

PLANE SEXTICS WITH A TYPE \mathbf{E}_6 SINGULAR POINT

ALEX DEGTYAREV

ABSTRACT. We give a classification up to equisingular deformation and compute the fundamental groups of maximizing plane sextics with a type \mathbf{E}_6 singular point.

1. INTRODUCTION

1.1. The subject. This paper concludes the series [11], [12], where we give a complete deformation classification and compute the fundamental groups of maximizing irreducible plane sextics with an \mathbf{E} type singular point. (With the common abuse of the language, by the *fundamental group* of a curve $B \subset \mathbb{P}^2$ we mean the group $\pi_1(\mathbb{P}^2 \setminus B)$ of its complement.) Here, we consider sextics $B \subset \mathbb{P}^2$ satisfying the following conditions:

- (*) B has simple (*i.e.*, $\mathbf{A-D-E}$) singularities only,
 B has a distinguished singular point P of type \mathbf{E}_6 , and
 B has no singular points of type \mathbf{E}_7 or \mathbf{E}_8 .

(Singular points of type \mathbf{E}_7 and \mathbf{E}_8 are excluded in order to reduce the lists. Sextics with a type \mathbf{E}_8 point are considered in [12], and irreducible sextics with a type \mathbf{E}_7 point are considered in [11]. Reducible sextics with a type \mathbf{E}_7 point, as well as the more involved case of a distinguished \mathbf{D} type point, may appear elsewhere.)

Recall that a plane sextic B with simple singularities only is called *maximizing* if the total Milnor number $\mu(B)$ assumes the maximal possible value 19. It is well known that maximizing sextics are defined over algebraic number fields (as they are related to singular $K3$ -surfaces). Furthermore, such sextics are rigid: two maximizing sextics are equisingular deformation equivalent if and only if they are related by a projective transformation.

Another important class is formed by the so called sextics of *torus type*, *i.e.*, those whose equation can be represented in the form $f_2^3 + f_3^2 = 0$, where f_2 and f_3 are certain homogeneous polynomials of degree 2 and 3, respectively. (This property turns out to be equisingular deformation invariant.) Each sextic B of torus type can be perturbed to Zariski's famous six cuspidal sextic [21], which is obtained when f_2 and f_3 above are sufficiently generic. Hence, the group $\pi_1(\mathbb{P}^2 \setminus B)$ factors to the *reduced braid group* $\mathbb{B}_3 := \mathbb{B}_3/(\sigma_1\sigma_2)^3$; in particular, it is never finite. (The existence of two distinct families of irreducible six cuspidal sextics, those of and not of torus type, was first stated by Del Pezzo and then proved by B. Segre, see

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e.g. [18, page 407]. Later Zariski [21] showed that the two families differ by the fundamental groups.)

A representation of the equation of B in the form $f_2^3 + f_3^2 = 0$ is called a *torus structure*. The points of intersection of the conic $\{f_2 = 0\}$ and cubic $\{f_3 = 0\}$ are always singular for B ; they are called *inner* (with respect to the given torus structure), whereas the other singular points are called *outer*. In the listing below, we indicate sextics of torus type by representing their sets of singularities in the form

$$(\text{inner singularities}) \oplus \text{outer singularities}.$$

An exception is the set of singularities $\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 4\mathbf{A}_2$, which is always of torus type and admits four distinct torus structures.

Formally, the deformation classification of plane sextics with simple singularities can be reduced to a purely arithmetical problem, see [6], and for maximizing sextics this latter problem has been completely solved, see [19] and [20], in the sense that all deformation classes have been enumerated. Unfortunately, this approach, based on the theory of $K3$ -surfaces and the global Torelli theorem, is not constructive and very little is known about the geometry of the curves. (Sporadic examples using explicit equations are scattered in the literature.) Here, we use another approach, suggested in [10] and [11]: a plane curve B with a sufficiently deep (with respect to the degree) singularity is reduced to a trigonal curve \bar{B} in an appropriate Hirzebruch surface. If B is a maximizing sextic (with a triple point), then \bar{B} is a maximal trigonal curve; hence it can be studied using Grothendieck's *dessin d'enfants* of its functional j -invariant. At the end, we obtain an explicit geometric description of \bar{B} and B (rather than equation); among other things, this description is suitable for computing the braid monodromy and hence the fundamental group of the curves.

1.2. Results. The principal results of the present paper are Theorems 1.2.1 (the classification) and 1.2.2, 1.2.3 (the computation of the fundamental group) below.

1.2.1. Theorem. *Up to projective transformation (equivalently, up to equisingular deformation) there are 93 maximizing plane sextics satisfying condition (*) above, realizing 71 combinatorial sets of singularities; of them, 53 sextics (40 sets of singularities) are irreducible, see Table 1 on page 10, and 40 sextics (32 sets of singularities) are reducible, see Table 2 on page 11.*

In Theorem 1.2.1, one set of singularities is common: $\mathbf{E}_6 \oplus \mathbf{A}_9 \oplus \mathbf{A}_4$ is realized by three irreducible and one reducible sextics. This theorem is proved in Section 2; for more details, see comments to the tables in Subsection 2.6.

Among the irreducible sextics in Theorem 1.2.1, twelve (eight sets of singularities) are of torus type. Seven of them (four sets of singularities) are 'new' in the sense that they have not been extensively studied before.

1.2.2. Theorem. *Let $B \subset \mathbb{P}^2$ be an irreducible maximizing sextic that satisfies condition (*) above and is not of torus type. If the set of singularities of B is $2\mathbf{E}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$ (nos. 4 and 5 in Table 1), then $\pi_1(\mathbb{P}^2 \setminus B) = SL(2, \mathbb{F}_5) \rtimes \mathbb{Z}_6$ (see (3.3.3) or (3.5.2) for the presentations). Otherwise, $\pi_1(\mathbb{P}^2 \setminus B) = \mathbb{Z}_6$.*

1.2.3. Theorem. *Let B be a sextic as in Theorem 1.2.1, and let B' be a proper irreducible perturbation of B that is not of torus type. Then $\pi_1(\mathbb{P}^2 \setminus B') = \mathbb{Z}_6$.*

Theorems 1.2.2 and 1.2.3 are proved in Section 3 and Subsection 4.4, respectively. The theorems substantiate my conjecture asserting that the fundamental

group of an irreducible plane sextic that has simple singularities only and is not of torus type is finite. Recall that, originally, the conjecture was motivated by a certain experimental evidence (which has now been extended) and the fact that the abelianization of the fundamental group of an irreducible sextic that is not of torus type is finite (which is a restatement of the proved part of the so called Oka conjecture, see [5]). At present, the conjecture is essentially settled for sextics with a triple singular point (the case of a \mathbf{D} type point is considered in a forthcoming paper).

In Section 3, we also write down presentations of the fundamental groups of all other sextics as in Theorem 1.2.1, and in Section 4, we consider their perturbations. In particular, we prove the following theorem, computing the groups of all ‘new’ irreducible sextics of torus type.

1.2.4. Theorem. *The fundamental group of a sextic with the set of singularities $(\mathbf{E}_6 \oplus \mathbf{A}_{11}) \oplus \mathbf{A}_2$, $(\mathbf{E}_6 \oplus \mathbf{A}_8 \oplus \mathbf{A}_2) \oplus \mathbf{A}_3$, or $(\mathbf{E}_6 \oplus \mathbf{A}_8 \oplus \mathbf{A}_2) \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$ (nos. 12, 13, 18, and 40 in Table 1) is isomorphic to the reduced braid group $\overline{\mathbb{B}}_3$. The groups of sextics with the set of singularities $(\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_4$ (nos. 9 and 41) are not isomorphic to $\overline{\mathbb{B}}_3$; their presentations are found in 3.3.9 and 3.6(3), respectively.*

1.2.5. Problem. Are the fundamental groups of the two sextics with the set of singularities $(\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_4$ (nos. 9 and 41 in Table 1) isomorphic to each other? (A similar question still stands for the sextics with the set of singularities $(2\mathbf{E}_6 \oplus \mathbf{A}_5) \oplus \mathbf{A}_2$, nos. 6 and 7 in Table 1, see [4] and [17].)

1.2.6. Theorem. *Let B be a sextic as in Theorem 1.2.1, and let B' be a proper irreducible perturbation of B that is of torus type. Then, with the following few exceptions:*

- a perturbation $\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 4\mathbf{A}_2 \rightarrow$ (a curve of weight 8), see [9], and
- a perturbation that can further be perturbed to a curve B'' with the set of singularities $(6\mathbf{A}_2) \oplus 4\mathbf{A}_1$, see [4] and [9],

the group $\pi_1(\mathbb{P}^2 \setminus B')$ is the reduced braid group $\overline{\mathbb{B}}_3$.

(Here, the *weight* of a sextic is understood in the sense of [5], as the total weight if all its singular points, where the weight $w(P)$ of a singular point P is defined via $w(\mathbf{A}_{3k-1}) = k$, $w(\mathbf{E}_6) = 2$, and $w(P) = 0$ otherwise. The fundamental group of a sextic of weight ≥ 8 is much larger than $\overline{\mathbb{B}}_3$, as it has a larger Alexander polynomial. In the second exceptional case in the statement, one has $\pi_1(\mathbb{P}^2 \setminus B') = \mathbb{B}_4 / \sigma_1^2 \sigma_2 \sigma_3^2 \sigma_2$, see [4] and [9].) Theorem 1.2.6 is proved in Subsection 4.5.

1.3. Acknowledgements. I am grateful to E. Artal Bartolo, who helped me to identify some of the groups of curves of torus type, thus making the statements more complete, and to I. Dolgachev for his enlightening remarks concerning history of six cuspidal sextics.

2. THE CLASSIFICATION

2.1. The settings. We remind briefly some of the results of [11], concerning the construction and the classification of plane sextics satisfying (*). For details on maximal trigonal curves and their skeletons, see [10] or [11]. We denote by Σ_k , $k > 0$, the geometrically ruled rational surface with an exceptional section E of self-intersection $-k$.

2.1.1. Proposition. *There is a natural bijection ϕ , invariant under equisingular deformations, between Zariski open (in each equisingular stratum) subsets of the following two sets:*

- (1) *plane sextics B with a distinguished type \mathbf{E}_6 singular point P , and*
- (2) *trigonal curves $\bar{B} \subset \Sigma_4$ with a distinguished type $\tilde{\mathbf{A}}_5$ singular fiber \bar{F} .*

A sextic B is irreducible if and only if so is $\bar{B} = \phi(B)$, and B is maximizing if and only if \bar{B} is maximal and stable, i.e., has no singular fibers of types $\tilde{\mathbf{A}}_0^$, $\tilde{\mathbf{A}}_1^*$, or $\tilde{\mathbf{A}}_2^*$ (these fibers are called unstable).*

Up to fiberwise equisingular deformation (equivalently, up to automorphism of Σ_k), maximal trigonal curves $\bar{B} \subset \Sigma_k$ are classified by their *skeletons* and *type specifications*. The skeleton $\text{Sk} = \text{Sk}_{\bar{B}} \subset S^2$ (which is defined as Grothendieck's *dessin d'enfants* of the functional j -invariant of \bar{B}) is an embedded connected bipartite graph with all \bullet -vertices of valency ≤ 3 and all \circ -vertices of valency ≤ 2 . The \bullet -vertices of valency ≤ 2 and \circ -vertices of valency 1 are called *singular*; they correspond to the unstable and type $\tilde{\mathbf{E}}$ singular fibers of \bar{B} . Besides, each n -gonal region of Sk (i.e., connected component of the complement $S^2 \setminus \text{Sk}$) contains a single singular fiber of \bar{B} , which is of type $\tilde{\mathbf{A}}_{n-1}$ ($\tilde{\mathbf{A}}_0^*$ if $n = 1$) or $\tilde{\mathbf{D}}_{n+4}$. The type specification is the function choosing, for each singular vertex and each region of Sk , whether the corresponding fiber is of type $\tilde{\mathbf{A}}$ or $\tilde{\mathbf{D}}$, $\tilde{\mathbf{E}}$.

The skeleton and the type specification of a maximal curve $\bar{B} \subset \Sigma_k$ are subject to the relation

$$\#\bullet + \#\circ(1) + \#\bullet(2) = 2(k - t),$$

where t is the number of triple singular points of the curve, $\#_*(n)$ is the number of $*$ -vertices of valency n , $*$ = \bullet or \circ , and $\#_*$ is the total number of $*$ -vertices. Any pair satisfying this relation gives rise to a unique curve.

Under the assumptions of this paper (\bar{B} has no unstable fibers or fibers of type $\tilde{\mathbf{E}}_7$ or $\tilde{\mathbf{E}}_8$), all \circ -vertices of Sk are of valency 2 and all its \bullet -vertices are of valency 3 or 1, the latter corresponding to the type $\tilde{\mathbf{E}}_6$ singular fibers of \bar{B} . Hence the vertex count above can be simplified to

$$(2.1.2) \quad \#\bullet = 2(k - t).$$

Furthermore, the \circ -vertices can be ignored, with the convention that a \circ -vertex is to be understood at the middle of each edge connecting two \bullet -vertices.

To summarize, proof of Theorem 1.2.1 reduces to the enumeration of all pairs $(\text{Sk}, \text{type specification})$, where $\text{Sk} \subset S^2$ is a connected graph with all vertices of valency 3 or 1 and with a distinguished hexagonal region.

2.2. The case of two type \mathbf{E}_6 points. The only maximizing sextic with three type \mathbf{E}_6 singular points (no. 1 in Table 1) is well known, see [4], [17]. Assume that B has two type \mathbf{E}_6 singular points. Then Sk has one monovalent \bullet -vertex and, in view of (2.1.2), it can be obtained by attaching the fragment $\bullet\text{---}\bullet$ at the center of an edge of a regular 3-graph Sk' with two or four vertices, see [7]. All possibilities resulting in a skeleton Sk with a hexagonal region are listed in Figure 1, where Sk' is shown in black and the possible position of the insertion, in grey. The sextics obtained are nos. 2–8 in Table 1; all curves are irreducible due to the existence of a type $\tilde{\mathbf{E}}_6$ singular fiber.

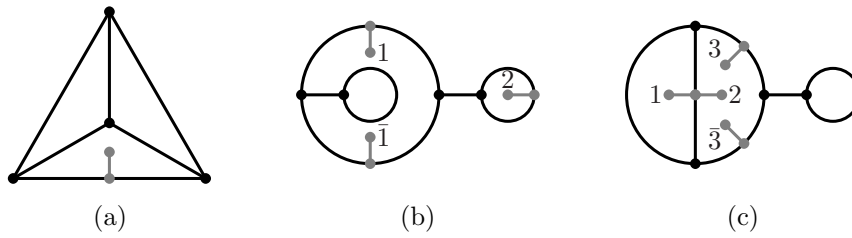


FIGURE 1. Two type E_6 points

2.2.1. Remark. For insertion 2 in Figure 1(c), the skeleton Sk has two hexagonal regions, resulting in two sextics (nos. 6 and 7 in Table 1). There are indeed two distinct deformation families of sextics of torus type with the set of singularities $(2E_6 \oplus A_5) \oplus A_2$, see [4], [17] and Remark 2.6.1 below.

2.3. The case of a hexagon with a loop. Till the rest of this section, assume that P is the only type E_6 singular point of B . Then Sk is a regular 3-graph with a distinguished hexagonal region \bar{H} . Such skeletons can be enumerated using [2]; however, we choose a more constructive descriptive approach.

Combinatorially, there are three possibilities for \bar{H} :

- hexagon with a loop, see Figure 2, left,
- hexagon with a double loop, see Figure 5, or
- genuine hexagon, see Subsection 2.5 below.

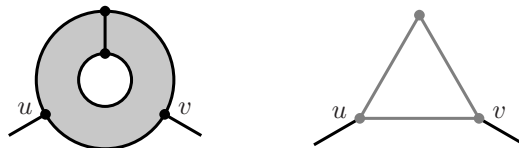


FIGURE 2. A hexagon with a loop

Assume that \bar{H} is a hexagon with a loop, see the shaded area in Figure 2, left. Removing a neighborhood of \bar{H} from Sk and patching vertices u, v in the figure to a single edge, one obtains another regular 3-graph Sk' with at most four vertices, see [7]. Conversely, Sk can be obtained from Sk' by inserting a fragment as in Figure 2, left, at the middle of an edge of Sk' . The essentially distinct possibilities for the position of the insertion are shown in Figures 3 (irreducible curves) and 4 (reducible curves; a reducibility criterion is found in [10]). To simplify the drawings, we represent the insertion by a grey triangle, as in Figure 2, right.

The resulting sextics are nos. 9–32 in Table 1 and nos. 1'–19' in Table 2.

2.3.1. Remark. The curves in pairs nos. 27, 28, nos. 31, 32, and nos. 15', 16' in the tables differ by their type specifications: the type \tilde{D}_5 fiber can be chosen either inside one of the 'free' loops of Sk' shown in the figure or inside the inner loop of the hexagon. We assume that the latter possibility corresponds to curves nos. 28, 32 in Table 1 and no. 16' in Table 2.

2.4. The case of a hexagon with a double loop. Now, assume that the distinguished hexagon \bar{H} looks like the outer region in Figure 5, left. Each of the

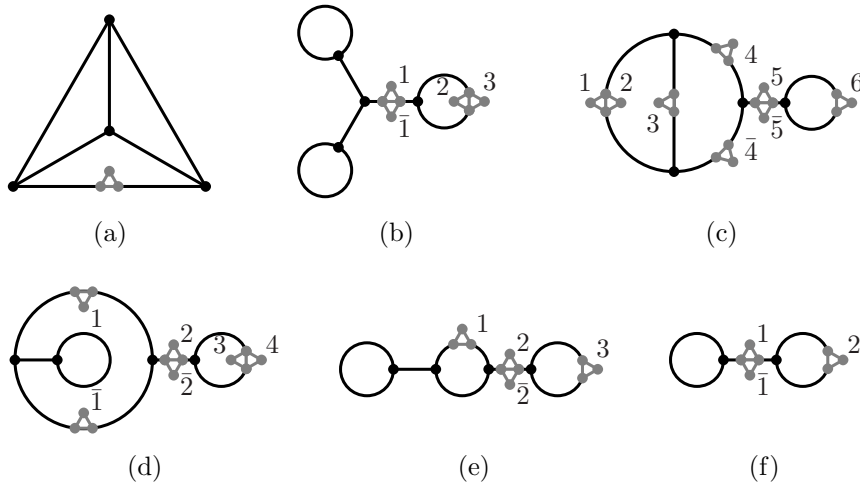


FIGURE 3. A hexagon with a loop: irreducible curves

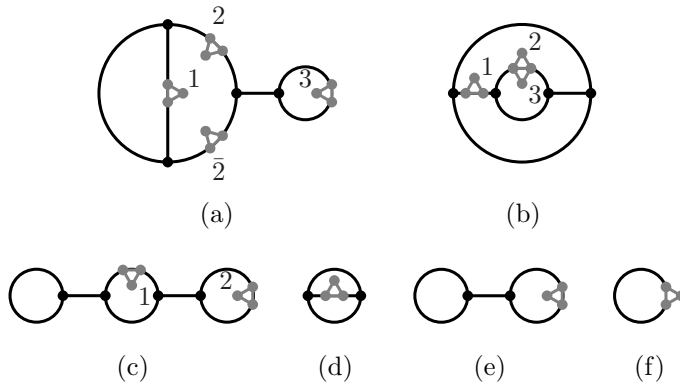


FIGURE 4. A hexagon with a loop: reducible curves

remaining fragments A , B of Sk has an odd number of vertices, and the total number of remaining vertices is at most four. Hence, one can assume that A has one vertex and B has at most three vertices. Then A is a single loop and the graph can be redrawn as shown in Figure 5, right, where \bar{H} is represented by the shaded area. In other words, Sk can be obtained from a regular 3-graph Sk' with two or four vertices and with a loop, see [7], by replacing a loop with the fragment shown in Figure 5, right. The five possibilities are listed in Figure 6; the resulting sextics are nos. 33–38 in Table 1. (Using [10], one can easily show that the existence of a

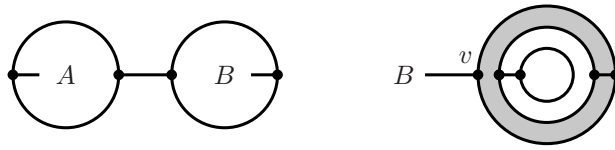


FIGURE 5. A hexagon with a double loop

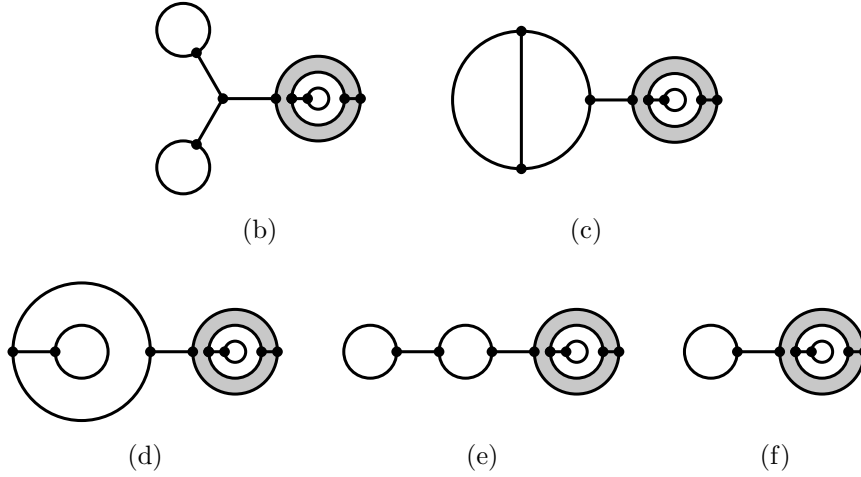


FIGURE 6. A hexagon with a double loop: the five skeletons

fragment as in Figure 5, right implies that the curve is irreducible.)

2.4.1. Remark. The skeleton in Figure 6(f) has a symmetry interchanging its two monogons and two pentagons. For this reason, unlike the case described in Remark 2.3.1, items nos. 37 and 38 are realized by one deformation family each.

2.5. The case of a genuine hexagon. Finally, assume that \bar{H} is a genuine hexagon, *i.e.*, all six vertices in the boundary $\partial\bar{H}$ are pairwise distinct. In other words, $\partial\bar{H}$ is the equator in S^2 , and Sk is obtained from $\partial\bar{H}$ by completing it to a regular 3-graph by inserting at most two vertices and connecting edges into one of the two hemispheres. All possibilities are listed in Figures 7 (irreducible curves) and 8 (reducible curves; a reducibility criterion is found in [10]); the resulting sextics are nos. 39–42 in Table 1 and nos. 20'–38' in Table 2.

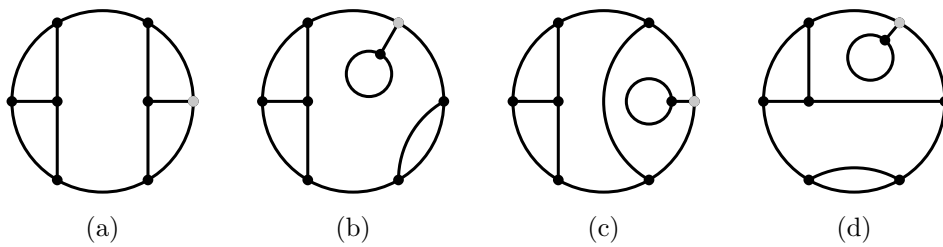


FIGURE 7. A genuine hexagon: irreducible curves

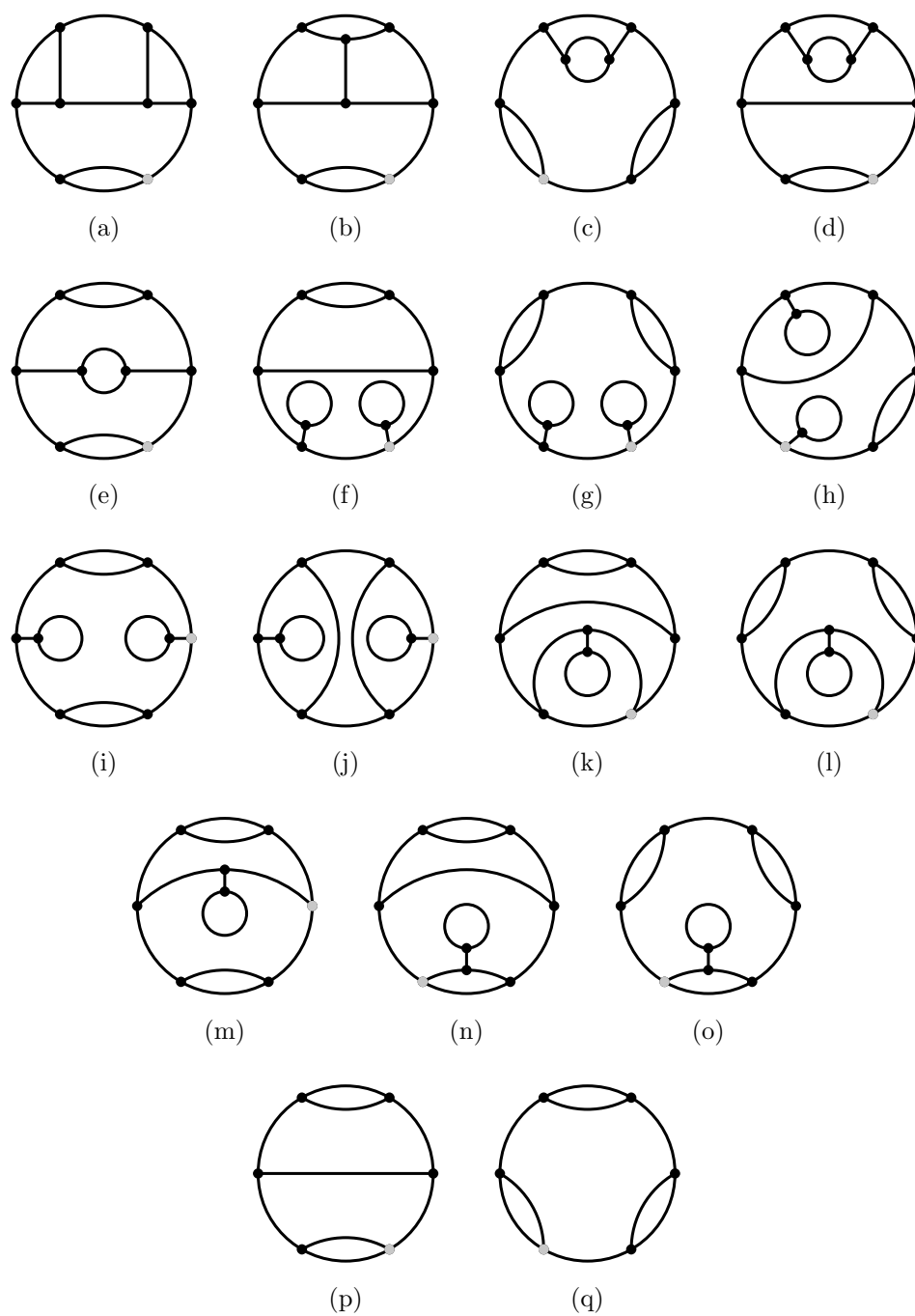


FIGURE 8. A genuine hexagon: reducible curves

2.6. The list. The results of the classification are collected in Tables 1 (irreducible curves) and 2 (reducible curves), where we list the combinatorial types of singularities and references to the corresponding figures. (For sextics of torus type,

the inner singularities are also indicated. Exception is no. 39 in Table 1, which admits four distinct torus structures.) Equal superscripts precede combinatorial types shared by several items in the tables. (One set of singularities, $\mathbf{E}_6 \oplus \mathbf{A}_9 \oplus \mathbf{A}_4$ marked with ¹⁰, is common for both tables.) The ‘Count’ column lists the numbers (n_r, n_c) of real curves and pairs of complex conjugate curves. The last two columns refer to the computation of the fundamental group and indicate the parameters used (explained below). Besides, the curves with nonabelian fundamental group $\pi_1 := \pi_1(\mathbb{P}^2 \setminus B)$ are prefixed with one of the following symbols:

- *: the group π_1 is not abelian;
- ? : the group π_1 is not known to be abelian;
- **: for curves of torus type, $\pi_1 \neq \mathbb{B}_3$.

2.6.1. Remark. For pairs nos. 4, 5 and nos. 6, 7, one can ask if the two curves remain non-equivalent if a permutation of the two type \mathbf{E}_6 points is allowed. For nos. 6 and 7, there still are two distinct equisingular deformation families, see [4] or [17]; for nos. 4 and 5, the two curves become equivalent, see [19]. (Alternatively, if nos. 4 and 5 were not equivalent, each of the curves would have a symmetry interchanging its two type \mathbf{E}_6 points. Since the curve is maximizing, the symmetry would necessarily be stable, contradicting to [7].)

2.6.2. Remark. The sets of singularities nos. 3 and 8 with $(n_r, n_c) = (0, 1)$ can be realized by real curves, see [19], but with respect to this real structure the two type \mathbf{E}_6 points must be complex conjugate.

3. THE COMPUTATION

3.1. Preliminaries and notation. To compute the fundamental groups, we use Zariski–van Kampen’s method [14], applying it to the ruling of Σ_4 . The principal steps of the computation are outlined below; for more details, see [11] and [12].

Fix a maximizing sextic B satisfying $(*)$ and let \bar{B} be the maximal trigonal curve given by Proposition 2.1.1. We take for the fiber at infinity F_∞ the distinguished type $\tilde{\mathbf{A}}_5$ fiber \bar{F} of \bar{B} , and for the reference fiber F , the fiber over an appropriate vertex v of the skeleton Sk of \bar{B} in the boundary $\partial\bar{H}$ of the hexagonal region \bar{H} containing \bar{F} . Choose a marking at v , see [10], so that the edges e_2 and e_3 at v belong to the boundary $\partial\bar{H}$, and let $\{\alpha_1, \alpha_2, \alpha_3\}$ be a canonical basis in F defined by this marking, see [10] or Figure 9. (The precise choice of the vertex v and the marking is explained below on a case by case basis.) Denote $\rho = \alpha_1\alpha_2\alpha_3$.

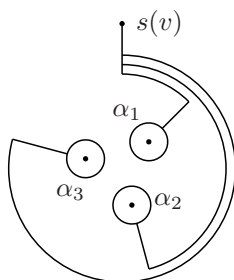


FIGURE 9. A canonical basis $\{\alpha_1, \alpha_2, \alpha_3\}$

TABLE 1. Maximal sets of singularities with a type \mathbf{E}_6 point represented by irreducible sextics

#	Set of singularities	Figure	Count	π_1	Parameters
**1	$(3\mathbf{E}_6) \oplus \mathbf{A}_1$		(1, 0)	see [4]	
**2	$(2\mathbf{E}_6 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_3$	1(a)	(1, 0)	see [4]	
3	$2\mathbf{E}_6 \oplus \mathbf{A}_7$	1(b)-1	(0, 1)	3.3	$(-, -, 1, -)$
*4	$^1 2\mathbf{E}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$	1(b)-2	(1, 0)	3.5	$l = 4$
*5	$^1 2\mathbf{E}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$	1(c)-1	(1, 0)	3.3	$(4, 5, -, -)$
**6	$^2 (2\mathbf{E}_6 \oplus \mathbf{A}_5) \oplus \mathbf{A}_2$	1(c)-2	(1, 0)	see [4]	
**7	$^2 (2\mathbf{E}_6 \oplus \mathbf{A}_5) \oplus \mathbf{A}_2$	1(c)-2	(1, 0)	see [4]	
8	$2\mathbf{E}_6 \oplus \mathbf{A}_6 \oplus \mathbf{A}_1$	1(c)-3	(0, 1)	3.2	
**9	$^3 (\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_4$	3(a)	(1, 0)	3.3.9	$(6, 5, 3, -)$
10	$\mathbf{E}_6 \oplus \mathbf{A}_{13}$	3(b)-1	(0, 1)	3.3	$(-, -, 1, -)$
11	$^4 \mathbf{E}_6 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_3$	3(b)-2	(1, 0)	3.3.4	
*12	$(\mathbf{E}_6 \oplus \mathbf{A}_{11}) \oplus \mathbf{A}_2$	3(b)-3	(1, 0)	3.3.7	
*13	$(\mathbf{E}_6 \oplus \mathbf{A}_8 \oplus \mathbf{A}_2) \oplus \mathbf{A}_3$	3(c)-1	(1, 0)	3.3.10	$(9, 4, 3, -)$
14	$^5 \mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	3(c)-2	(1, 0)	3.3	$(5, 8, 3, -)$
15	$^6 \mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_4$	3(c)-3	(1, 0)	3.3	$(5, 5, 6, -)$
16	$^7 \mathbf{E}_6 \oplus \mathbf{A}_8 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	3(c)-4	(0, 1)	3.3	$(-, -, -, 1)$
17	$^8 \mathbf{E}_6 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	3(c)-5	(0, 1)	3.3	$(-, -, 1, -)$
*18	$^9 (\mathbf{E}_6 \oplus \mathbf{A}_8 \oplus \mathbf{A}_2) \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	3(c)-6	(1, 0)	3.3.10	$(9, 3, -, 3)$
19	$\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_6$	3(d)-1	(0, 1)	3.3	$(-, -, -, 1)$
20	$^{10} \mathbf{E}_6 \oplus \mathbf{A}_9 \oplus \mathbf{A}_4$	3(d)-2	(0, 1)	3.3	$(-, -, 1, -)$
21	$\mathbf{E}_6 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$	3(d)-3	(1, 0)	3.3	$(4, 7, -, -)$
22	$^5 \mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	3(d)-4	(1, 0)	3.3	$(3, 8, -, 5)$
23	$^4 \mathbf{E}_6 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_3$	3(e)-1	(1, 0)	3.3	$(-, -, -, 1)$
24	$\mathbf{E}_6 \oplus \mathbf{A}_{12} \oplus \mathbf{A}_1$	3(e)-2	(0, 1)	3.3	$(-, -, 1, -)$
25	$^8 \mathbf{E}_6 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	3(e)-3	(1, 0)	3.3	$(11, 3, -, 2)$
26	$\mathbf{E}_6 \oplus \mathbf{D}_{13}$	3(f)-1	(1, 0)	3.3	$(-, -, 1, -)$
27	$^{11} \mathbf{E}_6 \oplus \mathbf{D}_5 \oplus \mathbf{A}_8$	3(f)-1	(0, 1)	3.3	$(-, -, 1, -)$
28	$^{11} \mathbf{E}_6 \oplus \mathbf{D}_5 \oplus \mathbf{A}_8$	3(f)-1	(1, 0)	3.3.6	
29	$\mathbf{E}_6 \oplus \mathbf{D}_{11} \oplus \mathbf{A}_2$	3(f)-2	(1, 0)	3.3	$(-, -, -, 1)$
30	$\mathbf{E}_6 \oplus \mathbf{D}_7 \oplus \mathbf{A}_6$	3(f)-2	(1, 0)	3.3	$(-, -, -, 1)$
31	$^{12} \mathbf{E}_6 \oplus \mathbf{D}_5 \oplus \mathbf{A}_6 \oplus \mathbf{A}_2$	3(f)-2	(1, 0)	3.3	$(7, 3, -, -)$
32	$^{12} \mathbf{E}_6 \oplus \mathbf{D}_5 \oplus \mathbf{A}_6 \oplus \mathbf{A}_2$	3(f)-2	(1, 0)	3.3.5	
33	$^{10} \mathbf{E}_6 \oplus \mathbf{A}_9 \oplus \mathbf{A}_4$	6(b)	(1, 0)	3.5	$l = 10$
34	$^{13} \mathbf{E}_6 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	6(c)	(1, 0)	3.5	$l = 7$
35	$^6 \mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_4$	6(d)	(1, 0)	3.5	$l = 6$
36	$^7 \mathbf{E}_6 \oplus \mathbf{A}_8 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	6(e)	(1, 0)	3.5	$l = 9$
37	$\mathbf{E}_6 \oplus \mathbf{D}_9 \oplus \mathbf{A}_4$	6(f)	(1, 0)	3.5	
38	$\mathbf{E}_6 \oplus \mathbf{D}_5 \oplus 2\mathbf{A}_4$	6(f)	(1, 0)	3.5	$l = 5$
**39	$\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 4\mathbf{A}_2$	7(a)	(1, 0)	3.6(1)	$(3, 3, 6, 3, 3, -)$
*40	$^9 (\mathbf{E}_6 \oplus \mathbf{A}_8 \oplus \mathbf{A}_2) \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	7(b)	(0, 1)	3.6(2)	$(9, -, 3, 3, -, 2)$
**41	$^3 (\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_4$	7(c)	(1, 0)	3.6(3)	$(6, -, 5, 3, 3, -)$
42	$^{13} \mathbf{E}_6 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	7(d)	(0, 1)	3.6	$(7, -, 3, 5, 2, -)$

TABLE 2. Maximal sets of singularities with a type E_6 point represented by reducible sextics

#	Set of singularities	Figure	Count	π_1	Parameters
**1'	$^{14}(\mathbf{E}_6 \oplus 2\mathbf{A}_5) \oplus \mathbf{A}_3$	4(a)-1	(1, 0)	3.4(1)	(6, 4, 6, -)
2'	$^{15}\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_5 \oplus \mathbf{A}_1$	4(a)-2	(0, 1)	3.4	(-, -, 2, -)
*3'	$^{16}\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	4(a)-3	(1, 0)	3.4(2)	(4, 8, -, 3)
4'	$^{17}\mathbf{E}_6 \oplus \mathbf{A}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_1$	4(b)-1	(1, 0)	3.4	(-, -, 2, -)
5'	$\mathbf{E}_6 \oplus \mathbf{A}_6 \oplus 2\mathbf{A}_3 \oplus \mathbf{A}_1$	4(b)-2	(1, 0)	3.4	(7, 4, 4, -)
*6'	$^{18}\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus \mathbf{A}_1$	4(b)-3	(1, 0)	3.4(3)	(5, 6, 4, -)
7'	$^{10}\mathbf{E}_6 \oplus \mathbf{A}_9 \oplus \mathbf{A}_4$	4(c)-1	(1, 0)	3.4	(-, -, -, 1)
8'	$^{19}\mathbf{E}_6 \oplus \mathbf{A}_9 \oplus \mathbf{A}_3 \oplus \mathbf{A}_1$	4(c)-2	(1, 0)	3.4	(4, 10, -, 2)
9'	$\mathbf{E}_6 \oplus \mathbf{D}_9 \oplus \mathbf{A}_3 \oplus \mathbf{A}_1$	4(d)	(1, 0)	3.4	(-, -, 2, -)
10'	$\mathbf{E}_6 \oplus \mathbf{D}_8 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	4(d)	(1, 0)	3.4	(-, -, 2, -)
11'	$\mathbf{E}_6 \oplus \mathbf{D}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$	4(d)	(1, 0)	3.4	(5, 4, -, -)
12'	$\mathbf{E}_6 \oplus \mathbf{D}_5 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus \mathbf{A}_1$	4(d)	(1, 0)	3.4.3	(-, -, 2, -)
13'	$\mathbf{E}_6 \oplus \mathbf{D}_{10} \oplus \mathbf{A}_3$	4(e)	(1, 0)	3.4	(-, -, -, 1)
14'	$\mathbf{E}_6 \oplus \mathbf{D}_8 \oplus \mathbf{A}_5$	4(e)	(1, 0)	3.4	(-, -, -, 1)
*15'	$^{20}\mathbf{E}_6 \oplus \mathbf{D}_5 \oplus \mathbf{A}_5 \oplus \mathbf{A}_3$	4(e)	(1, 0)	3.4(4)	(4, 6, -, -)
*16'	$^{20}\mathbf{E}_6 \oplus \mathbf{D}_5 \oplus \mathbf{A}_5 \oplus \mathbf{A}_3$	4(e)	(1, 0)	3.4.3	(4, 6, -, 1)
17'	$\mathbf{E}_6 \oplus \mathbf{D}_7 \oplus \mathbf{D}_6$	4(f)	(1, 0)	3.4.2	
18'	$\mathbf{E}_6 \oplus \mathbf{D}_7 \oplus \mathbf{D}_5 \oplus \mathbf{A}_1$	4(f)	(1, 0)	3.4.3	(-, 2, -, -)
19'	$\mathbf{E}_6 \oplus \mathbf{D}_6 \oplus \mathbf{D}_5 \oplus \mathbf{A}_2$	4(f)	(1, 0)	3.4.3	(3, -, -, -)
**20'	$(\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_3 \oplus \mathbf{A}_1$	8(a)	(1, 0)	3.7(1)	(2, 6, 3, 4, 3, -)
?21'	$\mathbf{E}_6 \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_3 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	8(b)	(1, 0)	3.7(2)	(2, 5, 4, 3, 4, -)
*22'	$\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_3 \oplus 3\mathbf{A}_1$	8(c)	(1, 0)	3.10(1)	(2, 8, 2, -, 4, -)
*23'	$\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_1$	8(d)	(1, 0)	3.10(2)	(2, 4, 6, 4, -, -)
**24'	$(\mathbf{E}_6 \oplus 2\mathbf{A}_5) \oplus 3\mathbf{A}_1$	8(e)	(1, 0)	3.10(3)	(2, 6, 6, 2, -, -)
25'	$^{19}\mathbf{E}_6 \oplus \mathbf{A}_9 \oplus \mathbf{A}_3 \oplus \mathbf{A}_1$	8(f)	(1, 0)	3.8	(10, -, 4, 2, -, -)
26'	$^{21}\mathbf{E}_6 \oplus \mathbf{A}_{11} \oplus 2\mathbf{A}_1$	8(g)	(1, 0)	3.8	(12, -, 2, -, 2, -)
27'	$^{15}\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_5 \oplus \mathbf{A}_1$	8(h)	(0, 1)	3.8	(8, -, 2, -, 6, -)
**28'	$^{21}(\mathbf{E}_6 \oplus \mathbf{A}_{11}) \oplus 2\mathbf{A}_1$	8(i)	(1, 0)	3.8	(12, -, 2, -, -, 2)
**29'	$^{14}(\mathbf{E}_6 \oplus 2\mathbf{A}_5) \oplus \mathbf{A}_3$	8(j)	(1, 0)	3.8	(6, -, 4, 6, -, -)
*30'	$^{18}\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus \mathbf{A}_1$	8(k)	(1, 0)	3.7(3)	(6, 5, 4, 2, -, -)
31'	$^{17}\mathbf{E}_6 \oplus \mathbf{A}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_1$	8(l)	(1, 0)	3.7	(6, 7, 2, -, 2, -)
32'	$\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_1$	8(m)	(1, 0)	3.7	(8, 5, 2, -, -, 2)
*33'	$^{16}\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	8(n)	(1, 0)	3.7(4)	(8, 3, -, 4, 2, -)
*34'	$\mathbf{E}_6 \oplus \mathbf{A}_9 \oplus \mathbf{A}_2 \oplus 2\mathbf{A}_1$	8(o)	(1, 0)	3.7(5)	(10, 3, -, 2, -, 2)
35'	$\mathbf{E}_6 \oplus \mathbf{D}_8 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_1$	8(p)	(1, 0)	3.9	(-, 2, -, 4, 2, -)
*36'	$\mathbf{E}_6 \oplus \mathbf{D}_6 \oplus 2\mathbf{A}_3 \oplus \mathbf{A}_1$	8(p)	(1, 0)	3.10(4)	(2, 4, 4, -, -, -)
*37'	$\mathbf{E}_6 \oplus \mathbf{D}_{10} \oplus 3\mathbf{A}_1$	8(q)	(1, 0)	3.10(5)	(2, -, 2, -, 2, -)
38'	$\mathbf{E}_6 \oplus \mathbf{D}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_1$	8(q)	(1, 0)	3.9	(-, 6, 2, -, 2, -)

According to [11], the generators $\alpha_1, \alpha_2, \alpha_3$ are subject to the so called *relation at infinity*

$$(3.1.1) \quad \rho^4 = (\alpha_2 \alpha_3)^3.$$

Let F_1, \dots, F_r be the singular fibers of \bar{B} other than \bar{F} . Dragging F about F_j and

keeping the base point in an appropriate section, one obtains an automorphism $m_j \in \mathbb{B}_3 \subset \text{Aut}\langle \alpha_1, \alpha_2, \alpha_3 \rangle$, called the *braid monodromy* about F_j . In this notation, the group $\pi_1 := \pi_1(\mathbb{P}^2 \setminus B)$ has a presentation of the form

$$(3.1.2) \quad \pi_1 = \langle \alpha_1, \alpha_2, \alpha_3 \mid m_j = \text{id}, j = 1, \dots, r, \text{ and (3.1.1)} \rangle,$$

where each *braid relation* $m_j = \text{id}$, $j = 1, \dots, r$, should be understood as the triple of relations $m_j(\alpha_i) = \alpha_i$, $i = 1, 2, 3$. Furthermore, in the presence of the relation at infinity, (any) one of the braid relations $m_j = \text{id}$, $j = 1, \dots, r$, can be omitted.

The braid monodromies m_j are computed using [10]; all necessary details are explained below in this section.

Throughout the paper, all finite groups/quotients are analyzed using GAP [13]; most infinite groups are handled by means of the following obvious lemma, which we state here for future references.

3.1.3. Lemma. *Let G be a group, and let $a \in G$ be a central element whose projection to the abelianization $G/[G, G]$ has infinite order. Then the projection $G \rightarrow G/a$ restricts to an isomorphism $[G, G] = [G/a, G/a]$. \square*

Given two elements α, β of a group and a nonnegative integer m , introduce the notation

$$\{\alpha, \beta\}_m = \begin{cases} (\alpha\beta)^k(\beta\alpha)^{-k}, & \text{if } m = 2k \text{ is even,} \\ ((\alpha\beta)^k\alpha)((\beta\alpha)^k\beta)^{-1}, & \text{if } m = 2k + 1 \text{ is odd.} \end{cases}$$

The relation $\{\alpha, \beta\}_m = 1$ is equivalent to $\sigma^m = \text{id}$, where σ is the generator of the braid group \mathbb{B}_2 acting on the free group $\langle \alpha, \beta \rangle$. Hence,

$$(3.1.4) \quad \{\alpha, \beta\}_m = \{\alpha, \beta\}_n = 1 \quad \text{is equivalent to} \quad \{\alpha, \beta\}_{\text{g.c.d.}(m,n)} = 1.$$

For the small values of m , the relation $\{\alpha, \beta\}_m = 1$ takes the following form:

- $m = 0$: tautology;
- $m = 1$: the identification $\alpha = \beta$;
- $m = 2$: the commutativity relation $[\alpha, \beta] = 1$;
- $m = 3$: the braid relation $\alpha\beta\alpha = \beta\alpha\beta$.

3.2. Two type \mathbf{E}_6 singular points. It suffices to consider the set of singularities $2\mathbf{E}_6 \oplus \mathbf{A}_6 \oplus \mathbf{A}_1$ (no. 8 in Table 1) only. The fundamental groups of all sextics of torus type with two type \mathbf{E}_6 singular points are found in [4], and the remaining curves that are not of torus type are covered by 3.3 and 3.5 below.

Thus, consider the skeleton Sk given by Figure 1(c)–3. Let u be the monovalent \bullet -vertex of Sk , and let v be the trivalent \bullet -vertex adjacent to u . Mark v so that $[u, v]$ is the edge e_2 at v . Then, in addition to (3.1.1), the group π_1 has the relations

$$\{\alpha_2\alpha_3\alpha_2^{-1}, \alpha_1\}_7 = \{\alpha_1\alpha_2\alpha_3\alpha_2^{-1}\alpha_1^{-1}, \alpha_2\}_2 = 1, \quad \rho\alpha_2\rho^{-1} = \alpha_2\alpha_3\alpha_2^{-1},$$

obtained from the heptagon, the bigon, and the monovalent \bullet -vertex u of Sk , respectively. The resulting group is abelian.

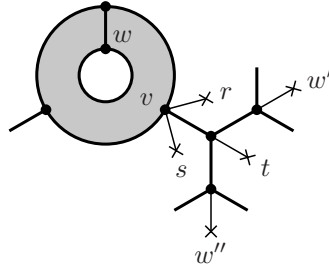


FIGURE 10. A hexagon with a loop: the regions

3.3. Hexagon with a loop: irreducible curves. Assume that \bar{H} is a hexagon with a loop, see 2.3 and Figure 10, and take for v the vertex shown in the figure.

The inner loop of \bar{H} (the monogonal region containing w in its boundary) gives the relation

$$(3.3.1) \quad \alpha_1 = \alpha_3^{-1} \alpha_2 \alpha_3.$$

Extend Sk to a dessin, see [10], and consider the \times -vertices $r, s, t,$ and w' shown in Figure 10. (We do not assert that all these vertices are distinct.) Assume that they are at the centers of l -, m -, n -, and k -gonal regions of Sk , respectively. Then the braid relations about the singular fibers of \bar{B} over these vertices are

$$(3.3.2) \quad \begin{aligned} r : \quad & \{\alpha_1, \alpha_2\}_l = 1, \\ s : \quad & \{\alpha_1, \alpha_2 \alpha_3 \alpha_2^{-1}\}_m = 1, \\ t : \quad & \{\alpha_2, \rho \alpha_3 \rho^{-1}\}_n = 1, \\ w' : \quad & \{\alpha_2^{-1} \alpha_1 \alpha_2, \rho \alpha_3 \rho^{-1}\}_k = 1, \end{aligned}$$

assuming that the fibers are of type $\tilde{\mathbf{A}}$. For a $\tilde{\mathbf{D}}$ type fiber, we omit the corresponding relation in (3.3.2) and indicate this fact by a ‘-’ in the parameter list. (Sometimes, we also omit a relation just because it is not necessary to prove that π_1 is abelian.) Using the values of (l, m, n, k) shown in Table 1, one concludes that the groups of most sextics that are not of torus type are abelian. The few exceptional cases are treated separately below.

The same arguments apply to the sets of singularities $2\mathbf{E}_6 \oplus \mathbf{A}_7$ (no. 3 in Table 1) and $2\mathbf{E}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$ (no. 5 in Table 1), as the corresponding skeletons can be represented as shown in Figure 11(a)–1, $\bar{1}$ and Figure 11(a)–2, respectively. The former fundamental group is abelian, the latter is the order 720 group given by

$$(3.3.3) \quad \pi_1 = \langle \alpha_1, \alpha_2, \alpha_3 \mid (3.1.1), (3.3.1), (3.3.2) \rangle$$

with $(l, m, n, k) = (4, 5, -, -)$; it splits into semidirect product $SL(2, \mathbb{F}_5) \rtimes \mathbb{Z}_6$. (Recall that the braid relation about the remaining type $\tilde{\mathbf{E}}_6$ singular fiber can be ignored.) An alternative presentation of this group is given by (3.5.2).



FIGURE 11

3.3.4. *The set of singularities $\mathbf{E}_6 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_3$ (no. 11 in Table 1).* In addition to (3.1.1), (3.3.1), and (3.3.2) with $(l, m, n, k) = (4, 11, -, -)$, one also has the relation

$$(\alpha_1 \alpha_2 \alpha_3 \alpha_2^{-1}) \alpha_1 (\alpha_1 \alpha_2 \alpha_3 \alpha_2^{-1})^{-1} = (\alpha_2^{-1} \alpha_1 \alpha_2) \alpha_3 (\alpha_2^{-1} \alpha_1 \alpha_2)^{-1}$$

from the lower left loop in Figure 3(b). The resulting group is abelian.

Alternatively, choosing a canonical basis $\{\alpha_1, \alpha_2, \alpha_3\}$ in the fiber over the upper left \bullet -vertex in Figure 3(b), one obtains the relations $\alpha_2 = \alpha_3$ (from the upper left loop) and $\alpha_2^{-1} \alpha_1 \alpha_2 = \rho^{-1} \alpha_2 \rho$ (from the lower left loop). In view of the former, the latter simplifies to the braid relation $\alpha_1 \alpha_2 \alpha_1 = \alpha_2 \alpha_1 \alpha_2$, *i.e.*, $\{\alpha_1, \alpha_2\}_3 = 1$. On the other hand, the 11-gonal outer region of the skeleton gives the relation $\{\alpha_1, \alpha_2\}_{11} = 1$. Hence $\{\alpha_1, \alpha_2\}_1 = 1$, see (3.1.4), and the group is cyclic.

3.3.5. *The set of singularities $\mathbf{E}_6 \oplus \mathbf{D}_5 \oplus \mathbf{A}_6 \oplus \mathbf{A}_2$ (no. 32 in Table 1).* In this case we assume that the type \mathbf{D}_5 singular fiber is chosen inside the inner loop of the insertion, see Remark 2.3.1. Hence the group has no relation (3.3.1). However, relations (3.1.1) and (3.3.2) with $(l, m, n, k) = (7, 3, -, 1)$ suffice to show that the group is abelian.

3.3.6. *The set of singularities $\mathbf{E}_6 \oplus \mathbf{D}_5 \oplus \mathbf{A}_8$ (no. 28 in Table 1).* As above, the type \mathbf{D}_5 fiber is chosen inside the inner loop of the insertion, see Remark 2.3.1. Hence the group has no relation (3.3.1). Still, it has relations (3.1.1) and (3.3.2) with $(l, m, n, k) = (9, -, 1, -)$ and, in addition, the relation

$$\alpha_2 \alpha_3 \alpha_2^{-1} = (\alpha_1 \alpha_2 \alpha_3) \alpha_2 (\alpha_1 \alpha_2 \alpha_3)^{-1}$$

resulting from the left loop in Figure 3(f). These relations suffice to show that the group is abelian.

3.3.7. *The set of singularities $(\mathbf{E}_6 \oplus \mathbf{A}_{11}) \oplus \mathbf{A}_2$ (no. 12 in Table 1).* As explained in 3.3.4, the group π_1 is a quotient of the braid group \mathbb{B}_3 (for the braid relations, only the two left loops in Figure 3(b) are used); hence $\pi_1 = \bar{\mathbb{B}}_3$ due to the following simple lemma (see, *e.g.*, Lemma 3.6.1 in [8]).

3.3.8. Lemma. *Let B be an irreducible plane sextic of torus type. Then any epimorphism $\mathbb{B}_3 \twoheadrightarrow \pi_1(\mathbb{P}^2 \setminus B)$ factors through an isomorphism $\bar{\mathbb{B}}_3 = \pi_1(\mathbb{P}^2 \setminus B)$.*

3.3.9. *The set of singularities $(\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_4$ (no. 9 in Table 1).* The group π_1 has presentation (3.3.3) with $(l, m, n, k) = (6, 5, 3, -)$. Using GAP [13], one can see that the quotient π_1/α_1^5 is a perfect group of order 7680, whereas $\bar{\mathbb{B}}_3/\sigma_1^5 = \mathbb{A}_5$ has order 60. Hence the natural epimorphism $\pi_1 \twoheadrightarrow \bar{\mathbb{B}}_3$ is proper.

3.3.10. *Other curves of torus type.* The remaining sextics of torus type appearing in this section are nos. 13 and 18 in Table 1. The group π_1 has presentation (3.3.3) with the values of the parameters (l, m, n, k) given in the table. Both groups factor to $\bar{\mathbb{B}}_3$. Entering the presentations to GAP [13] and simplifying them *via*

```
P := PresentationNormalClosure(g, Subgroup(g, [g.1/g.2]));
SimplifyPresentation(P);
```

one finds that the commutant $[\pi_1, \pi_1]$ is a free group on two generators. Since all groups involved are residually finite, hence Hopfian, one concludes that both epimorphisms $\pi_1 \twoheadrightarrow \bar{\mathbb{B}}_3$ are isomorphisms. (This approach was suggested to me by E. Artal Bartolo.)

3.4. Hexagon with a loop: reducible curves. Choose a basis $\{\alpha_1, \alpha_2, \alpha_3\}$ as in Subsection 3.3. Then (3.1.1) and (3.3.1) imply that $\pi_1/[\pi_1, \pi_1] = \mathbb{Z}$ and the projection $\pi_1 \rightarrow \mathbb{Z}$ is given by $\alpha_1, \alpha_2 \mapsto 1, \alpha_3 \mapsto -5$. In particular, it follows that the sextic splits into an irreducible quintic and a line.

In addition to (3.1.1) and (3.3.1), consider the relations

$$(3.4.1) \quad \begin{aligned} r &: \{\alpha_1, \alpha_2\}_l = 1, \\ s &: \{\alpha_1, \alpha_2 \alpha_3 \alpha_2^{-1}\}_m = 1, \\ t &: \{\alpha_2, \rho \alpha_3 \rho^{-1}\}_n = 1, \\ w'' &: \{\alpha_2^{-1} \alpha_1 \alpha_2, \rho^{-1} \alpha_2 \rho\}_k = 1 \end{aligned}$$

arising from the \times -vertices r, s, t, w'' in Figure 10 (assuming that the fibers over these vertices are of type $\tilde{\mathbf{A}}$; if a fiber is of type $\tilde{\mathbf{D}}$, the corresponding relation is omitted.)

In order to analyze the group using GAP [13], observe that relation (3.1.1) implies $[\alpha_1, (\alpha_2 \alpha_3)^3] = 1$; then, in view of (3.3.1), the element $(\alpha_2 \alpha_3)^3$ commutes with α_3 and hence with α_2 . Thus, $(\alpha_2 \alpha_3)^3 \in \pi_1$ is a central element and its projection to the abelianization of π_1 is the element -12 of infinite order. Due to Lemma 3.1.3, it suffices to study the commutant of the quotient $\pi_1/(\alpha_2 \alpha_3)^3$. The abelianization of the latter quotient is \mathbb{Z}_{12} .

The sets of parameters (l, m, n, k) used in the calculation are listed in Table 2. The curves with the following sets of singularities have nonabelian groups:

- (1) $(\mathbf{E}_6 \oplus 2\mathbf{A}_5) \oplus \mathbf{A}_3$ (no. 1'): the curve is of torus type, hence $\text{ord}[\pi_1, \pi_1] = \infty$. Note that $\pi_1/\alpha_1^2 = GL(2, \mathbb{F}_3)$ has order 48; in particular, π_1 is not isomorphic to any braid group \mathbb{B}_n .
- (2) $\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$ (no. 3'): one has $\pi_1 = SL(2, \mathbb{F}_7) \times \mathbb{Z}$.
- (3) $\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus \mathbf{A}_1$ (no. 6'): one has $\pi_1 = SL(2, \mathbb{F}_5) \rtimes \mathbb{Z}$.
- (4) $\mathbf{E}_6 \oplus \mathbf{D}_5 \oplus \mathbf{A}_5 \oplus \mathbf{A}_3$ (no. 15'): one has $\pi_1 = ((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}$.

(In Items (2), (3), and (4), the centralizer of $[\pi_1, \pi_1]$ projects to a subgroup of index 1, 2, and 4, respectively, in the abelianization. In the former case, it follows that the product is direct.) The group in Item (2) was first computed in [1].

3.4.2. The set of singularities $\mathbf{E}_6 \oplus \mathbf{D}_7 \oplus \mathbf{D}_6$ (no. 17' in Table 2). The curve has no $\tilde{\mathbf{A}}$ -type singular fibers outside the insertion, and we replace (3.4.1) with the relations

$$(\rho \alpha_1 \alpha_2 \alpha_1) \alpha_2 (\rho \alpha_1 \alpha_2 \alpha_1)^{-1} = \alpha_1, \quad (\rho \alpha_1 \alpha_2) \alpha_1 (\rho \alpha_1 \alpha_2)^{-1} = \alpha_2$$

resulting from the type $\tilde{\mathbf{D}}_7$ singular fiber over r . The resulting group is abelian.

3.4.3. A $\tilde{\mathbf{D}}$ type fiber inside the insertion. If the singular fiber of \bar{B} in the loop inside the insertion is of type $\tilde{\mathbf{D}}_5$, relation (3.3.1) should be replaced with

$$(3.4.4) \quad \rho \beta_2 \beta_3 \beta_2^{-1} \rho^{-1} = \beta_2, \quad \rho \beta_2 \rho^{-1} = \beta_3,$$

where $\{\beta_1, \beta_2, \beta_3\}$ is an appropriate canonical basis over the \bullet -vertex w in Figure 10. Using [10], one has $\beta_1 = \alpha_1 \alpha_3 \alpha_1^{-1}$, $\beta_2 = \alpha_1$, and $\beta_3 = \alpha_3^{-1} \alpha_2 \alpha_3$ (in particular, $\beta_1 \beta_2 \beta_3 = \rho$). From (3.4.4) it follows that $\delta := \rho^2 \beta_2 \beta_3$ is a central element of π_1 ;

since the projection of δ to the abelianization of π_1 is the element -4 of infinite order, one can use Lemma 3.1.3 and study the commutant of the quotient π_1/δ .

The sets of parameters (l, m, n, k) used in the calculation are listed in Table 2. The only nonabelian group in this series is the one corresponding to the set of singularities $\mathbf{E}_6 \oplus \mathbf{D}_5 \oplus \mathbf{A}_5 \oplus \mathbf{A}_3$ (no. 16'); it can be represented as a semidirect product $((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}$ and is isomorphic to the one described in 3.4(4).

3.5. Hexagon with a double loop. Assume that \bar{H} is a hexagon with a double loop, see 2.4, and choose for v the vertex shown in Figure 5, right. We can assume that the singular fibers inside the insertion are all of type $\tilde{\mathbf{A}}$, see Remark 2.4.1. Then, the braid relations resulting from the inner pentagon and monogon are

$$(3.5.1) \quad \{\alpha_1, \alpha_3^{-1} \alpha_2 \alpha_3\}_5 = 1, \quad \alpha_1 \alpha_3^{-1} \alpha_2 \alpha_3 \alpha_1^{-1} = \alpha_3^{-1} \alpha_2^{-1} \alpha_3 \alpha_2 \alpha_3.$$

In addition, for all curves except no. 37 in Table 1, there is a relation $\{\alpha_1, \alpha_2\}_l = 1$, where $l = 10, 7, 6, 9$, or 5 (in the order of appearance in Table 1). For no. 37, one has the commutativity relation $[\alpha_3, \alpha_1 \alpha_2] = 1$ from the $\tilde{\mathbf{D}}_9$ type fiber. In all cases, using GAP [13], one concludes that the group is abelian.

The same arguments apply to the set of singularities $2\mathbf{E}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$ (no. 4 in Table 1), as the corresponding skeleton can be represented as shown in Figure 11(b) (so that one has $l = 4$). The resulting group has order 720, and its presentation is

$$(3.5.2) \quad \pi_1 = \langle \alpha_1, \alpha_2, \alpha_3 \mid (3.1.1), (3.5.1), \{\alpha_1, \alpha_2\}_4 = 1 \rangle.$$

This group is isomorphic to (3.3.3), see Remark 2.6.1.

3.6. Genuine hexagon: irreducible curves. Consider one of the four skeletons shown in Figure 7 and take for v any vertex in $\partial\bar{H}$. Let $v_0 = v, v_1, \dots, v_5$ be the vertices in $\partial\bar{H}$ numbered starting from v in the *clockwise* direction (which is the *counterclockwise* direction in the figures, which represent the complementary hexagon $S^2 \setminus \bar{H}$). Mark each vertex similar to $v_0 = v$ and denote by R_i the region of Sk whose boundary contains the edges e_1 and e_2 at v_i , $i = 0, \dots, 5$. Let n_i be the number of vertices in ∂R_i , $i = 0, \dots, 5$, *i.e.*, assume that R_i is an n_i -gon. Then, in addition to the common relation at infinity (3.1.1), the group π_1 has the relations

$$(3.6.1) \quad \{\sigma_2^i(\alpha_2), \alpha_1\}_{n_i} = 1, \quad i = 0, \dots, 5,$$

resulting from the singular fibers in R_i . If R_i and \bar{H} are all but at most one regions of Sk (which is always the case for irreducible curves, see Figure 7), then (3.1.1) and (3.6.1) form a complete set of relations for π_1 . Furthermore, in the sequence R_0, \dots, R_5 , some of the regions coincide; for each region, it suffices to consider only one instance in the sequence and ignore the other relations by letting the corresponding parameters n_i equal 0; these relations would follow from the others.

The values of the parameters (n_0, \dots, n_5) used in the calculation are listed in Table 1, and the initial vertex $v = v_0$ is shown in Figure 7 in grey. For the set of singularities $\mathbf{E}_6 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$ (no. 42 in Table 1), the resulting group is abelian. The other three curves are of torus type, hence their groups factor to \mathbb{B}_3 . Below is some information on these groups.

- (1) $\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 4\mathbf{A}_2$ (no. 39): the curve is a sextic of torus type of weight 8 in the sense of [5]; hence π_1 is much larger than \mathbb{B}_3 : its Alexander module is

a direct sum of *two* copies of $\mathbb{Z}[t]/(t^2 - t + 1)$. An alternative presentation of this group is found in [9].

- (2) $(\mathbf{E}_6 \oplus \mathbf{A}_8 \oplus \mathbf{A}_2) \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$ (no. 40): similar to 3.3.10, one can use GAP [13] to show that the natural epimorphism $\pi_1 \twoheadrightarrow \mathbb{B}_3$ is an isomorphism.
- (3) $(\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_4$ (no. 41): the quotient π_1/α_1^5 is a perfect group of order 7680, whereas $\mathbb{B}_3/\alpha_1^5 = \mathbf{A}_5$. Hence the natural epimorphism $\pi_1 \twoheadrightarrow \mathbb{B}_3$ is proper. (The values of the parameters actually used are $(n_0, \dots, n_5) = (3, 3, 5, 6, 6, 5)$.) I do not know whether this group is isomorphic to the one considered in 3.3.9.

3.6.2. Remark. In Items (2) and (3), if the reference fiber is chosen as shown in Figure 7, the group has also relation $(\alpha_1\alpha_2)^{-1}\alpha_2(\alpha_1\alpha_2) = \alpha_3$, see (3.8.1) below, which may simplify its analysis. (Formally, this relation follows from the others.)

3.7. Genuine hexagon: reducible curves. The approach of Subsection 3.6 applies to reducible curves \bar{B} as well, see Figure 8, provided that the skeleton of \bar{B} has at most one region or $\bar{\mathbf{D}}$ type singular fiber strictly inside the complementary hexagon $S^2 \setminus \bar{H}$, *i.e.*, to all skeletons except those shown in Figures 8(f)–(j). (In the case of a $\bar{\mathbf{D}}$ -type fiber, Figures 8(p) and (q), the region R_i containing this fiber should be ignored in (3.6.1).)

The first two curves (Figures 8(a) and (b)) do not seem to have any extra features that would facilitate the study of their groups. (Each of these curves splits into an irreducible quintic and a line; hence $\pi_1/[\pi_1, \pi_1] = \mathbb{Z}$.) The values of the parameters (n_0, \dots, n_5) are listed in Table 2 (assuming that $v = v_0$ is the vertex shown in the figures in grey), and the resulting groups are as follows:

- (1) $(\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_3 \oplus \mathbf{A}_1$ (no. 20'): the curve is of torus type, hence π_1 factors to the braid group \mathbb{B}_3 ; in particular, $[\pi_1, \pi_1]$ is infinite.
- (2) $\mathbf{E}_6 \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_3 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$ (no. 21'): the commutant $[\pi_1, \pi_1]$ is perfect (one can compute the Alexander module $A = 0$ similar to Subsection 3.10 below); it appears to be infinite, but at present I do not even know if it is nontrivial.

In the rest of this subsection, we consider the skeletons with one monogonal region inside $\mathbb{P}^1 \setminus H$, *i.e.*, those shown in Figures 8(k)–(o). Choose for the initial vertex $v = v_0$ the one shown in the figures in grey. Then, the monogonal region gives an extra relation

$$(3.7.1) \quad (\alpha_2\alpha_1\alpha_2)^{-1}\alpha_1(\alpha_2\alpha_1\alpha_2) = \alpha_3.$$

(Strictly speaking, this relation follows from the others, but its presence simplifies the calculations. In particular, since each sextic B in question is known to be reducible, it follows that it splits into an irreducible quintic and a line, the projection $\pi_1 \twoheadrightarrow \pi_1/[\pi_1, \pi_1] = \mathbb{Z}$ being given by $\alpha_1, \alpha_3 \mapsto 1, \alpha_2 \mapsto -5$.) On the other hand, relation (3.6.1) implies that $\delta := (\alpha_1\alpha_2)^{n_0}$ commutes with α_1 and α_2 ; hence δ is a central element and, using Lemma 3.1.3, one can study the commutant of the quotient π_1/δ .

The parameters (n_0, \dots, n_5) are listed in Table 2. The following three sets of singularities result in nonabelian fundamental groups.

- (3) $\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus \mathbf{A}_1$ (no. 30'): one has $\pi_1 = SL(2, \mathbb{F}_5) \rtimes \mathbb{Z}$.
- (4) $\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$ (no. 33'): one has $\pi_1 = SL(2, \mathbb{F}_7) \rtimes \mathbb{Z}$.

- (5) $\mathbf{E}_6 \oplus \mathbf{A}_9 \oplus \mathbf{A}_2 \oplus 2\mathbf{A}_1$ (no. 34'): the commutant of π_1 is a perfect group; it appears infinite, but I do not know a proof. The commutants of π_1/α_1^2 and π_1/α_1^3 have orders 60 and 51840, respectively.

(In Items (3) and (4), the centralizer of $[\pi_1, \pi_1]$ projects to a subgroup of index 2 and 1, respectively, in $\pi_1/[\pi_1, \pi_1]$. The groups are isomorphic to those in 3.4(3) and (2), respectively.) The group in Item (4) was first computed in [1], where it is also shown that sextics nos. 3' and 33' in Table 2 are Galois conjugate over $\mathbb{Q}(\sqrt{2})$.

3.8. Genuine hexagon: two monogons inside $S^2 \setminus \bar{H}$. In this subsection, we consider a skeleton with two monogonal regions strictly inside $S^2 \setminus \bar{H}$, *i.e.*, one of those shown in Figures 8(f)–(j). Take for $v = v_0$ the vertex shown in the figures in grey. Then, in addition to (3.1.1) and (3.6.1), the group has an extra relation

$$(3.8.1) \quad (\alpha_1\alpha_2)^{-1}\alpha_2(\alpha_1\alpha_2) = \alpha_3$$

resulting from the monogon closest to v . In particular, it follows that the curve splits into an irreducible quartic and irreducible conic. Furthermore, since $\delta := (\alpha_1\alpha_2)^{n_0}$ commutes with α_1 and α_2 , it is a central element and one can use Lemma 3.1.3 and study the commutant of the quotient π_1/δ .

The parameters (n_0, \dots, n_5) are listed in Table 2. For the first three curves, the groups are abelian. (As a consequence, the curve defined by the skeleton in Figure 8(g), no. 26' in Table 2, is not of torus type.) The other two curves are of torus type; an alternative way to construct these curves and to compute their fundamental groups is found in [8]. (To prove that these curves are of torus type, one can argue that the existence of such curves is shown in [8], and nos. 28, 29 are the only candidates left.)

3.9. Genuine hexagon with a $\tilde{\mathbf{D}}$ -type fiber. Consider one of the two skeletons shown in Figure 8(p) or (q) and choose the initial vertex $v = v_0$ so that R_0 is the region containing the $\tilde{\mathbf{D}}$ -type fiber F of \tilde{B} . Then, as above, the defining relations for π_1 are (3.1.1) and (3.6.1), with the contribution of R_0 ignored in the latter.

However, we do make use of the region R_0 in order to find central elements in π_1 . Let $n = n_0$. Then F is of type $\tilde{\mathbf{D}}_{n+4}$, and the braid relations about F are

$$\alpha_3^{-1}\alpha_i\alpha_3 = \sigma_1^{n+2}(\alpha_i), \quad i = 1, 2.$$

As a consequence,

$$(3.9.1) \quad [\alpha_3, \alpha_1\alpha_2] = 1, \quad \text{hence} \quad [\alpha_3, \rho] = [\alpha_1\alpha_2, \rho] = 1,$$

and $\delta := \alpha_3(\alpha_1\alpha_2)^{1+n/2}$ is a central element of π_1 . (Note that in all cases n is even.) Since $[\alpha_2\alpha_3, \rho^4] = 1$, see (3.1.1), from (3.9.1) one has $[\alpha_2, \rho^4] = 1$ and then $[\alpha_1, \rho^4] = 1$. Thus, $\rho^4 = (\alpha_2\alpha_3)^3$ is also a central element of π_1 , and Lemma 3.1.3 applied twice implies that the commutant $[\pi_1, \pi_1]$ of π_1 is isomorphic to that of the quotient $\pi_1/\langle \delta, \rho^4 \rangle$. (It is worth mentioning that $n \geq 2$ and hence the images of δ and ρ in the abelianization of π_1 are linearly independent.)

The values of the parameters (n_0, \dots, n_5) are listed in Table 2. (Recall that the initial vertex v_0 is determined by the position of the $\tilde{\mathbf{D}}$ -type fiber, which depends on the curve.) Using GAP [13], one concludes that, for the sets of singularities

$$\mathbf{E}_6 \oplus \mathbf{D}_8 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_1 \quad \text{and} \quad \mathbf{E}_6 \oplus \mathbf{D}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_1$$

(nos. 35' and 38' in Table 2), the group π_1 is abelian, whereas for the other two curves (nos. 36' and 37') it has infinite commutant. For a more precise statement, see Subsection 3.10 below, where we compute the Alexander modules of these and a few other groups.

3.9.2. Remark. The fact that the fundamental groups of the reducible sextics with the sets of singularities $\mathbf{E}_6 \oplus \mathbf{D}_6 \oplus 2\mathbf{A}_3 \oplus \mathbf{A}_1$ and $\mathbf{E}_6 \oplus \mathbf{D}_{10} \oplus 3\mathbf{A}_1$ have infinite commutants can also be explained as follows. Each curve splits into an irreducible quartic B_4 with a type \mathbf{E}_6 singular point and a pair of lines. One of the lines B_1 either is double tangent to B_4 or has a single point of 4-fold intersection with B_4 . Hence, even patching back in the other line (which corresponds to letting one of the canonical generators of π_1 equal to 1), one obtains a curve with large fundamental group (which is, respectively, \mathbb{B}_3 or $\mathbb{T}_{3,4} = \langle \alpha, \beta \mid \alpha^3 = \beta^4 \rangle$, see [3]).

3.10. Other sextics with two linear components. With the exception of the two curves mentioned in the previous section, all maximizing sextics splitting into an irreducible quartic (with a type \mathbf{E}_6 singular point) and two lines have fundamental groups with infinite commutants. In order to prove this statement and make it more precise, we compute the so called Alexander modules of the groups, see [16].

3.10.1. Definition. Let G be a group, and let $G' = [G, G]$ be its commutant. The *Alexander module* of G is the abelian group $G'/[G', G']$ regarded as a $\mathbb{Z}[G/G']$ -module *via* the conjugation action $(a, x) \mapsto a^{-1}xa$, $a \in G/G'$, $x \in G'/[G', G']$.

Abbreviate $\pi'_1 = [\pi_1, \pi_1]$ and denote the Alexander module of π_1 by A .

The sextics in question are represented by the skeletons shown in Figures 8(c)–(e), (p), and (q), and the defining relations for π_1 are (3.1.1) and (3.6.1). (As usual, if a $\tilde{\mathbf{D}}$ type fiber is present, the corresponding relation in (3.6.1) should be ignored.) The abelianization $\pi_1/\pi'_1 = \mathbb{Z} \oplus \mathbb{Z}$ is generated by the images s, t of α_1, α_2 , respectively, and the group ring $\mathbb{Z}[\pi_1/\pi'_1]$ can be identified with the ring $\Lambda := \mathbb{Z}[s, s^{-1}, t, t^{-1}]$ of Laurent polynomials in s, t .

Each skeleton in question has a bigonal region not containing a $\tilde{\mathbf{D}}$ -type fiber, and we choose the initial vertex v_0 so that this region is R_0 (see the grey vertex in the figures; note that, for Figures 8(p) and (q), this choice of v_0 differs from that used in Subsection 3.9). Then $[\alpha_1, \alpha_2] = 1$ and, using the Reidemeister–Schreier method (see, *e.g.*, [15]), one can see that A is generated over Λ by a single element $a := \alpha_1^4 \alpha_2 \alpha_3$. The relation at infinity (3.1.1) transforms into $(s^{-4} + s^{-8})a = (s^{-3} + s^{-6} + s^{-9})a$, or

$$(3.10.2) \quad Q_\infty(s)a = 0, \quad \text{where} \quad Q_\infty(s) := (s^2 - s + 1)(s^4 - s^2 + 1).$$

Alternatively, (3.10.2) can be rewritten in the form $(s - 1)s(s^4 + s + 1)a = -a$, which means that the multiplication by $(s - 1)$ is invertible in A . For this reason, we cancel the factor $(s - 1)$ in all other relations.

For an integer $m \geq 0$, denote $P_m(x) = (x^m - 1)/(x - 1)$. (In particular, $P_0 \equiv 0$ and $P_1 \equiv 1$.) Observe that, for curves with two linear components, all integers n_i in (3.6.1) are even, see Table 2. A relation $\{\sigma_2^i(\alpha_2), \alpha_1\}_{2r} = 1$ results in the following relation for A :

$$\begin{cases} (t - 1)P_r(st)P_j(s^4)a = 0, & \text{if } i = 2j \text{ is even, or} \\ P_r(s^3t)[(1 - s^4t)P_j(s^4) + s^{4j}t]a = 0, & \text{if } i = 2j - 1 \text{ is odd.} \end{cases}$$

(Recall that we cancel all factors $(s - 1)$.) Using the values of (n_0, \dots, n_5) listed in Table 2, one arrives at the following Alexander modules.

- (1) $\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_3 \oplus 3\mathbf{A}_1$ (no. 22'): $A = \mathbb{Z}[s]/Q_\infty(s)$ and $ta = a$.
- (2) $\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_1$ (no. 23'): $A = \mathbb{Z}[s]/Q_\infty(s)$ and $ta = -s^{-3}a$.

- (3) $(\mathbf{E}_6 \oplus 2\mathbf{A}_5) \oplus 3\mathbf{A}_1$ (no. 24'): $A = \mathbb{Z}[s]/(s^2 - s + 1)$ and $ta = (1 + s^{-4})a$.
(4) $\mathbf{E}_6 \oplus \mathbf{D}_6 \oplus 2\mathbf{A}_3 \oplus \mathbf{A}_1$ (no. 36'): $A = \mathbb{Z}[s]/(s^2 - s + 1)$ and $ta = -s^{-3}a$.
(5) $\mathbf{E}_6 \oplus \mathbf{D}_{10} \oplus 3\mathbf{A}_1$ (no. 37'): $A = \mathbb{Z}[s]/Q_\infty(s)$ and $ta = a$.

(Note that $Q_\infty \mid (s^{12} - 1)$; hence s is invertible in $\mathbb{Z}[s]/Q_\infty$ and one does not need to consider Laurent polynomials explicitly.)

3.10.3. Remark. In Items (1) and (5) above, one has $n_2 = 2$, hence $(t - 1)a = 0$. In Items (2) and (4), one has $n_1 = 4$, hence $(1 + s^3t)a = 0$. In each case, t is a Laurent polynomial in s and one can represent A as a quotient of the Λ -module $\mathbb{Z}[s]/Q_\infty$ (with an appropriate action of t); then, in most cases, all extra relations follow from relation at infinity (3.10.2).

Denote $Q(s) = s^2 - s + 1$ and $R(s) = s^4 - s^2 + 1$, so that $Q_\infty(s) = Q(s)R(s)$. In Item (4), in addition to (3.10.2), one has the relation $Q(s)S_1(s)a = 0$, where $S_1(s) := (s + 1)^2$. Since $R(s) - (s - 1)^2S_1(s) = s^2$ is an invertible element, the relation ideal (in $\mathbb{Z}[s]$) is generated by a single element $s^2 - s + 1$.

In Item (3), the additional relations are

$$P_3(s^3t)a = (t - 1)P_3(st)a = (1 - s^4t + s^4)a = 0.$$

From the last relation, one obtains $ta = (1 + s^{-4})a$. Hence, again t is a polynomial in s and, substituting $t = 1 + s^{-4}$ to the first two relations, one has

$$(s^2 - s + 1)S_1(s)a = (s^2 - s + 1)S_2(s)a = 0,$$

where

$$S_1(s) := s^6 + s^5 + 2s^2 + 2s + 1, \quad S_2(s) := s^6 + 2s^5 + 2s^4 + s + 1.$$

One can easily check that

$$s(s^3 - s - 2)(s^4 + s^3 + s^2 - s + 1)R(s) - s^2(s^4 + s^3 + s^2 - s + 1)S_1(s) \\ + (s^5 + s^4 + s^3 - s^2 + s + 1)S_2(s) = 1.$$

Hence again the relation ideal is generated by a single element $s^2 - s + 1$.

3.10.4. Remark. In Items (1) and (2) above, the fact that the groups are infinite can be explained similar to Remark 3.9.2: each curve splits into an irreducible quartic B_4 and a pair of lines, one of them either being double tangent to B_4 or having a single point of 4-fold intersection with B_4 .

4. PERTURBATIONS

We fix a maximizing plane sextic B satisfying condition (*) in the introduction and consider a perturbation B' of B . Throughout this section, \bar{B} stands for the maximal trigonal curve corresponding to B via Proposition 2.1.1.

4.1. Perturbations of the type \mathbf{E}_6 point P . Let U be a Milnor ball about the distinguished type \mathbf{E}_6 singular point P of B . The group $\pi_1(U \setminus B)$ is generated by three elements $\beta_1, \beta_2, \beta_3$ subject to the relations

$$(4.1.1) \quad \beta_3 = (\beta_1\beta_2\beta_3)\beta_2(\beta_1\beta_2\beta_3)^{-1}, \quad \beta_2 = (\beta_1\beta_2\beta_3)\beta_1(\beta_1\beta_2\beta_3)^{-1}.$$

According to [11], the inclusion homomorphism $\pi_1(U \setminus B) \rightarrow \pi_1(\mathbb{P}^2 \setminus B)$ is given by

$$\beta_1 \mapsto \rho \alpha_1 \rho^{-1}, \quad \beta_2 \mapsto \alpha_1, \quad \beta_3 \mapsto \rho^{-1} \alpha_1 \rho,$$

where $\{\alpha_1, \alpha_2, \alpha_3\}$ is any basis for $\pi_1(\mathbb{P}^2 \setminus B)$ as in Subsection 3.1. Note that a type \mathbf{E}_6 singularity has an order 3 automorphism inducing the automorphism

$$\beta_1 \mapsto (\beta_1 \beta_2) \beta_3 (\beta_1 \beta_2)^{-1}, \quad \beta_2 \mapsto \beta_1, \quad \beta_3 \mapsto \beta_2$$

of the local fundamental group. However, on the image in $\pi_1(\mathbb{P}^2 \setminus B)$, this latter automorphism is inner, *viz.* it is induced by the conjugation $g \mapsto \rho g \rho^{-1}$. (For proof, one needs to use the commutativity relation $[\alpha_1, (\alpha_2 \alpha_3)^3] = 1$, which follows from (3.1.1).) Hence, the local automorphisms at P can be ignored when studying the extra relations in $\pi_1(\mathbb{P}^2 \setminus B')$ resulting from a perturbation $B \rightarrow B'$.

The fundamental groups $\pi_1(U \setminus B')$ of small perturbations $B \rightarrow B'$ are found in [4]. They are as follows:

- (1) $\mathbf{E}_6 \rightarrow 2\mathbf{A}_2 \oplus \mathbf{A}_1$: one has $\pi_1(U \setminus B) = \mathbb{B}_4$, the additional relations being $[\beta_1, \beta_3] = \{\beta_1, \beta_2\}_3 = \{\beta_2, \beta_3\}_3 = 1$;
- (2) $\mathbf{E}_6 \rightarrow 2\mathbf{A}_2$ (as a further perturbation of (1)) and $\mathbf{E}_6 \rightarrow \mathbf{A}_5$: one has $\pi_1(U \setminus B) = \mathbb{B}_3$, the additional relations being $\beta_1 = \beta_3$ and $\{\beta_1, \beta_2\}_3 = 1$;
- (3) all others: one has $\pi_1(U \setminus B) = \mathbb{Z}$ and the relations are $\beta_1 = \beta_2 = \beta_3$.

Note that, in the presence of (4.1.1), the additional relations in Items (1) and (2) follow from the first relation, *viz.* $[\beta_1, \beta_3] = 1$ and $\beta_1 = \beta_3$, respectively. The first two perturbations are of *torus type*, *i.e.*, they preserve torus structures of B ; the other perturbations destroy any torus structure with respect to which P is an inner singularity.

Combining (1)–(3) above with the inclusion homomorphism, we obtain the following extra relations for the perturbed group $\pi_1(\mathbb{P}^2 \setminus B')$:

- (1) $\mathbf{E}_6 \rightarrow 2\mathbf{A}_2 \oplus \mathbf{A}_1$: the extra relation is $[\alpha_1, \rho^{-2} \alpha_1 \rho^2] = 1$;
- (2) $\mathbf{E}_6 \rightarrow 2\mathbf{A}_2$ and $\mathbf{E}_6 \rightarrow \mathbf{A}_5$: the extra relation is $[\alpha_1, \rho^2] = 1$;
- (3) all others: the extra relation is $[\alpha_1, \rho] = 1$.

As a consequence, if P is perturbed as in (3), one has $[\alpha_1, \alpha_2 \alpha_3] = 1$ and (3.1.1) becomes

$$(4.1.2) \quad \alpha_1^4 \alpha_2 \alpha_3 = 1.$$

In particular, π_1 is generated by α_1 and α_2 (or by α_1 and α_3) in this case.

4.1.3. Lemma. *Assume that the distinguished hexagon \bar{H} of Sk is genuine, one of the regions R_i , see Subsection 3.6, is a bigon, and the point P is perturbed as in (3) above. Then the group $\pi_1(\mathbb{P}^2 \setminus B')$ is abelian.*

Proof. Taking R_i for R_0 , one concludes that $[\alpha_1, \alpha_2] = 1$; together with (4.1.2), this relation implies that the group is abelian. \square

4.1.4. Remark. Note that almost all skeletons with a genuine hexagon have a bigonal region; exceptions are Figure 7(a), (c) and Figure 8(j).

4.2. Perturbations of A type points. Extend the skeleton Sk of B to a dessin by inserting a \circ -vertex at the middle of each edge, inserting a \times -vertex v_R at the center of each region R , and connecting v_R to the vertices in the boundary ∂R by appropriate edges, see [10]. Let Q be a type \mathbf{A}_p singular point of B ; it is located in a type $\tilde{\mathbf{A}}_p$ singular fiber F of \tilde{B} at the center of a certain $(p+1)$ -gonal region R of Sk . If Q is perturbed, F splits into singular fibers of types $\tilde{\mathbf{A}}_{s_i}$ ($\tilde{\mathbf{A}}_0^*$ if $s_i = 0$), $i = 1, \dots, r$, with $\sum(s_i + 1) = p + 1$. Geometrically, the $2(p+1)$ -valent \times -vertex v_R splits into several \times -vertices of valencies $2(s_i + 1)$.

Assume that the braid relation in $\pi_1(\mathbb{P}^2 \setminus B)$ resulting from the region R above is of the form $\{\delta_1, \delta_2\}_{p+1} = 1$, where δ_1, δ_2 are certain words in $\alpha_1, \alpha_2, \alpha_3$, cf. (3.3.2), (3.4.1), (3.6.1). (In other words, δ_1, δ_2 are the first two elements of a canonical basis over a vertex $u \in \partial R$ defined by a marking at u , see [10], with respect to which e_1 and e_2 belong to ∂R .)

4.2.1. Lemma. *In the notation above, the relation for the new group $\pi_1(\mathbb{P}^2 \setminus B')$ resulting from R is $\{\delta_1, \delta_2\}_s = 1$, where $s = \text{g.c.d.}(s_i + 1)$, $i = 1, \dots, r$.*

Proof. The statement follows from the description of the braid monodromy found in [10] and from (3.1.4). \square

4.2.2. Corollary. *If Q is perturbed, the relation $\{\delta_1, \delta_2\}_{p+1} = 1$ changes to the relation $\{\delta_1, \delta_2\}_s = 1$, where $s < p + 1$ is a divisor of $(p + 1)$. In particular, one has $s = 1$ if $(p + 1)$ is a prime. \square*

4.2.3. Corollary. *If Q is a point of intersection of two components of B (hence p is odd) and the perturbation is to be irreducible in a Milnor ball about Q , then s in Corollary 4.2.2 is an odd divisor of $(p + 1)$. \square*

4.2.4. Corollary. *If B is of torus type, Q is an inner singularity (hence $p + 1 = 0 \pmod{3}$), and the torus structure is to be destroyed, then s in Corollary 4.2.2 is not divisible by 3. \square*

4.3. Perturbations of D type points. Let Q be a type \mathbf{D}_p , $p \geq 5$, singular point of B ; it is located in a type $\tilde{\mathbf{D}}_p$ singular fiber F of \tilde{B} at the center of a $(p-4)$ -gonal region R of Sk .

4.3.1. Lemma. *If a perturbation $B \rightarrow B'$ is irreducible in a Milnor ball U_Q about Q , the group $\pi_1(\mathbb{P}^2 \setminus B')$ is abelian.*

Proof. Under the assumptions, the group $\pi_1(U_Q \setminus B')$ is abelian, see [12]; on the other hand, the inclusion homomorphism $\pi_1(U_Q \setminus B') \rightarrow \pi_1(\mathbb{P}^2 \setminus B')$ is onto, as Q is a triple point of a trigonal curve. \square

Let δ_1, δ_2 be the first two elements of a canonical basis over a vertex $u \in \partial R$ defined by a marking at u , see [10], with respect to which e_1 and e_2 belong to ∂R . (In other words, if R contained an $\tilde{\mathbf{A}}$ type fiber, the resulting braid relation would be $\{\delta_1, \delta_2\}_{p-4} = 1$, cf. (3.3.2), (3.4.1), (3.6.1).)

4.3.2. Lemma. *Assume that a proper perturbation $B \rightarrow B'$ is still reducible in a Milnor ball U_Q about Q . Then, in the notation above, the new group $\pi_1(\mathbb{P}^2 \setminus B)$ has an extra relation $\{\delta_1, \delta_2\}_s = 1$ for some integer s , $1 \leq s \leq p - 2$. If $B \cap U_Q$ has three components (p is even) but $B' \cap U_Q$ has two components, then s is odd.*

Proof. The statement follows from the computation of the fundamental group $\pi_1(U_Q \setminus B')$ found in [12]. \square

4.4. Proof of Theorem 1.2.3. We skip sextics of torus type with two or more type \mathbf{E}_6 singular points (nos. 1, 2, 6, and 7 in Table 1; they are considered in details in [4]) and the maximizing sextic of weight 8 (no. 39 in Table 1; considered in [9]).

All other perturbations with nonabelian group $\pi_1 := \pi_1(\mathbb{P}^2 \setminus B)$ can be handled on a case by case basis, using GAP [13] and modifying the presentation for π_1 found in Section 3 according to Subsection 4.1, Corollaries 4.2.2–4.2.4, and Lemmas 4.3.1 and 4.3.2. (Recall that, due to Proposition 5.1.1 in [9], all singular points of B can be perturbed arbitrarily and independently.) A few details are given in 4.4.1–4.4.5 below. In some cases, the same approach shows that any proper perturbation, including reducible ones, that is not of torus type has abelian fundamental group:

- no. 1': all perturbations not of torus type are abelian;
- no. 3': all proper perturbations are abelian, see 4.4.3;
- nos. 6', 15', 16': all proper perturbations are abelian;
- no. 20': all perturbations not of torus type are abelian, see 4.4.4;
- no. 24': all perturbations not of torus type are abelian, see 4.4.5;
- nos. 28', 29': all perturbations not of torus type are abelian;
- nos. 30', 33': all proper perturbations are abelian;
- no. 34': all proper perturbations except $\mathbf{E}_6 \rightarrow 2\mathbf{A}_2 \oplus \mathbf{A}_1$ are abelian; for the (reducible) nonabelian one, the perturbation epimorphism appears to be an isomorphism.

4.4.1. The set of singularities $2\mathbf{E}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$ (nos. 4 and 5). The two curves are equivalent, see Remark 2.6.1, and it is easier to use no. 5 and its model given in Figure 11(a)–2, see Subsection 3.3.

4.4.2. The sets of singularities $(\mathbf{E}_6 \oplus \mathbf{A}_{11}) \oplus \mathbf{A}_2$, $(\mathbf{E}_6 \oplus \mathbf{A}_8 \oplus \mathbf{A}_2) \oplus \mathbf{A}_3$, and $(\mathbf{E}_6 \oplus \mathbf{A}_8 \oplus \mathbf{A}_2) \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$ (nos. 12, 13, 18, and 40). One has $\pi_1 = \mathbb{B}_3$, and the statement on the perturbations follows from Lemma 3.12 in [8].

4.4.3. The set of singularities $\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$ (no. 3'). The \mathbf{A}_1 type point is not involved in the computation of the original group π_1 , see Subsection 3.4. Using Corollary 4.2.2 and computing the braid monodromy as explained in [10], one can easily see that the additional relation resulting from perturbing this point is $(\alpha_2^{-1}\alpha_1\alpha_2)\alpha_3(\alpha_2^{-1}\alpha_1\alpha_2)^{-1} = \rho^{-1}\alpha_2\rho$.

4.4.4. The set of singularities $(\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_3 \oplus \mathbf{A}_1$ (no. 20'). The perturbations of the distinguished type \mathbf{E}_6 point P are covered by Lemma 4.1.3. If a type \mathbf{A}_5 point Q is perturbed, one can number the regions R_i , see Subsection 3.6, so that R_0 is a bigon and Q is over R_1 . According to Corollary 4.2.4, the new group π'_1 has relations $[\alpha_1, \alpha_2] = [\alpha_1, \alpha_3] = 1$; then (3.1.1) turns into (4.1.2) and π'_1 is generated by two commuting elements α_1, α_2 .

Suppose that a type \mathbf{A}_2 point Q is perturbed; since the skeleton is symmetric, one can assume that Q is over the left triangle in Figure 8(a). Taking this triangle for R_0 , see Subsection 3.6, one obtains the relations $\alpha_1 = \alpha_2$ (Corollary 4.2.2) and $[\alpha_2, \alpha_3\alpha_2\alpha_3^{-1}] = 1$ (from the bigon R_2); hence also $[\alpha_2, \alpha_3^{-1}\alpha_2\alpha_3] = 1$. On the other hand, (3.1.1) implies $[\alpha_2, (\alpha_2\alpha_3)^3] = 1$ (in view of $\alpha_1 = \alpha_2$). Combining these relations, one arrives at $[\alpha_2, \alpha_3^3] = 1$, *i.e.*, α_3^3 is a central element. Now, using Lemma 3.1.3 and GAP [13], one concludes that the perturbed group is abelian.

4.4.5. The set of singularities $(\mathbf{E}_6 \oplus 2\mathbf{A}_5) \oplus 3\mathbf{A}_1$ (no. 24'). Perturbation of the type \mathbf{E}_6 or type \mathbf{A}_5 points are handled similar to 4.4.4. \square

4.5. Proof of Theorem 1.2.6. As in the previous subsection, we ignore the sets of singularities with two or more type \mathbf{E}_6 points, see [4], or the set of singularities $\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 4\mathbf{A}_2$ of weight 8, see [9]. These sets of singularities give rise to the exceptions mentioned in the statement.

Consider a dessin with one of the two fragments shown in Figure 12. We assume that a region of the skeleton may contain several \times -vertices, of valencies $2(s_i + 1)$, $i = 1, \dots, r$, and parameter s shown in the figures stands for the greatest common divisor $\text{g.c.d.}(s_i + 1)$, cf. Lemma 4.2.1.



FIGURE 12. Two special fragments

4.5.1. Lemma. *Let B' be an irreducible sextic of torus type and with a type \mathbf{E}_6 singular point, and assume that the dessin of the corresponding trigonal curve \bar{B}' has one of the two fragments shown in Figure 12. Then $\pi_1(\mathbb{P}^2 \setminus B') = \bar{\mathbb{B}}_3$.*

Proof. The statement follows immediately from Lemma 4.2.1. In the first case, in an appropriate canonical basis $\{\alpha_1, \alpha_2, \alpha_3\}$ over the \bullet -vertex shown in the figure, one has $\alpha_1 = \alpha_2$ and $\{\alpha_2, \alpha_3\}_3 = 1$. The second case (the right fragment in the figure) is essentially considered in 3.3.4. In both cases, one obtains an epimorphism $\mathbb{B}_3 \twoheadrightarrow \pi_1(\mathbb{P}^2 \setminus B')$, and Lemma 3.3.8 applies. \square

Now, considering sextics of torus type one by one and perturbing their \mathbf{A} type points taking into account Corollaries 4.2.2 and 4.2.3, one finds only three perturbations not covered by Lemma 4.5.1:

- $(\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_4 \rightarrow (\mathbf{E}_6 \oplus 4\mathbf{A}_2) \oplus \mathbf{A}_4$ (no. 9);
- $(\mathbf{E}_6 \oplus \mathbf{A}_8 \oplus \mathbf{A}_2) \oplus \mathbf{A}_2 \oplus \mathbf{A}_1 \rightarrow (\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$ (no. 18);
- $(\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_3 \oplus \mathbf{A}_1 \rightarrow (\mathbf{E}_6 \oplus 4\mathbf{A}_2) \oplus \mathbf{A}_3 \oplus \mathbf{A}_1$ (no. 20').

Using equivalence of dessins, see [10], one can easily see that the curves obtained are deformation equivalent to perturbations of the set of singularities $\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 4\mathbf{A}_2$ (no. 39 in Table 1). Their fundamental groups are found in [9]. Alternatively, the second case (as well as any perturbation of nos. 12, 13, 18, and 40) is covered by Corollary 3.10 in [8], and in the first case one can show that $\pi_1 = \bar{\mathbb{B}}_3$ similar to 3.3.10. (In the last case, the group is different: $\pi_1 = \mathbb{B}_4/\sigma_1^2\sigma_2\sigma_3^2\sigma_2$, see [9], which agrees with the statement.)

It remains to consider perturbations of the distinguished \mathbf{E}_6 type point P . The only ‘new’ (*i.e.*, not considered in [9] or [4]) set of singularities realized by irreducible curves of torus type with $\pi_1 \neq \bar{\mathbb{B}}_3$ is $(\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_4$ (nos. 9 and 41 in the table), and the ‘worst’ perturbation is the one in 4.1(1), resulting in the extra relation $[\alpha_1, \rho^{-2}\alpha_1\rho^2] = 1$. Adding this relation to the presentations of π_1 , see 3.3.9 and 3.6(3), respectively, and using GAP [13] similar to 3.3.10, one finds that the resulting groups are isomorphic to $\bar{\mathbb{B}}_3$. (For no. 41, to make the approach work, one also needs to add the relation $(\alpha_1\alpha_2)^{-1}\alpha_2(\alpha_1\alpha_2) = \alpha_3$, which is known to hold in the group, see Remark 3.6.2.) \square

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DEPARTMENT OF MATHEMATICS, BILKENT UNIVERSITY, 06800 ANKARA, TURKEY
E-mail address: degt@fen.bilkent.edu.tr