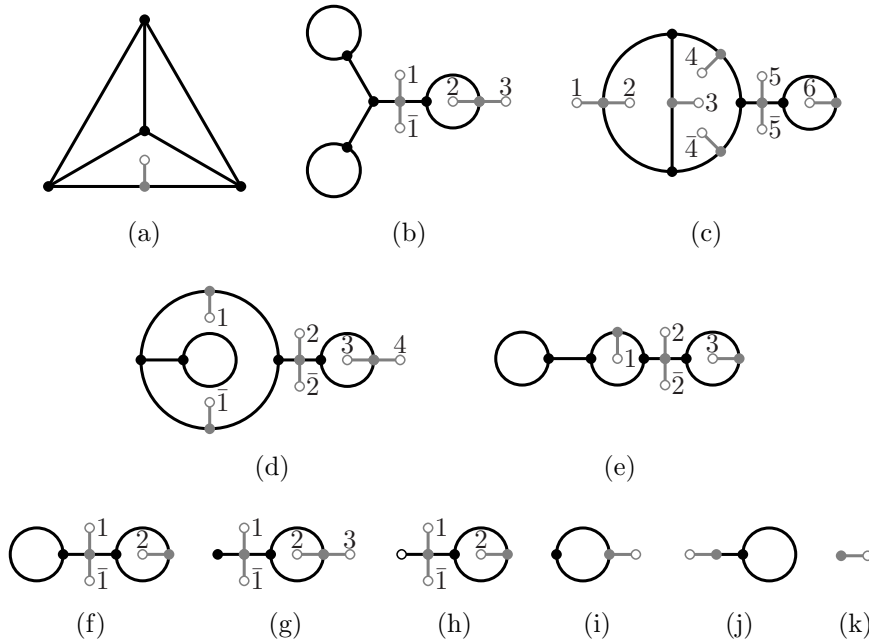


FIGURE 1. An insertion resulting in (a) irreducible and (b) reducible trigonal curve \bar{B}

Proof. In view of condition 2.4.1(2), a splitting marking of Sk restricts to a splitting marking of Sk' . Conversely, due to 2.4.1(2) again, a splitting marking of Sk' extends to Sk if and only if the insertion is as shown in Figure 1(b). \square

2.4.3. Corollary. In the notation of Lemma 2.4.2, if Sk' admits more than one splitting marking (equivalently, if the corresponding trigonal curve \bar{B}' splits into three distinct linear components), then \bar{B} is reducible.

Proof. A splitting marking is uniquely determined by its restriction to a single vertex. Hence, the restrictions of the three splitting markings of Sk' to any given oriented edge e are pairwise distinct, and, since there are only three types (see condition 2.4.1(1)), e can be made of any given type under one of the markings. \square

FIGURE 2. Skeletons of irreducible curves \bar{B}

2.5. Proof of Theorem 1.1.1. Due to Proposition 2.2.1, we need to enumerate all irreducible maximal trigonal curves $\bar{B} \subset \Sigma_3$ with a unique unstable fiber, which is of type $\tilde{\mathbf{A}}_1^*$. The study of such curves reduces, in its turn, to the study of the skeletons containing an insertion and satisfying the vertex count (2.2.3). With two exceptions, described in Lemmas 2.3.1 and 2.3.2 (see Figure 2(j) and (k); these two skeletons obviously define irreducible curves), each skeleton Sk as above is obtained by attaching an insertion to the skeleton Sk' of a stable maximal trigonal curve $\bar{B}' \subset \Sigma_2$ (see Lemma 2.3.3). Note that the exceptional skeleton in Lemma 2.3.3 admits a splitting marking and thus defines a reducible curve.

The classification of stable maximal trigonal curves in Σ_2 is found in [5]. In view of Corollary 2.4.3, one should take for Sk' a skeleton admitting none (Figure 2(a), (b), (d), (g), and (i)) or exactly one (Figure 2(c), (e), (f), and (h)) splitting marking. In the former case, the insertion can be attached arbitrarily at the middle of any edge. In the latter case, Lemma 2.4.2 implies that the edge e that the insertion is attached to is either of type $[1, 1]$ (with respect to the only splitting marking) or of type $[2, 3]$ and oriented with respect to the insertion as shown in Figure 1(a). In Figure 2, we list and number the resulting possibilities for Sk' (shown in black) and the attaching of the insertion (shown in gray), up to symmetries of Sk' , i.e., orientation preserving auto-diffeomorphisms of the sphere preserving Sk' . A pair of indices n, \bar{n} designates a pair of insertions that differ by an orientation reversing automorphism of Sk' ; they result in a pair of complex conjugate sextics.

The set of singularities of B is almost determined by the skeleton Sk (see Lemma 2.2.2 and Remark 2.2.4). The only indeterminacy is the one caused by the choice of the region containing a $\tilde{\mathbf{D}}$ -type singular fiber (see Remark 2.2.4). The configurations obtained are listed in Table 1. The table also contains a reference to the fragment in Figure 2 representing the curve and the number of equisingular deformation classes

TABLE 1. Maximal sets of singularities with a type E_8 point represented by irreducible sextics

#	Set of singularities	Figure	Count	π_1	(l, m, n)
1	$E_8 \oplus A_4 \oplus A_3 \oplus 2A_2$	2(a)	(1, 0)	3.5	(5, 4, 3)
2	$E_8 \oplus A_{11}$	2(b)-1, $\bar{1}$	(0, 1)	3.3.1	(-, -, 1)
3	¹ $E_8 \oplus A_9 \oplus A_2$	2(b)-2	(1, 0)	3.3.2	(3, -, -)
4	² $E_8 \oplus A_{10} \oplus A_1$	2(b)-3	(1, 0)	3.3.3	
5	$E_8 \oplus A_7 \oplus 2A_2$	2(c)-1	(1, 0)	3.2.3	(8, 3, 3)
6	$E_8 \oplus A_6 \oplus A_3 \oplus A_2$	2(c)-2	(1, 0)	3.2.3	(4, 7, 3)
7	³ $E_8 \oplus A_5 \oplus A_4 \oplus A_2$	2(c)-3	(1, 0)	3.2.3	(5, 3, 6)
8	⁴ $E_8 \oplus A_6 \oplus A_4 \oplus A_1$	2(c)-4, $\bar{4}$	(0, 1)	3.2.3	(5, 7, 2)
9	⁵ $E_8 \oplus A_8 \oplus A_2 \oplus A_1$	2(c)-5, $\bar{5}$	(0, 1)	3.3.1	(-, -, 1)
10	$E_8 \oplus A_6 \oplus 2A_2 \oplus A_1$	2(c)-6	(1, 0)	3.3.2	(3, -, -)
11	$E_8 \oplus A_6 \oplus A_5$	2(d)-1, $\bar{1}$	(0, 1)	3.2.3	(7, -, -)
12	$E_8 \oplus A_7 \oplus A_4$	2(d)-2, $\bar{2}$	(0, 1)	3.3.1	(-, -, 1)
13	³ $E_8 \oplus A_5 \oplus A_4 \oplus A_2$	2(d)-3	(1, 0)	3.3.2	(3, -, -)
14	⁴ $E_8 \oplus A_6 \oplus A_4 \oplus A_1$	2(d)-4	(1, 0)	3.3.3	
15	$E_8 \oplus A_8 \oplus A_3$	2(e)-1	(1, 0)	3.2.3	(4, 9, 9)
16	² $E_8 \oplus A_{10} \oplus A_1$	2(e)-2, $\bar{2}$	(0, 1)	3.3.1	(-, -, 1)
17	⁵ $E_8 \oplus A_8 \oplus A_2 \oplus A_1$	2(e)-3	(1, 0)	3.3.2	(3, -, -)
18	$E_8 \oplus D_{11}$	2(f)-1	(1, 0)	3.3.1	(-, -, 1)
19	$E_8 \oplus D_5 \oplus A_6$	2(f)-1, $\bar{1}$	(0, 1)	3.3.1	(-, -, 1)
20	$E_8 \oplus D_9 \oplus A_2$	2(f)-2	(1, 0)	3.3.2	(3, -, -)
21	$E_8 \oplus D_7 \oplus A_4$	2(f)-2	(1, 0)	3.2.3	(-, 5, 5)
22	$E_8 \oplus D_5 \oplus A_4 \oplus A_2$	2(f)-2	(1, 0)	3.3.2	(3, -, -)
23	$E_8 \oplus E_6 \oplus A_5$	2(g)-1, $\bar{1}$	(0, 1)	3.3.1	(-, -, 1)
24	$E_8 \oplus E_6 \oplus A_3 \oplus A_2$	2(g)-2	(1, 0)	3.3.2	(3, -, -)
25	$E_8 \oplus E_6 \oplus A_4 \oplus A_1$	2(g)-3	(1, 0)	3.3.3	
26	$E_8 \oplus E_7 \oplus A_4$	2(h)-1, $\bar{1}$	(0, 1)	3.3.1	(-, -, 1)
27	$E_8 \oplus E_7 \oplus 2A_2$	2(h)-2	(1, 0)	3.3.2	(3, -, -)
28	$2E_8 \oplus A_2 \oplus A_1$	2(i)	(1, 0)	3.3.3	
29	$2E_8 \oplus A_3$	2(j)	(1, 0)	3.4.3	
30	$E_8 \oplus E_6 \oplus D_5$	2(k)	(1, 0)	3.4.4	

of curves, shown in the form (n_r, n_c) , where n_r is the number of real curves and n_c is the number of pairs of complex conjugate curves, so that the total number is $n_r + 2n_c$. The last two columns are related to the computation of the fundamental group: the column ' π_1 ' refers to the section where the group is computed, and (l, m, n) are the values of the parameters used in the computation (see 3.2.3 for details).

Note that some sets of singularities appear from several distinct skeletons. In Table 1, we prefix the corresponding lines with an index ¹, ², etc., equal indices corresponding to the same set of singularities. The set of singularities prefixed with ¹ is also realized by a reducible sextic (see Table 2). Summarizing, one obtains 26 sets of singularities realized by 39 sextics, 21 real ones and 9 pairs of complex conjugate ones. This proves Theorem 1.1.1. \square

2.5.1. Remark. The skeleton Sk' in Figure 2(f) has a symmetry interchanging

fragments 1 and $\bar{1}$. However, this symmetry also interchanges the two monogonal regions of the corresponding skeleton Sk . Hence, if one of these regions contains a type $\bar{\mathbf{D}}_5$ fiber of \bar{B} (the set of singularities $\mathbf{E}_8 \oplus \mathbf{D}_5 \oplus \mathbf{A}_6$, see No. 19 in Table 1), this symmetry does not lift to a symmetry of $\bar{B} \subset \Sigma_3$ and one obtains two distinct complex conjugate families.

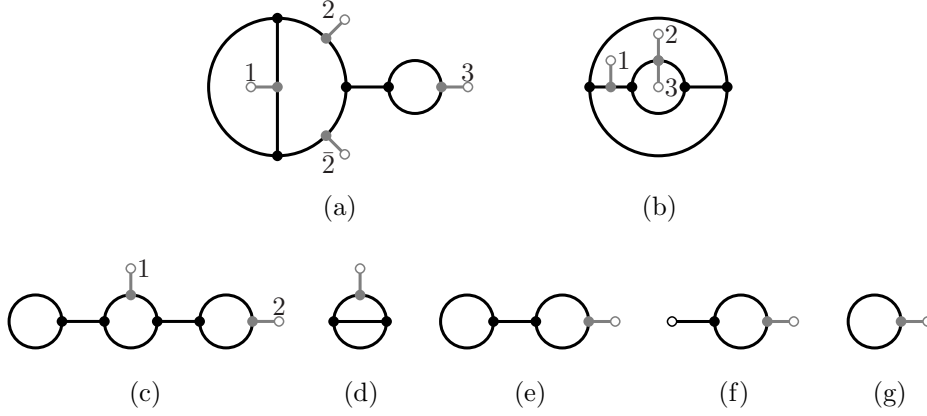


FIGURE 3. Skeletons of reducible curves \bar{B}

2.6. Proof of Theorem 1.1.2. The proof repeats, almost literally, the proof of Theorem 1.1.1. Since the two exceptional curves given by Lemmas 2.3.1 and 2.3.2 are irreducible, it suffices to consider a skeleton Sk obtained from a graph Sk' as in Lemma 2.3.3. Furthermore, unless Sk' is a single circle (see Figure 3(g), the exceptional case in Lemma 2.3.3), Lemma 2.4.2 and Corollary 2.4.3 imply that Sk' is the skeleton of a *reducible* curve $B' \subset \Sigma_2$ and either B' splits into three linear components (Figure 3(b) and (d)), and then the insertion can be attached arbitrarily, or Sk' has exactly one splitting marking (Figure 3(a), (c), (e), and (f)) and, with respect to this marking, the insertion is attached to an edge of type $[2, 3]$ and is oriented as shown in Figure 1(a).

The classification of reducible stable maximal trigonal curves in Σ_2 is found in [5]; their skeletons are shown (in black) in Figure 3. The possible positions of the insertion, up to symmetries of Sk' , are shown in gray, and the corresponding sets of singularities (see Lemma 2.2.2 and Remark 2.2.4) are listed in Table 2. The notation in the figure and the columns in the table are the same as in the previous section. It turns out that each set of singularities is obtained from a unique skeleton. The set of singularities $\mathbf{E}_8 \oplus \mathbf{A}_9 \oplus \mathbf{A}_2$ prefixed with 1 in the table is also realized by an irreducible sextic (see Table 1). Adding the (unique) isotrivial trigonal model realizing the set of singularities $\mathbf{E}_8 \oplus \mathbf{E}_7 \oplus \mathbf{D}_4$ (see Proposition 2.2.1) and summarizing, one obtains 17 sets of singularities realized by 18 curves, 16 real ones and one pair of complex conjugate ones. \square

3. THE FUNDAMENTAL GROUP

In this section, we compute the fundamental group $\pi_1 := \pi_1(\mathbb{P}^2 \setminus B)$ of a maximal sextic B satisfying (*). Most of the time, we assume that the trigonal model \bar{B} of B is not isotrivial, and we denote by Sk the skeleton of \bar{B} and keep the notation u and v for, respectively, the \circ - and \bullet -vertices spanning the insertion (see 2.2).

TABLE 2. Maximal sets of singularities with a type E_8 point represented by reducible sextics

#'	Set of singularities	Figure	Count	π_1	(l, m, n)
1'	$E_8 \oplus A_5 \oplus 2A_3$	3(a)-1	(1, 0)	3.2.4	(4, 4, 6)
2'	$E_8 \oplus A_7 \oplus A_3 \oplus A_1$	3(a)-2, $\bar{2}$	(0, 1)	3.2.4	(8, 4, 2)
3'	$E_8 \oplus A_7 \oplus A_2 \oplus 2A_1$	3(a)-3	(1, 0)	3.3.3	
4'	$E_8 \oplus A_5 \oplus A_4 \oplus 2A_1$	3(b)-1	(1, 0)	3.2.4	(6, 5, 2)
5'	$E_8 \oplus A_5 \oplus A_3 \oplus A_2 \oplus A_1$	3(b)-2	(1, 0)	3.2.4	(6, 3, 4)
6'	$E_8 \oplus A_4 \oplus 2A_3 \oplus A_1$	3(b)-3	(1, 0)	3.2.4	(4, 5, 4)
7'	$E_8 \oplus A_9 \oplus A_2$	3(c)-1	(1, 0)	3.2.4	(10, 3, -)
8'	$E_8 \oplus A_9 \oplus 2A_1$	3(c)-2	(1, 0)	3.3.3	
9'	$E_8 \oplus D_8 \oplus A_2 \oplus A_1$	3(d)	(1, 0)	3.2.4	(-, 3, 2)
10'	$E_8 \oplus D_7 \oplus A_3 \oplus A_1$	3(d)	(1, 0)	3.2.4	(4, -, 2)
11'	$E_8 \oplus D_6 \oplus A_3 \oplus A_2$	3(d)	(1, 0)	3.6	(4, 3, -)
12'	$E_8 \oplus D_{10} \oplus A_1$	3(e)	(1, 0)	3.3.3	
13'	$E_8 \oplus D_6 \oplus A_5$	3(e)	(1, 0)	3.4.1	(6, -, 6)
14'	$E_8 \oplus D_5 \oplus A_5 \oplus A_1$	3(e)	(1, 0)	3.3.3	
15'	$E_8 \oplus E_7 \oplus A_3 \oplus A_1$	3(f)	(1, 0)	3.3.3	
16'	$E_8 \oplus D_6 \oplus D_5$	3(g)	(1, 0)	3.4.2	
17'	$E_8 \oplus E_7 \oplus D_4$	isotrivial	(1, 0)	3.7	

3.1. The presentation. Assume that v is trivalent, choose the marking at v so that the edge $[v, u]$ is e_2 at v , and let $\{\alpha_1, \alpha_2, \alpha_3\}$ be a canonical basis in the fiber F_v over v defined by this marking (Figure 4; see [3] or [6] for details). According to [6], the basis elements are subject to the following relations, called the *relations at infinity*:

$$(3.1.1) \quad \rho^3 = \alpha_1 \alpha_2^2,$$

$$(3.1.2) \quad \alpha_3 = \alpha_2 \alpha_1 \alpha_2^{-1},$$

$$(3.1.3) \quad [\alpha_1, \alpha_2^3] = 1,$$

where $\rho = \alpha_1 \alpha_2 \alpha_3$. In particular, it follows that α_2^3 is a central element.

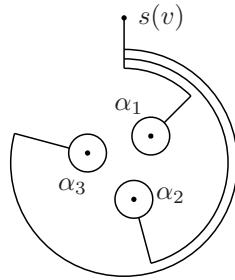


FIGURE 4. The canonical basis

Note that one can eliminate α_3 and simplify (3.1.1) to the following relation:

$$(3.1.4) \quad (\alpha_1 \alpha_2^{-1})^5 \alpha_2^6 = 1.$$

We will use (3.1.4) in the final presentations of the groups.

Let F_1, \dots, F_r be the singular fibers of \bar{B} other than F , and let $m_j \subset \mathbb{B}_3$ be the braid monodromy about F_j , $j = 1, \dots, r$ (see [6] for the choice of the reference section and other details). Then, the Zariski–van Kampen theorem [10] states that π_1 has the presentation

$$(3.1.5) \quad \pi_1 = \langle \alpha_1, \alpha_2, \alpha_3 \mid m_j = \text{id}, j = 1, \dots, r, \text{ and (3.1.1)–(3.1.3)} \rangle,$$

where each *braid relation* $m_j = \text{id}$, $j = 1, \dots, r$, should be understood as the triple of relations $m_j(\alpha_i) = \alpha_i$, $i = 1, 2, 3$. Furthermore, in the presence of the relations at infinity, (any) one of the braid relations $m_j = \text{id}$, $j = 1, \dots, r$, can be omitted.

The braid monodromy m_j can easily be computed using the skeleton (or dessin) of \bar{B} , see [3] or [6] for an exposition more tailored for the problem in question. We omit all details and merely indicate the results.

3.2. The monodromy to be considered. Given two elements α, β of a group and a nonnegative integer m , introduce the notation

$$\{\alpha, \beta\}_m = \begin{cases} (\alpha\beta)^k(\beta\alpha)^{-k}, & \text{if } m = 2k \text{ is even,} \\ ((\alpha\beta)^k\alpha)((\beta\alpha)^k\beta)^{-1}, & \text{if } m = 2k + 1 \text{ is odd.} \end{cases}$$

The relation $\{\alpha, \beta\}_m = 1$ is equivalent to $\sigma^m = \text{id}$, where σ is the generator of the braid group \mathbb{B}_2 acting on the free group $\langle \alpha, \beta \rangle$. Hence,

$$(3.2.1) \quad \{\alpha, \beta\}_m = \{\alpha, \beta\}_n = 1 \quad \text{is equivalent to} \quad \{\alpha, \beta\}_{\text{g.c.d.}(m,n)} = 1.$$

For the small values of m , the relation $\{\alpha, \beta\}_m = 1$ takes the following form:

- $m = 0$: tautology;
- $m = 1$: the identification $\alpha = \beta$;
- $m = 2$: the commutativity relation $[\alpha, \beta] = 1$;
- $m = 3$: the braid relation $\alpha\beta\alpha = \beta\alpha\beta$.

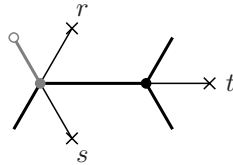


FIGURE 5. The braid monodromy to be considered

Recall that the skeleton Sk of a maximal trigonal curve \bar{B} is introduced in [3] as a simplified version of its *dessin* Γ , which is defined as the pull-back under $j_{\bar{B}}$ of the real part $\mathbb{P}_{\mathbb{R}}^1 \subset \mathbb{P}^1$, with the three special values 0, 1, and ∞ taken into account. Thus, in addition to \bullet - and \circ -vertices, Γ also has \times -vertices (the pull-backs of ∞). The \times -vertices are connected to the \bullet - and \circ -vertices by, respectively, solid and dotted edges, whereas the edges of Sk are regarded bold. If \bar{B} is maximal, then Γ is uniquely recovered from Sk : one inserts a single \times -vertex a_R inside each region R of Sk and connects it to all vertices in ∂R in the star like fashion. This vertex a_R

corresponds to the unique singular fiber of \bar{B} inside R , cf. Remark 2.2.4. The braid monodromy about a_R is described in [6].

In most cases, in order to compute the fundamental group, it suffices to consider the braid monodromy about the three \times -vertices r, s, t shown in Figure 5. Here, r is the \times -vertex immediately adjacent to v , s is the vertex ‘opposite’ to v , and t has an extra bold edge in the path connecting it to v . Note that we do *not* assume that all three vertices are distinct. Assume that r, s , and t are the centers of, respectively, an l -, m -, and n -gonal region of the skeleton. Then the relations resulting from the braid monodromy about these vertices are

$$(3.2.2) \quad \begin{aligned} r &: \{\alpha_1, \alpha_2\}_l = 1, \\ s &: \{\alpha_1, \alpha_s\}_m = 1, \quad \text{where } \alpha_s = \alpha_2\alpha_3\alpha_2^{-1}, \\ t &: \{\alpha_2, \alpha_t\}_n = 1, \quad \text{where } \alpha_t = (\alpha_1\alpha_2)\alpha_3(\alpha_1\alpha_2)^{-1}, \end{aligned}$$

provided that the singular fiber of \bar{B} over the corresponding vertex is of type \tilde{A} . (If the fiber is of type \tilde{D} , we usually ignore the corresponding relation.) The additional relations (3.2.2), as well as the original relations at infinity (3.1.1)–(3.1.3), are easily programmable in GAP [9] (see Figure 6).

```
gap> a := FreeGroup(3);;
gap> ## Common relations (at infinity)
gap> r1 := (a.1*a.2*a.3)^3/(a.1*a.2^2);;
gap> r2 := a.2*a.1*a.2^-1/a.3;;
gap> r3 := a.1*a.2^3/(a.2^3*a.1);;
gap> ## Common elements to be used in other relations
gap> as := a.2*a.3*a.2^-1;;
gap> at := (a.1*a.2)*a.3*(a.1*a.2)^-1;;
gap> ## The variable relation (n=0 means no relation)
gap> r := function(a,b,n)
> if (n mod 2) = 0 then
> return (a*b)^(n/2)/(b*a)^(n/2);
> else
> return (a*b)^((n-1)/2)*a/((b*a)^((n-1)/2)*b);
> fi;
> end;;
gap> ## Most fundamental groups are handled by this
gap> size := function(l, m, n)
> return Size(a/[r1, r2, r3, r(a.1,a.2,l), r(a.1,as,m), r(a.2,at,n)]);
> end;;
gap> size2 := function(l, m, n)
> return Size(a/[r1, r2, r3, r(a.1,a.2,l), r(a.1,as,m), r(a.2,at,n), a.2^3]);
> end;;
```

FIGURE 6. The GAP input

Note that any of relations (3.2.2) can easily be ignored: one should just let the corresponding parameter l, m , or n to be equal to zero. To emphasize the fact that a relation is omitted, we will use ‘-’ instead of ‘0’ in the references to (3.2.2).

3.2.3. For most irreducible maximal curves, the group π_1 is computed using GAP [9]: the function `size(l,m,n)` in Figure 6 returns 6 and, since $\pi_1/[\pi_1, \pi_1] = \mathbb{Z}_6$, this implies that π_1 is abelian. The values (l, m, n) used are listed in Table 1. These values are easily read from the skeletons of the curves (see Figure 2). The \tilde{D} -type singular fibers, if present, are ignored.

3.2.4. For reducible maximal curves, we use the following obvious observation: Let G be a group, and let $H \subset G$ be a central subgroup such that the projection $H \rightarrow G/[G, G]$ is a monomorphism. Then the commutants of G and of G/H are isomorphic. We apply this statement to the subgroup $H \subset \pi_1$ generated by the central element α_2^3 and compute the quotient π_1/α_2^3 using GAP [9]. Note that, if the curve is known to be reducible, then α_2^3 projects to an element of infinite order in $\pi_1/[\pi_1, \pi_1]$ (hence, the statement above does apply), and the abelianization of π_1/α_2^3 is \mathbb{Z}_{15} . Thus, if the function `size2(1,m,n)` in Figure 6 returns 15, we conclude that π_1 is abelian. The values (l, m, n) used are listed in Table 2. These values are read from the skeletons of the curves (see Figure 3), with the $\tilde{\mathbf{D}}$ -type singular fibers ignored.

In the sequel, without further references, we assume that all statements like ‘ $\text{ord } \pi_1 = 6$ ’ or ‘ $\text{ord}(\pi_1/\alpha_2^3) = 15$ ’ are proved using GAP [9].

3.3. Insertion close to a loop. Here, we consider a few special positions of the insertion with respect to a loop (i.e., a monogonal region) of Sk' .

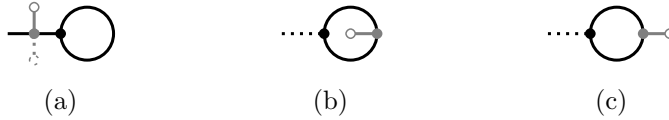


FIGURE 7. Fragments resulting in abelian π_1

3.3.1. Assume that the skeleton of \bar{B} has a fragment shown in Figure 7(a), i.e., the insertion is right next to a loop. Then π_1 has relation (3.2.2) with $(l, m, n) = (-, -, 1)$, and one concludes that $\pi_1 = \mathbb{Z}_6$ (as `size(0,0,1)` returns 6). If the insertion is as shown in Figure 7(a) by the dotted lines, then still $\pi_1 = \mathbb{Z}_6$: one can use either a similar calculation or just the symmetry arguments. As a consequence, the curve B is necessarily irreducible in this case.

3.3.2. Assume that the skeleton of \bar{B} has a fragment shown in Figure 7(b), i.e., the insertion is inside a loop of Sk' . The rightmost \bullet -vertex in the figure can be either bi- or trivalent; it is not used. Then π_1 has relation (3.2.2) with $(l, m, n) = (3, -, -)$, and one concludes that $\pi_1 = \mathbb{Z}_6$ (as `size(3,0,0)` returns 6). This fact implies, in particular, that the curve B is irreducible.

3.3.3. Assume that the skeleton of \bar{B} has a fragment shown in Figure 7(c), i.e., the insertion is right outside a loop of Sk' . The rightmost \bullet -vertex in the figure can be either bi- or trivalent; it is not used. Then the group π_1 has relation (3.2.2) with $(l, m, n) = (-, 2, -)$, i.e., $[\alpha_1, \alpha_2 \alpha_3 \alpha_2^{-1}] = 1$. Using (3.1.2) and (3.1.3), this relation can be rewritten in the form $[\alpha_1, \alpha_2^{-1} \alpha_1 \alpha_2] = 1$. Hence, also $[\alpha_1, \alpha_2 \alpha_1 \alpha_2^{-1}] = 1$. Spelling (3.1.1) out and eliminating α_3 with the help of (3.1.2), after the cancellation one has

$$\alpha_1 \cdot \alpha_2^{-1} \alpha_1 \alpha_2 \cdot \alpha_2 \alpha_1 \alpha_2^{-1} \cdot \alpha_1 \cdot \underline{\alpha_2} \cdot \alpha_2 \alpha_1 \alpha_2^{-1} = 1.$$

Thus, α_2 (the underlined instance) is a product of elements commuting with α_1 , and the group is abelian.

Observe that these arguments apply to both reducible and irreducible curves.

3.4. A few special cases. Below, we treat the few special cases that are not covered by 3.2 directly.

3.4.1. *The set of singularities $\mathbf{E}_8 \oplus \mathbf{D}_6 \oplus \mathbf{A}_5$* (No. 13' in Table 2). The type $\tilde{\mathbf{D}}_6$ fiber over s prevents one from using relation (3.2.2) with $m = 2$. However, the relations at infinity (3.1.1) through (3.1.3), relations (3.2.2) with $(l, m, n) = (6, -, 6)$, and the relation

$$\alpha_3 = (\alpha_2 \alpha_1 \alpha_2)^{-1} \alpha_1 (\alpha_2 \alpha_1 \alpha_2)$$

resulting from the braid monodromy about the leftmost loop in Figure 3(e) suffice to show that $\text{ord}(\pi_1/\alpha_3^3) = 15$. Hence, π_1 is abelian (see 3.2.4).

3.4.2. *The set of singularities $\mathbf{E}_8 \oplus \mathbf{D}_6 \oplus \mathbf{D}_5$* (No. 16' in Table 2). The $\tilde{\mathbf{D}}$ -type fibers prevent one from using (3.2.2). However, the type $\tilde{\mathbf{D}}_6$ fiber over r gives a relation $[\alpha_3, \alpha_1 \alpha_2] = 1$. Then, $[\alpha_3, \alpha_2] = 1$ (from (3.1.1)) and $\alpha_3 = \alpha_1$ (from (3.1.2)). Hence, the group is abelian.

3.4.3. *The set of singularities $2\mathbf{E}_8 \oplus \mathbf{A}_3$* (No. 29 in Table 1). The vertex v is bivalent, and we choose the reference fiber over an inner point of the edge $[u, v]$. Relations at infinity (3.1.1) through (3.1.3) still hold, and the type $\tilde{\mathbf{E}}_8$ fiber over v gives, among others, the relation $\alpha_2 = \rho^2 \alpha_3 \rho^{-2}$. Now, one can see that $\pi_1 = \mathbb{Z}_6$.

3.4.4. *The set of singularities $\mathbf{E}_8 \oplus \mathbf{E}_6 \oplus \mathbf{D}_5$* (No. 30 in Table 1). The vertex v is monovalent. As in the previous case, choose the reference fiber over an inner point of $[u, v]$. Then, it suffices to add to (3.1.1) through (3.1.3) the relation $\alpha_3 = \rho \alpha_2 \rho^{-1}$ arising from the monodromy about the type $\tilde{\mathbf{E}}_6$ fiber over v to conclude that $\pi_1 = \mathbb{Z}_6$.

3.5. The set of singularities $\mathbf{E}_8 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$. This is No. 1 in Table 1. As explained in 3.1, one of the four braid relations corresponding to the four singular fibers other than F can be ignored. Ignoring the upper left triangle in Figure 2(a), one arrives at the presentation

$$(3.5.1) \quad \langle \alpha_1, \alpha_2, \alpha_3 \mid (3.1.1) \text{ through } (3.1.3) \text{ and } (3.2.2) \text{ with } (l, m, n) = (5, 4, 3) \rangle.$$

Denote this group by G_6 . Using GAP [9], see Figure 8, one can see that:

- (1) one has $\text{ord } G_6 = 720$, and $[G_6, G_6]$ is a perfect group of order 120;
- (2) the only perfect group of order $120 = \text{ord}[G_6, G_6]$ is $SL(2, \mathbb{F}_5)$;
- (3) the last two relations in (3.5.1) follow from the others (as dropping these relations does not change the order of the group);
- (4) G_6 is a semidirect product of $G_6/[G_6, G_6] = \mathbb{Z}_6$ and $[G_6, G_6] = SL(2, \mathbb{F}_5)$;
- (5) the order of each generator α_i , $i = 1, 2, 3$, in G equals 12;
- (6) the group is generated by $\alpha_2 \alpha_3 \alpha_2^{-1}$, α_1 , and α_3 ;
- (7) the centralizer C of $[G_6, G_6]$ is isomorphic to \mathbb{Z}_{12} and $C \cap [G_6, G_6] = \mathbb{Z}_2$; in particular, the canonical projection $C \rightarrow G_6/[G_6, G_6]$ is onto.

The presentation of G_6 stated in Theorem 1.1.3 is obtained from (3.5.1) by dropping the last two relations, see statement (3) above, eliminating the last generator α_3 via (3.1.2), and replacing (3.1.1) with (3.1.4). The central product description of the group given in Remark 1.1.4 follows from statement (7) above: one has

$$G_6 = (C \times [G_6, G_6]) / (C \cap [G_6, G_6]).$$

3.6. The set of singularities $\mathbf{E}_8 \oplus \mathbf{D}_6 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$. This is No. 11' in Table 2. As explained in 3.1, one of the three braid relations corresponding to the three singular

```

gap> ## l=5, m=4, n=3 (no.1)
gap> g := a / [r1, r2, r3, r(a.1,a.2,5), r(a.1,as,4), r(a.2,at,3)];
gap> Size(g);
720
gap> Size(a / [r1, r2, r3, r(a.1,a.2,5)]);          ## drop two relations
720
gap> List(DerivedSeriesOfGroup(g), Size);          ## size of [G,G]
[ 720, 120 ]
gap> List(DerivedSeriesOfGroup(g), AbelianInvariants); ## [G,G] is perfect
[ [ 2, 3 ], [ ] ]
gap> NumberPerfectGroups(120); PerfectGroup(120);
1
A5 2^1
gap> StructureDescription(DerivedSubgroup(g));      ## [G,G]
"SL(2,5)"
gap> StructureDescription(g);                      ## semidirect product
"C3 x (SL(2,5) : C2)"
gap> Size(Subgroup(g, [g.1]));                      ## order of g.1
12
gap> Index(g, Subgroup(g, [g.2*g.3*g.2^-1, g.1, g.3])); ## generating subset
1
gap> h := DerivedSubgroup(g); c := Centralizer(g, h); ## centralizer of G'
gap> Size(Intersection(c, h));
2
gap> StructureDescription(c);
"C12"

```

FIGURE 8. The GAP output, the set of singularities $\mathbf{E}_8 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$

fibers other than F can be ignored. Ignoring the type $\tilde{\mathbf{D}}_6$ fiber over t , one arrives at the presentation

$$(3.6.1) \quad \langle \alpha_1, \alpha_2, \alpha_3 \mid (3.1.1) \text{ through } (3.1.3) \text{ and } (3.2.2) \text{ with } (l, m, n) = (4, 3, -) \rangle.$$

Denote this group by G_∞ , and analyze the quotient $G = G_\infty/\alpha_2^3$ using GAP [9] (see Figure 9). One has:

- (1) $\text{ord } G = 1800$, and $[G, G]$ is a perfect group of order 120;
- (2) any of the last two relations in (3.6.1) follows from the other relations (as dropping a relation does not change the order of the group);
- (3) the group G is generated by $\alpha_2\alpha_3\alpha_2^{-1}$, α_1 , and α_3 ; hence, G_∞ is generated by $\alpha_2\alpha_3\alpha_2^{-1}$, α_1 , α_3 , and the central element α_2^3 ;
- (4) the centralizer C of $[G, G]$ is isomorphic to \mathbb{Z}_{30} and $C \cap [G, G] = \mathbb{Z}_2$; in particular, the canonical projection $C \rightarrow G/[G, G]$ is onto.

Hence, one has $[G_\infty, G_\infty] = [G, G] = SL(2, \mathbb{F}_5)$, cf. 3.2.4 and 3.5(2), and, since the abelianization $G_\infty/[G_\infty, G_\infty] = \mathbb{Z}$ is free, G_∞ splits into a semi-direct product. From statement (4) above it follows that the generator of the abelianization lifts to an element commuting with $[G_\infty, G_\infty]$. Hence, the product is in fact direct (see Remark 1.1.4). The presentation of G_∞ stated in Theorem 1.1.3 is obtained from (3.6.1) by dropping the last relation $\{\alpha_1, \alpha_s\}_3 = 1$ (see statement (2) above), eliminating the last generator α_3 via (3.1.2), and replacing (3.1.1) with (3.1.4).

3.7. Isotrivial curves. The trigonal model \bar{B} of a plane sextic B with the set of singularities $\mathbf{E}_8 \oplus \mathbf{E}_7 \oplus \mathbf{D}_4$ (No. 17' in Table 2) is isotrivial, and it is the only isotrivial trigonal model of a maximal sextic (see Proposition 2.2.1).


```

gap> g := a/[r1, r2, r3, r(a.1,a.2,4), r(a.1,as,3), a.2^3];
gap> Size(g);
1800
gap> List(DerivedSeriesOfGroup(g), Size);          ## size of [G,G]
[ 1800, 120 ]
gap> List(DerivedSeriesOfGroup(g), AbelianInvariants); ## [G,G] is perfect
[ [ 3, 5 ], [ ] ]
gap> Size(a/[r1, r2, r3, r(a.1,a.2,4), a.2^3]);      ## drop a relation
1800
gap> Size(a/[r1, r2, r3, r(a.1,as,3), a.2^3]);      ## drop a relation
1800
gap> Index(g, Subgroup(g, [g.2*g.3*g.2^-1, g.1, g.3]));## generating subset
1
gap> h := DerivedSubgroup(g);; c := Centralizer(g, h);## centralizer of G'
gap> Size(Intersection(c, h));
2
gap> StructureDescription(c);
"C30"

```

FIGURE 9. The GAP output, the set of singularities $\mathbf{E}_8 \oplus \mathbf{D}_6 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$

One has $j_{\bar{B}} \equiv 1$ and, in appropriate affine coordinates (x, y) in Σ_3 , the Weierstraß equation of \bar{B} has the form

$$(3.7.1) \quad y^3 + p(x)y = 0,$$

where $\deg p = 5$. We assume that the distinguished fiber F , corresponding to a simple root of p , is at $x = \infty$.

In general, for any curve \bar{B} given by (3.7.1), the braid monodromy is abelian: the singular fibers of \bar{B} are in a one-to-one correspondence with the roots of p , and the monodromy m_j about the fiber F_j corresponding to a s_j -fold root of p is $(\sigma_1 \sigma_2 \sigma_1)^{s_j}$ (for an appropriate basis in the reference fiber). Hence, the braid relations are equivalent to a single relation $(\sigma_1 \sigma_2 \sigma_1)^s = \text{id}$, where s is the greatest common divisor of the multiplicities of all roots of p .

If \bar{B} as above is the trigonal model of a plane sextic (not necessarily maximal) given by Proposition 2.2.1, then $\deg p = 5$ (a simple root at infinity) and the multiplicity of each root of p is at most 3 (simple singularities only). Hence, the greatest common divisor s above equals 1, and the resulting relation $\sigma_1 \sigma_2 \sigma_1 = \text{id}$ yields $\alpha_1 = \alpha_3$, $[\alpha_1, \alpha_2] = 1$. Thus, the fundamental group is abelian.

4. PERTURBATIONS

We start with a description of the fundamental groups of the perturbations of a singular point of type \mathbf{E}_8 or \mathbf{D}_m . We apply these results to the perturbations of maximal sextics satisfying (*), proving Theorems 1.1.3 and 1.1.5.

4.1. Perturbations of a type \mathbf{E}_8 singular point. Consider a type \mathbf{E}_8 singular point P of a plane curve B and let M be a Milnor ball about P . Let B_t be a perturbation of $B = B_0$ transversal to the boundary ∂M . We are interested in the perturbation epimorphism $\pi_1(M \setminus B) \rightarrow \pi_1(M \setminus B_1)$.

A local normal form of B at P is $\{y^3 + x^5 = 0\}$. Consider the line $L_\epsilon = \{x = \epsilon^3\}$, where $\epsilon > 0$ is a real number with ϵ^3 sufficiently small compared to the radius of M . The intersection $L_\epsilon \cap B$ consists of the vertices $y_k = -\epsilon^5 \exp(2\pi ki/3)$, $k = 0, 1, 2$,

of an equilateral triangle, and we denote by $\{\beta_1, \beta_2, \beta_3\}$ a corresponding canonical basis for the group $\pi_1(L_\epsilon \setminus B)$ (cf. Figure 4). Clearly, $\{\beta_1, \beta_2, \beta_3\}$ is also a basis for $\pi_1(M \setminus B)$, and one has

$$\pi_1(M \setminus B) = \langle \beta_1, \beta_2, \beta_3 \mid \beta_1 \rho^2 = \rho^2 \beta_2, \beta_2 \rho^2 = \rho^2 \beta_3, \beta_3 \rho = \rho \beta_1 \rangle,$$

where $\rho = \beta_1 \beta_2 \beta_3$.

4.1.1. Proposition. *Up to deformation, there are three proper perturbations of a type \mathbf{E}_8 singularity with nonabelian fundamental group. They are as follows:*

- $\mathbf{A}_4 \oplus \mathbf{A}_3$: $\{\beta_1, \beta_2\}_4 = \{\beta_1, \beta_3\}_5 = 1$ and $\beta_1 = \beta_3 \beta_1 \beta_2 \beta_1^{-1} \beta_3^{-1}$;
- $\mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$: $\{\beta_1, \beta_2\}_5 = \{\beta_1, \beta_3\}_3 = [\beta_1 \beta_2 \beta_1^{-1}, \beta_3] = 1$;
- $\mathbf{D}_5 \oplus \mathbf{A}_2$: $\{\beta_1, \beta_2\}_3 = [\beta_2, \beta_3 \beta_1] = 1$ and $\beta_3 = (\beta_1 \beta_2 \beta_3) \beta_1 (\beta_1 \beta_2 \beta_3)^{-1}$,

where listed are the sets of singularities of the perturbed curves B_t , $t \neq 0$, and the relations in the group $\pi_1(M \setminus B_t)$ in the basis $\{\beta_1, \beta_2, \beta_3\}$ described above.

Proof. According to E. Looijenga [11], the deformation classes of perturbations of a simple singularity are in a one-to-one correspondence with the induced subgraphs of its Dynkin graph. In particular, there are eight maximal, i.e., those with the total Milnor number $\mu = 7$, perturbations of \mathbf{E}_8 . We assert that each of these eight perturbations can be realized by a maximal trigonal curve $\bar{B} \subset \Sigma_2$ with a type \mathbf{A}_0^{**} singular fiber F (which we place at infinity and cut off) and all other fibers stable.

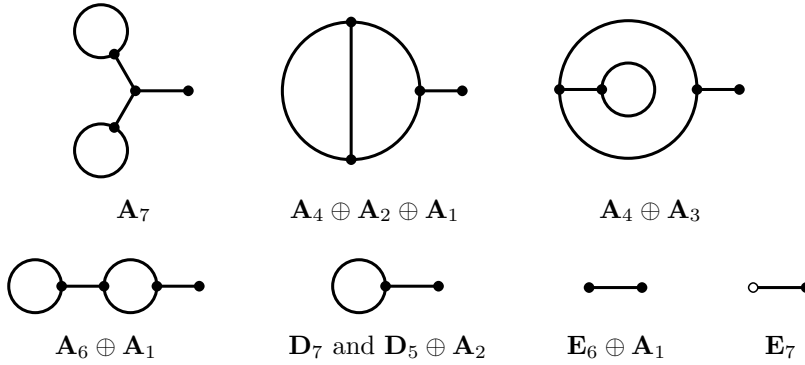


FIGURE 10. The perturbations of \mathbf{E}_8

Indeed, there are eight such maximal curves, defined by the seven skeletons shown in Figure 10, and their sets of singularities are exactly those predicted by [11]. On the other hand, the Weierstraß equation of any trigonal curve $\bar{B} \subset \Sigma_2$ with a type $\tilde{\mathbf{A}}_0^{**}$ singular fiber F at infinity has the form

$$f(x, y) := y^3 + p(x)y + q(x) = 0,$$

where $\deg p = 3$ and $\deg q = 5$, i.e., both p and q have a simple root at infinity. This equation can be renormalized to make the leading coefficient of q equal to one, and then the family \bar{B}_t given by the polynomial $f_t(x, y) = t^{15} f(x/t^3, y/t^5)$ defines a degeneration of $\bar{B} = \bar{B}_1$ to the isotrivial curve $\bar{B}_0 = \{y^3 + x^5 = 0\}$ with a type \mathbf{E}_8 singular point at the origin. The family is indeed a degeneration as the curves \bar{B}_1 and \bar{B}_t , $t \neq 0$, are related by the automorphism $(x, y) \mapsto (t^3 x, t^5 y)$ of Σ_2 . Note that this automorphism preserves the type $\tilde{\mathbf{A}}_0^{**}$ fiber F at infinity.

4.1.2. Remark. The skeletons in Figure 10 are characterized by the fact that they have a monovalent \bullet -vertex. Formally, any such skeleton can be obtained by shrinking a loop in the skeleton of a stable maximal curve in Σ_2 (see [5]). This observation gives one a complete classification.

Pick a Milnor ball \bar{M} about the type \mathbf{E}_8 singular point of \bar{B} and consider the pair (\bar{M}, \bar{B}) instead of (M, B) . Since \bar{B} has no singular fibers except 0 and ∞ , there is a diffeomorphism $\bar{M} \setminus \bar{B} \cong \Sigma_2 \setminus (\bar{B} \cup E \cup F)$. Then, since the perturbation \bar{B}_t constructed above is ‘constant’ in a neighborhood of infinity, there also are diffeomorphisms $\bar{M} \setminus \bar{B}_t \cong \Sigma_2 \setminus (\bar{B}_t \cup E \cup F)$. Here, we assume that all curves \bar{B}_t are transversal to $\partial\bar{M}$.

Thus, it remains to compute the groups $\pi_1(\Sigma_2 \setminus (\bar{B}_1 \cup E \cup F))$. The computation is very similar to §3; it uses Zariski–van Kampen’s approach [10]. Pick a trivalent \bullet -vertex v of Sk . In most cases, we take for v the vertex adjacent to the distinguished monovalent \bullet -vertex u corresponding to F . The two exceptional cases, when Sk has no trivalent \bullet -vertices, are treated separately. Let F_v be the fiber over v , and let $\{\delta_1, \delta_2, \delta_3\}$ be a canonical basis for the group $\pi_1(F_v \setminus (\bar{B}_1 \cup E))$ defined by the marking with respect to which $[u, v]$ is the edge e_1 at v (cf. Figure 4). Then the group $\pi_1(\Sigma_2 \setminus (\bar{B}_1 \cup E \cup F))$ has a presentation

$$\pi_1(\Sigma_2 \setminus (\bar{B}_1 \cup E \cup F)) = \langle \delta_1, \delta_2, \delta_3 \mid m_j = \text{id}, j = 1, \dots \rangle,$$

and the braid monodromies m_j about the singular fibers of \bar{B}_1 are computed as explained in [6] or [3]. Note that, since the fiber F at infinity remains removed, the presentation above has no relation at infinity.

Below, we consider the eight maximal perturbations one by one. By default, v is the vertex adjacent to u . To shorten the notation, we abbreviate the fundamental group $\pi_1(\Sigma_2 \setminus (\bar{B}_1 \cup E \cup F))$ in question by G .

4.1.3. The perturbation \mathbf{A}_7 . Take for v the corner of the upper monogonal region in Figure 10. Then the relations are $\delta_1 = \delta_3$, $\delta_2 = \delta_3^{-1} \delta_2^{-1} \delta_1 \delta_2 \delta_3$, and $\{\delta_1, \delta_2\}_8 = 1$. Eliminating δ_3 , one can simplify the second relation to $\{\delta_1, \delta_2\}_3 = 1$. Hence, G is abelian due to (3.2.1).

4.1.4. The perturbation $\mathbf{A}_6 \oplus \mathbf{A}_1$. The relations are $[\delta_2, \delta_3] = \{\delta_1, \delta_2\}_7 = 1$ and $\delta_3 = \delta_1 \delta_2 \delta_1^{-1}$. Thus, G is generated by δ_1, δ_2 , and the second relation implies that $(\delta_1 \delta_2)^7$ is a central element. Using GAP [9], one can see that $\text{ord}(G/(\delta_1 \delta_2)^7) = 14$. Hence, this quotient is abelian, and so is G (cf. 3.2.4).

4.1.5. The perturbation \mathbf{D}_7 . Among other relations, one has $\delta_2 = \delta_3$ (from the vertical tangent) and $[\delta_3, \delta_1 \delta_2] = 1$ (from the type $\tilde{\mathbf{D}}_7$ fiber). Eliminating δ_3 , one concludes that G is abelian.

4.1.6. The perturbation $\mathbf{E}_6 \oplus \mathbf{A}_1$. The skeleton Sk has no trivalent \bullet -vertices, and we choose the reference fiber F_v over a point v in the solid edge of the dessin of the curve \bar{B}_1 connecting its type $\tilde{\mathbf{E}}_6$ and type $\tilde{\mathbf{A}}_1$ singular fibers. Let $\{\delta_1, \delta_2, \delta_3\}$ be an appropriate ‘canonical’ basis in F_v , such that the generators δ_2 and δ_3 are brought together when the fiber approaches the node. Then the relations are $[\delta_2, \delta_3] = 1$ (from the node), $\delta_2 = \rho \delta_1 \rho^{-1}$, and $\delta_3 = \rho \delta_2 \rho^{-1}$, where $\rho = \delta_1 \delta_2 \delta_3$. Conjugating the first relation by ρ and taking into account the last two relations, one obtains $[\delta_1, \delta_2] = 1$. Hence, δ_2 is a central element, and the last relation implies $\delta_2 = \delta_3$. Thus, G is abelian.

4.1.7. The perturbation \mathbf{E}_7 . This time, we choose the reference fiber F_v over a point v in the dotted edge of the dessin of the curve \bar{B}_1 connecting its type $\tilde{\mathbf{E}}_7$ and type $\tilde{\mathbf{A}}_0^*$ singular fibers, and take for $\{\delta_1, \delta_2, \delta_3\}$ an appropriate ‘linear’ basis in F_v , so that δ_1 and δ_2 are brought together when the point approaches the vertical tangent $\tilde{\mathbf{A}}_0^*$. Among other relations, one has $\delta_1 = \delta_2$ (from the vertical tangent) and $[\delta_2, \delta_1 \delta_2 \delta_3 \delta_1] = 1$. Eliminating δ_2 , one concludes that G is abelian.

4.1.8. Other maximal perturbations. For the remaining three maximal perturbations, we merely list the relations for G . They are as follows:

- $\mathbf{A}_4 \oplus \mathbf{A}_3$: $\{\delta_1, \delta_2\}_4 = \{\delta_2, \delta_3\}_5 = 1$ and $\delta_2 = \delta_3 \delta_1 \delta_3^{-1}$;
- $\mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$: $\{\delta_1, \delta_2\}_5 = \{\delta_2, \delta_3\}_3 = [\delta_1, \delta_3] = 1$;
- $\mathbf{D}_5 \oplus \mathbf{A}_2$: $\{\delta_1, \delta_2\}_3 = [\delta_1, \delta_2 \delta_3] = 1$ and $\delta_3 = (\delta_1 \delta_2 \delta_3) \delta_2 (\delta_1 \delta_2 \delta_3)^{-1}$.

All three groups are nonabelian. Indeed, in the order of appearance, one has $\text{ord}(G/\delta_1^3) = 360$, $\text{ord}(G/\delta_1^2) = 120$, and $\text{ord}(G/\delta_1^5) = 600$. On the other hand, if G were abelian, for any integer $n > 0$ one would have $G/\delta_1^n = \mathbb{Z}_n$. Note that, for the last group (the perturbation $\mathbf{D}_5 \oplus \mathbf{A}_2$), one also has

$$(4.1.9) \quad \text{ord}(G/\delta_1^{12}) = 12, \quad \text{hence} \quad G/\delta_1^{12} = \mathbb{Z}_{12}.$$

To complete the proof for the maximal perturbations, it remains to notice that, from the point of view of the trigonal implementation of the type \mathbf{E}_8 singularity, the line L_ϵ introduced at the beginning of this section (the line carrying the basis $\{\beta_1, \beta_2, \beta_3\}$) should be regarded as a fiber ‘close to infinity’, e.g., the fiber over a point in the edge $[u, v]$ of Sk close to u . It is related to the fiber F_v used in the computation via the monodromy through the \circ -vertex at the middle of the edge. Hence, the two bases are related via $\delta_1 = \beta_1 \beta_2 \beta_1^{-1}$, $\delta_2 = \beta_1$, $\delta_3 = \beta_3$. Substituting, one obtains the presentations announced in the statement.

4.1.10. Non-maximal perturbations. We assert that the fundamental group of any perturbation with the total Milnor number $\mu = 6$, and hence of any non-maximal perturbation, is abelian. For a proof, one can list all such perturbations, which are obtained by removing two vertices from the Dynkin graph \mathbf{E}_8 , and show, e.g., using trigonal curves and their dessins, that each of them degenerates to a maximal one with abelian group. Alternatively, if a perturbation appears to degenerate to a maximal one with nonabelian group, one can analyze the extra relations (using the dessins again) and show that the perturbed group is abelian. We omit the details. \square

4.2. Perturbations of D-type singular points. Consider a type \mathbf{D}_m , $m > 4$, singular point Q of a plane curve B . According to Looijenga [11], its perturbations are classified by the induced subgraphs of the Dynkin graph \mathbf{D}_m . In particular, maximal are the perturbation \mathbf{A}_{m-1} (removing a short end of the diagram) or \mathbf{D}_{m-1} and $\mathbf{D}_p \oplus \mathbf{A}_{m-p-1}$, $2 \leq p \leq m-2$ (removing a vertex from the long end or the trivalent vertex). In the latter case, in order to emphasize the perturbation, we let $\mathbf{D}_2 = 2\mathbf{A}_1$ and $\mathbf{D}_3 = \mathbf{A}_3$.

All perturbations can be realized by trigonal curves. Thus, we assume that Q is a singular point of a trigonal curve \bar{B} and consider a perturbation $\bar{B} \rightarrow \bar{B}'$ in the class of trigonal curves. We are interested in the perturbation epimorphism $\pi_1(M \setminus \bar{B}) \rightarrow \pi_1(M \setminus \bar{B}')$, where M is a Milnor ball about Q . We take for M the union of the affine fibers over a small disk Δ about the type $\tilde{\mathbf{D}}_m$ fiber of \bar{B} ; then

the groups can be computed using van Kampen's method (cf. 3.1 or 4.1), with only the monodromy within Δ taken into account.

Let u be the $(m-4)$ -valent \times -vertex of the dessin of \bar{B} representing the singular fiber containing Q . Pick a trivalent \bullet -vertex v adjacent to u and let $\{\beta_1, \beta_2, \beta_3\}$ be a canonical basis in the fiber F_v over v (cf. Figure 4), defined by the marking such that $[u, v]$ is the solid edge opposite to e_3 at v . In other words, the generators β_1 and β_2 are brought together when the fiber approaches u . Then, the braid monodromy about u is $\sigma_1^{m-4}(\sigma_1\sigma_2)^3$, and letting $\sigma_1^{m-4}(\sigma_1\sigma_2)^3 = \text{id}$ results in the relations $[\beta_3, \beta_1\beta_2] = 1$ and $\sigma_1^{m-4}(\beta_i) = (\beta_1\beta_2\beta_3)^{-1}\beta_i(\beta_1\beta_2\beta_3)$, $i = 1, 2$. Since the restriction of σ_1^{-2} to the subgroup $\langle \beta_1, \beta_2 \rangle$ is the conjugation by $(\beta_1\beta_2)$, one obtains

$$\pi_1(M \setminus \bar{B}) = \langle \beta_1, \beta_2 \rangle \rtimes \langle \beta_3 \rangle, \quad \beta_3^{-1}\beta_i\beta_3 = \sigma_1^{m-2}(\beta_i), \quad i = 1, 2.$$

4.2.1. Lemma. *For the perturbation $\mathbf{D}_m \rightarrow \mathbf{A}_{m-1}$ or any further perturbation thereof, the group $\pi_1(M \setminus \bar{B}')$ is abelian.*

Proof. The perturbation is realized as follows: The vertex u is replaced with a new \times -vertex u' of valency m and the fragment shown in Figure 11(a). For clarity, we keep omitting \circ - and \times -vertices in the drawings. The new vertex u' is in the outer region in Figure 11(a). The appearance of the two new \bullet -vertices is due to the fact that the perturbation increases the degree of $j_{\bar{B}}$. Choosing a new canonical basis $\{\delta_1, \delta_2, \delta_3\}$ over one of the \bullet -vertices in the figure, from the two loops one obtains the relations $\delta_2 = \delta_3$ and $\delta_2 = \delta_1\delta_2\delta_3\delta_2^{-1}\delta_1^{-1}$. Hence, the group is abelian. \square

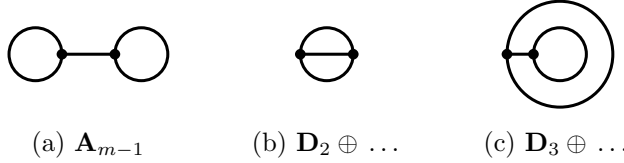


FIGURE 11. The perturbations of \mathbf{D}_m

4.2.2. Lemma. *Consider the perturbation*

$$\mathbf{D}_m \rightarrow \mathbf{D}_p \oplus \bigoplus_{i=1}^k \mathbf{A}_{s_i}, \quad p \geq 2, \quad d := m - p - \sum_{i=1}^k (s_i + 1) \geq 0.$$

If $d = 0$, let $s = \text{g. c. d.}(s_i + 1)$, $1 \leq i \leq k$; otherwise, let $s = 1$. Then

$$\pi_1(M \setminus \bar{B}') = \mathbb{T}_{2,s} \rtimes \langle \beta_3 \rangle, \quad \beta_3^{-1}\beta_i\beta_3 = \sigma_1^{m-2}(\beta_i), \quad i = 1, 2,$$

where $\mathbb{T}_{2,s} = \langle \beta_1, \beta_2 \mid \{\beta_1, \beta_2\}_s = 1 \rangle$.

Proof. The perturbation is realized as follows: The original $(m-4)$ -valent vertex u is replaced with:

- (1) k \times -vertices of valencies $s_1 + 1, \dots, s_k + 1$,
- (2) d monovalent \times -vertices, and
- (3) either the fragment shown in Figure 11(b) (if $p = 2$) or (c) (if $p = 3$), or a $(p-4)$ -valent \times -vertex u' (if $p > 4$).

Note that, if $p \leq 3$, all singular fibers are of type $\tilde{\mathbf{A}}$ and the perturbation increases the degree of $j_{\tilde{B}}$, introducing two new \bullet -vertices. If $p > 4$, the fiber over u' is of type $\tilde{\mathbf{D}}$. If $p = 4$, there also is a type $\tilde{\mathbf{D}}_4$ singular fiber of \tilde{B}' that does not correspond to any vertex of the dessin.

The braid monodromy about the i -th vertex in (1) is $\sigma_1^{s_{i+1}}$, and the monodromy about each vertex in (2), if any, is σ_1 . Thus, the braid relations resulting from (1) and (2) simplify to $\sigma_1^s = \text{id}$, or $\{\beta_1, \beta_2\}_s = 1$. The monodromy about the original type $\tilde{\mathbf{D}}_m$ singular fiber (the ‘monodromy at infinity’) is $\sigma_1^{m-4}(\sigma_1\sigma_2)^3$. As explained at the beginning of this section, it results in the braid relations

$$(4.2.3) \quad \beta_3^{-1}\beta_i\beta_3 = \sigma_1^{m-2}(\beta_i), \quad i = 1, 2,$$

as stated.

If $p \geq 4$, there are no other relations, as the remaining type $\tilde{\mathbf{D}}_p$ fiber can be ignored in the presence of the ‘relation at infinity’. Otherwise, if $p = 2$ or 3 , the braid $\sigma_1^{p-4}(\sigma_1\sigma_2)^3$ above is the monodromy along the outer contour of the insertion shown in Figure 11. The braid relations resulting from the two regions separately are $[\beta_1, \beta_3] = [\beta_2, \beta_3] = 1$ (if $p = 2$, Figure 11(b)) or $\{\beta_2, \beta_3\}_4 = 1$ and $\beta_2 = \beta_3^{-1}\beta_2^{-1}\beta_1\beta_2\beta_3$ (if $p = 3$, Figure 11(c)). They are equivalent to (4.2.3). \square

4.2.4. Remark. Formally, one has $\pi_1(M \setminus \tilde{B}) = \mathbb{T}_{2,0} \rtimes \mathbb{Z}$ and $\pi_1(M \setminus \tilde{B}')$ is obtained from $\pi_1(M \setminus \tilde{B})$ by adding an extra relation $\{\beta_1, \beta_2\}_s = 1$.

4.2.5. Remark. If s is odd, σ_1 is an inner automorphism of $\mathbb{T}_{2,s}$ and, in any case, σ_1^2 is an inner automorphism of $\mathbb{T}_{2,s}$. Hence, if s is odd or m is even, the group in Lemma 4.2.2 splits into *direct* product $\mathbb{T}_{2,s} \times \mathbb{Z}$.

4.2.6. Corollary. For a perturbation as in Lemma 4.2.2, the group $\pi_1(M \setminus \tilde{B}')$ is abelian if and only if either $s = 1$ or $s = 2$ and m is even. \square

4.2.7. Corollary. The only perturbations of a type \mathbf{D}_6 singular point that have nonabelian fundamental groups are $\mathbf{D}_3 \oplus \mathbf{A}_2$ and $\mathbf{D}_2 \oplus \mathbf{A}_3$. \square

4.3. Perturbations of maximal sextics. According to the results of §3, the fundamental group of a *maximal* sextic B satisfying (*) is abelian unless the set of singularities of B is $\mathbf{E}_8 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$ or $\mathbf{E}_8 \oplus \mathbf{D}_6 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$. In this section, we show that the only perturbations of these two sextics that have nonabelian groups are those listed in Theorem 1.1.5.

4.3.1. Lemma. Let B be the irreducible plane sextic with the set of singularities $\mathbf{E}_8 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$. The only proper perturbation $B \rightarrow B'$ that has nonabelian fundamental group is given by $\mathbf{E}_8 \rightarrow \mathbf{A}_4 \oplus \mathbf{A}_3$. In this case, the perturbation epimorphism is an isomorphism.

Proof. Any perturbation of the \mathbf{A} -type points of B can be realized on the level of the trigonal model \tilde{B} : In the language of the skeletons, the \times -vertex at the center of a region of Sk splits into several \times -vertices of smaller valencies. Assume that a vertex of valency l splits into vertices of valencies l_1, \dots, l_k , so that $l_1 + \dots + l_k = l$. Under an appropriate choice of the basis $\{\alpha'_1, \alpha'_2, \alpha'_3\}$, the braid relation about the original \times -vertex is $\{\alpha'_1, \alpha'_2\}_l = 1$. After the splitting, this relation simplifies to $\{\alpha'_1, \alpha'_2\}_{l'} = 1$, where $l' = \text{g. c. d.}(l_i)$, $1 \leq i \leq k$ (see (3.2.1)).

Applying this observation to the set of singularities $\mathbf{E}_8 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$, one can see that, if the point perturbed is \mathbf{A}_4 , \mathbf{A}_3 , or \mathbf{A}_2 , so that l above is 5, 4, or 3,

respectively, the new parameter l' divides 1, 2, or 1, respectively. Hence, instead of $(l, m, n) = (5, 4, 3)$ (see 3.2), one has $(l, m, n) = (1, 4, 3)$, $(5, 2, 3)$, or $(5, 4, 1)$, respectively. Note that the two cusps of B are permuted by the complex conjugation; hence, if a cusp is perturbed, one can assume that it is the one over t (see Figure 5). Using GAP [9] (see 3.2.3 and also 3.3.1 for the last case), one concludes that all three groups are abelian.

Assume that perturbed is the type \mathbf{E}_8 point P . Let M be a Milnor ball about P , and let $\{\beta_1, \beta_2, \beta_3\}$ be the basis for $\pi_1(M \setminus B)$ introduced at the beginning of 4.1. As shown in [6], the inclusion homomorphism $\pi_1(M \setminus B) \rightarrow \pi_1(\mathbb{P}^2 \setminus B)$ is given by

$$(4.3.2) \quad \beta_1 \mapsto (\alpha_1 \alpha_2) \alpha_3 (\alpha_1 \alpha_2)^{-1}, \quad \beta_2 \mapsto \alpha_1, \quad \beta_3 \mapsto \alpha_3.$$

In particular, it follows from (4.3.2) and 3.5(6) that it is an epimorphism. Hence, if $\pi_1(M \setminus B')$ is abelian, so is $\pi_1(\mathbb{P}^2 \setminus B')$. In the remaining three cases, one adds to (3.5.1) the relations between (the images of) $\beta_1, \beta_2, \beta_3$ given by Proposition 4.1.1 and computes the sizes of the new groups. The results are 720 (the group does not change), 6, and 6 (the group is abelian). \square

4.3.3. Lemma. *Let B be the reducible plane sextic with the set of singularities $\mathbf{E}_8 \oplus \mathbf{D}_6 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$. The only proper perturbation $B \rightarrow B'$ that has nonabelian fundamental group is given by $\mathbf{E}_8 \rightarrow \mathbf{D}_5 \oplus \mathbf{A}_2$. In this case, the perturbation epimorphism is an isomorphism.*

Proof. If the type \mathbf{A}_4 or type \mathbf{A}_2 point is perturbed, then, as explained in the previous proof, one replaces $(l, m, n) = (4, 3, -)$ in (3.6.1) with $(l, m, n) = (2, 3, -)$ or $(4, 1, -)$, respectively. Computing the size of the quotient by α_2^3 , one concludes that the group is abelian.

Assume that the type \mathbf{D}_6 singular point Q is perturbed and let M be a Milnor ball about Q . The inclusion homomorphism $\pi_1(M \setminus B) \rightarrow \pi_1(\mathbb{P}^2 \setminus B)$ is an epimorphism, as the three generators of $\pi_1(\mathbb{P}^2 \setminus B)$, when chosen in a fiber close to Q , ‘fit’ into M . (Note that the latter assertion holds for any triple point of any trigonal curve.) Thus, if $\pi_1(M \setminus B')$ is abelian, so is $\pi_1(\mathbb{P}^2 \setminus B')$ and, in view of Corollary 4.2.7, it remains to consider the perturbations (back to the conventional notation) $2\mathbf{A}_1 \oplus \mathbf{A}_3$ and $\mathbf{A}_3 \oplus \mathbf{A}_2$, which result in extra relations $\{\alpha_2, \alpha_t\}_s = 1$, $s = 4$ or 3 , respectively, cf. Lemma 4.2.2, Figure 3(d), and (3.2.2). In other words, the values $(l, m, n) = (4, 3, -)$ (see 3.6) are replaced with $(4, 3, 4)$ or $(4, 3, 3)$, respectively. Both groups are abelian.

Finally, the perturbations of the type \mathbf{E}_8 point P are studied similar to the previous proof, using (4.3.2). In view of 3.6(3), whenever $\pi_1(M \setminus B')$ is abelian, so is $\pi_1(\mathbb{P}^2 \setminus B')$. For the three nonabelian perturbations of P in Proposition 4.1.1, the sizes of the quotient $\pi_1(\mathbb{P}^2 \setminus B')/\alpha_1^3$ are 15, 15 (the group is abelian), and 1800 (the group does not change, cf. 3.6(1)). \square

4.4. Degenerations of sextics. In this section, we show that each plane sextic B satisfying (*) degenerates to a maximal one.

4.4.1. Lemma. *Any irreducible plane sextic B with a type \mathbf{E}_8 singular point and at least two more triple points degenerates to a maximal sextic with the set of singularities $\mathbf{E}_8 \oplus \mathbf{E}_6 \oplus \mathbf{D}_5$ (No. 30 in Table 1).*

Proof. Consider the triangle Cremona transformation $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with the centers at the type \mathbf{E}_8 point and two other triple points of B . The transform of B is a cuspidal

cubic $C \subset \mathbb{P}^2$, and the three exceptional divisors E_i , $i = 1, 2, 3$, are positioned as follows:

- E_1 is tangent to C at the cusp, and
- none of the intersection points $E_i \cap E_j$, $1 \leq i < j \leq 3$, belongs to C .

It is clear that one can keep E_1 and degenerate the pair (E_2, E_3) to an ‘extremal’ position, so that, up to reordering, E_2 is inflection tangent to C and E_3 is tangent to C . (Recall that C has a unique inflection point.) The new inverse transform of C has the set of singularities $\mathbf{E}_8 \oplus \mathbf{E}_6 \oplus \mathbf{D}_5$. \square

4.4.2. Lemma. *Any reducible plane sextic B with a type \mathbf{E}_8 singular point and at least two more triple points degenerates to a maximal sextic with the set of singularities $\mathbf{E}_8 \oplus \mathbf{D}_6 \oplus \mathbf{D}_5$ or $\mathbf{E}_8 \oplus \mathbf{E}_7 \oplus \mathbf{D}_4$ (respectively, No. 16' or 17' in Table 2).*

Proof. We proceed as in the previous proof. This time, B splits into an irreducible quintic B_5 and a line B_1 . The linear component B_1 passes through two double points P_2, P_3 of B_5 , and we will deform B_5 , keeping B_1 passing through these points. Apply the triangular Cremona transformation $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ centered at the type \mathbf{E}_8 point, P_2 , and P_3 . The transform of B_5 is a cuspidal cubic C , and the exceptional divisors E_i , $i = 1, 2, 3$, are as follows:

- E_1 is tangent to C at the cusp,
- $E_2 \cap E_3 \in C$, and
- the other two points $E_1 \cap E_2, E_1 \cap E_3$ do *not* belong to C .

Now, one can keep E_1 and degenerate (E_2, E_3) to one of the following ‘extremal’ configurations: Either E_2 is inflection tangent to C , necessarily at $E_2 \cap E_3$, or E_2 is tangent to C at $E_2 \cap E_3$ and E_3 is tangent to C at another point (see Remark 4.4.3 below). The new inverse transform of C , with the line $B_1 = (P_2, P_3)$ added, has the set of singularities $\mathbf{E}_8 \oplus \mathbf{E}_7 \oplus \mathbf{D}_4$ or $\mathbf{E}_8 \oplus \mathbf{D}_6 \oplus \mathbf{D}_5$, respectively. \square

4.4.3. Remark. Note that the two configurations considered at the end of the previous proof are indeed extremal, as the cuspidal cubic C has a unique inflection point Q_0 and, from each smooth point $Q \neq Q_0$ of C , there is a unique tangent to C other than that at Q .

4.4.4. Proposition. *Each plane sextic B satisfying (*) degenerates to a maximal sextic satisfying (*).*

Proof. Due to Lemmas 4.4.1 and 4.4.2, it suffices to consider the case when B has at most one triple point other than P . Then the trigonal model \bar{B} of B is not isotrivial and has at most one triple point. Hence, this trigonal model is obtained by at most one elementary transformation from a trigonal curve \bar{B}' with double singular points only. According to [3], there is a degeneration \bar{B}'_t , $t \in [0, 1]$, of $\bar{B}' = \bar{B}'_1$ to a maximal curve \bar{B}'_0 . It is followed by a degeneration, in the class of trigonal models of sextics satisfying (*), \bar{B}_t of $\bar{B} = \bar{B}'_1$ to a maximal trigonal model \bar{B}_0 and, hence, by a degeneration B_t of $B = B_1$ to a maximal sextic B_0 . \square

4.4.5. Remark. If B has three triple points then, in the proof of Proposition 4.4.4, the trigonal model \bar{B} is obtained from $\bar{B}' \subset \Sigma_1$ by two elementary transformation and, during the degeneration \bar{B}'_t , the two fibers contracted could merge to a single fiber, resulting in a non-simple singular point of \bar{B} . To exclude this possibility, we treat the case of three triple points separately in Lemmas 4.4.1 and 4.4.2.

4.5. Proof of Theorem 1.1.3. According to Proposition 4.4.4, any plane sextic satisfying $(*)$ degenerates to a maximal one. Any perturbation $B \rightarrow B'$ induces an epimorphism $\pi_1(\mathbb{P}^2 \setminus B) \twoheadrightarrow \pi_1(\mathbb{P}^2 \setminus B')$ of the fundamental groups (see [14]). Hence, due to §3, nonabelian can only be the groups of the perturbations of the sextics with the sets of singularities $\mathbf{E}_8 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$ or $\mathbf{E}_8 \oplus \mathbf{D}_6 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$. According to Lemmas 4.3.1 and 4.3.3, there are only two proper perturbations with nonabelian fundamental groups. None of them has a type \mathbf{E}_8 singular point. \square

4.6. Proof of Theorem 1.1.5. The statement follows immediately from the list of sextics with nonabelian groups (Theorem 1.1.3) and the description of their perturbations (Lemmas 4.3.1 and 4.3.3). \square

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