CLASSICAL ZARISKI PAIRS

ALEX DEGTYAREV

Abstract. We compute the fundamental groups of all irreducible plane sextics constituting classical Zariski pairs

1. Introduction

A classical Zariski pair is a pair of irreducible plane sextics that share the same combinatorial type of singularities but differ by the Alexander polynomial [10]. The first example of such a pair was constructed by O. Zariski [13]. Then, it was shown in [2] that the curves constituting a classical Zariski pair have simple singularities only and, within each pair, the Alexander polynomial of one of the curves is $t^2 - t + 1$, whereas the polynomial of the other curve is trivial. The former curve is called abundant, and the latter non-abundant. The abundant curve is necessarily of torus type, i.e., its equation can be represented in the form $f_2^2 + f_3^3 = 0$, where $f_2$ and $f_3$ are homogeneous polynomials of degree 2 and 3, respectively.

A complete classification of classical Zariski pairs up to equisingular deformation was recently obtained by A. Özgür [11]. Altogether, there are 51 pairs, one of them being in fact a triple: the non-abundant curves with the set of singularities $E_6 \oplus A_{11} \oplus A_1$ form two distinct complex conjugate deformation families. The purpose of this note is to compute the fundamental groups of (the complements of) the curves constituting classical Zariski pairs. We prove the following theorem.

1.1. Theorem. Within each classical Zariski pair, the fundamental group of the abundant (respectively, non-abundant) curve is $\mathbb{Z}_3/\langle \sigma_1 \sigma_2 \rangle^3$ (respectively, $\mathbb{Z}_6$).

This theorem is proved in Section 4, using the list of [11] and a case by case analysis. In fact, most groups are already known, see [1], [5], [6], [7], and [8], and the few missing curves can be obtained by perturbing the set of singularities $A_{17} \oplus 2A_1$. The construction and the computation of the fundamental group are found in Sections 2 (the non-abundant curves) and 3 (the abundant curves).

2. The curve not of torus type

2.1. Up to projective transformation, there is a unique curve $C \subset \mathbb{P}^2$ with the set of singularities $A_{17} \oplus 2A_1$ and not of torus type, see [12]; its transcendental lattice is $[2, 10]$. After nine blow-ups, the curve transforms to the union of two of the

2000 Mathematics Subject Classification. Primary: 14H45; Secondary: 14H30, 14H50.

Key words and phrases. Plane sextic, Zariski pair, torus type, fundamental group, elliptic surface.

Typeset by Ams-TEX
three type $\tilde{A}_8^*$ fibers in a Jacobian rational elliptic surface with the combinatorial type of singular fibers $\tilde{A}_8 \oplus 3\tilde{A}_0^*$. For the equation, consider the pencil of cubics given by
\[ f_b(x, y) := b(-x^2 - xy^2 + y) + (x^3 - xy + y^3) = 0, \quad b \in \mathbb{P}^1, \]
and take two fibers corresponding to two distinct roots of $b^3 = 1/27$. (All three roots give rise to nodal cubics, which are the three type $\tilde{A}_8^*$ fibers in the elliptic pencil above. The curve corresponding to $b = \epsilon/3$, $\epsilon^3 = 1$, has a node at $x = (2/5)\epsilon^{-1}$, $y = (1/5)\epsilon$. The type $\tilde{A}_8$ fiber blows down to the nodal cubic \{\(f_0 = 0\)\}.

2.2. Lemma. For the curve $C$ as in 2.1 one has
\[ \pi_1(\mathbb{P}^2 \setminus C) = \langle p, \gamma \mid p^9 = 1, \gamma^{-1}p\gamma = p^4 \rangle. \]

Proof. Consider the trigonal curve $\tilde{B} \subset \Sigma_2$ with a type $A_8$ singular point. Its skeleton $\operatorname{Sk}$, see [3], is shown in Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1}
\caption{The skeleton $\operatorname{Sk}$ of $\tilde{B}$}
\end{figure}

Let $F_1, F_\pm$ be the type $\tilde{A}_8^*$ singular fibers of $\tilde{B}$, and let $F_\infty$ be the type $\tilde{A}_8$ fiber. (Recall that $F_1, F_\pm$ are located inside the small loops in Figure 1 whereas $F_\infty$ is inside the outer region.) Consider the minimal resolution of the double covering $\tilde{X} \to \Sigma_2$ ramified at $\tilde{B}$ and the exceptional section $E \subset \Sigma_2$, and denote by tildes the pull-backs of the fibers in $\tilde{X}$.

Consider the nonsingular fiber $F$ over the $\bullet$-vertex $v$ of $\operatorname{Sk}$ next to $F_1$ (shown in grey in Figure 1), denote $\pi_F := \pi_1(F \setminus (\tilde{B} \cup E))$, and pick a canonical basis \{\(\alpha_1, \alpha_2, \alpha_3\)\} for $\pi_F$ defined by the marking of $\operatorname{Sk}$ at $v$ shown in Figure 1 see [3]. Then the fundamental group $\tilde{\pi}_F := \pi_1(\tilde{F} \setminus E)$ of the punctured torus $\tilde{F} \setminus E$ is obtained from $\pi_F$ by adding the relations $\alpha_1^3 = \alpha_2^3 = \alpha_3^3 = 1$ and passing to the kernel of the homomorphism $\pi_F \to \mathbb{Z}_2$, $\alpha_1, \alpha_2, \alpha_3 \mapsto 1$. Hence, $\tilde{\pi}_F$ is the free group generated by
\[ p := \alpha_1\alpha_2 = (\alpha_2\alpha_1)^{-1} \quad \text{and} \quad q := (\alpha_3\alpha_2) = (\alpha_2\alpha_3)^{-1}. \]

Start with the group
\[ G_1 = \pi_1(\tilde{X} \setminus (E \cup \tilde{F}_+ \cup \tilde{F}_- \cup \tilde{F}_\infty)) \]
and compute it applying Zariski–van Kampen’s approach [9] to the elliptic pencil on $\tilde{X}$. Let $\gamma_1, \gamma_\pm$ be the generators of the free group
\[ \pi_1(\mathbb{P}^1 \setminus (F_1 \cup F_+ \cup F_- \cup F_\infty), F) \]
represented by the shortest loops in $S^3$ starting at $v$ and circumventing the corresponding fibers in the counterclockwise direction. (We identify fibers of the ruling and their projections to the base.) Using a proper section, see [3], one can lift these generators to $\Sigma_2 \backslash (\bar{B} \cup E)$ and to $\bar{X} \backslash E$. Using the same proper section, define the braid monodromies $m_1, m_\pm \in \text{Aut } \pi$ and their lifts $\bar{m}_1, \bar{m}_\pm \in \text{Aut } \bar{\pi}_F$. In this notation, the group $G_1$ has the following presentation, cf. [9]:

$$G_1 = \langle p, q, \gamma_+, \gamma_- \mid p = \bar{m}_1(p), q = \bar{m}_1(q), \gamma_+^{-1} p \gamma_+ = \bar{m}_\pm(p), \gamma_-^{-1} q \gamma_- = \bar{m}_\pm(q) \rangle.$$  

The braid monodromy is computed as explained in [3]; for $B$ it is

$$m_1 = \sigma_2, \quad m_+ = \sigma_1^{-3} \sigma_2 \sigma_1^3, \quad m_- = \sigma_1^{-1} \sigma_2 \sigma_1^{-2} \sigma_1,$$

where $\sigma_1, \sigma_2$ are the Artin generators of $\mathbb{B}_3$ (we assume that the braid group $\mathbb{B}_3$ acts on $\pi_F$ from the left), and in terms of $p$ and $q$ it takes the form

$$\bar{m}_1 : p \mapsto pq, \quad q \mapsto q;$$
$$\bar{m}_+ : p \mapsto pq^3, \quad q \mapsto p^{-4} q^{-1} p q^{-1} p^{-1};$$
$$\bar{m}_- : p \mapsto (pq)^2 (p^2 q)^3 p, \quad q \mapsto p^{-1} q^{-1} (p^{-2} q^{-1})^3 p^{-1} q^{-1} p^{-1}.$$

The very first relation $p = pq$ implies $q = 1$. Hence also $\bar{m}_\pm(q) = 1$ and $p^9 = 1$. Thus, one has

$$(2.3) \quad G_1 = \langle p, \gamma_+, \gamma_- \mid p^9 = 1, \gamma_+^{-1} p \gamma_+ = p^3, \gamma_-^{-1} p \gamma_- = p^7 \rangle.$$  

In order to pass to the group $\pi_1(\mathbb{P}^2 \backslash B)$, we need to patch back in one of the nine irreducible components of the type $\bar{A}_3$ fiber $E_\infty$. (The component to be patched in is the proper transform of the nodal curve $\{f_0(x, y) = 0\}$.) This operation adds to (2.3) an additional relation $[\partial \bar{\Gamma}] = 1$, where $\bar{\Gamma}$ is a small holomorphic disk in $\bar{X}$ transversal to the component in question. Using a proper section again, one can see that in $G_1$ there is a relation $[\partial \bar{\Gamma}]^{-1} p^7 = \gamma_- \gamma_+$, where $p^7$ is merely an element of the group $\bar{\pi}_F$ of the fiber (modulo the relations in $G_1$), which we do not bother to compute. Adding the extra relation $[\partial \bar{\Gamma}] = 1$ to (2.3) and eliminating $\gamma_-$, one arrives at the presentation announced in the statement. (Note that $7 = 4^{-1}$ mod $9$, hence the order of $p$ remains $9$.) □

2.4. **Corollary.** The commutant of the group $\pi_1(\mathbb{P}^2 \backslash C)$ as in Lemma 2.2 is a central subgroup of order $3$.

**Proof.** The commutant is normally generated by the commutator $p^{-1} \gamma_+^{-1} p \gamma_+ = p^3$; it is a central element of order $3$. □

2.5. **Corollary.** For any irreducible perturbation $C'$ of the curve $C$ as in 2.1, one has $\pi_1(\mathbb{P}^2 \backslash C') = \mathbb{Z}_6$.

**Proof.** Any central extension $\{1\} \to \mathbb{Z}_3 \to G \to \mathbb{Z}_6 \to \{1\}$ would be abelian. □
3. The curve of torus type

3.1. Up to projective transformation, there is a unique torus type curve $C \subset \mathbb{P}^2$ with the set of singularities $\mathbb{A}_{17} \oplus 2\mathbb{A}_1$, see [12]; its transcendental lattice is $\left[ \begin{array}{c} 2 \\ 0 \\ 2 \end{array} \right]$. Similar to [21], this curve blows up to the union of the two type $\tilde{\mathbb{A}}_8^2$ fibers in a Jacobian rational elliptic surface with the combinatorial type of singular fibers $E_8 \oplus 2\mathbb{A}_1^2$. The curve can be given by the equation

$$f(x,y) := (y^3 + y^2 + x^2)\left(y^3 + y^2 + x^2 - \frac{4}{27}\right) = 0,$$

and its torus structure is

$$f(x,y) = \left(y^3 + y^2 + x^2 - \frac{2}{27}\right)^2 + \left(\frac{\sqrt{7}}{9}\right)^3.$$

3.2. Lemma. Let $C$ be a curve as in 3.1 and let $U$ be a Milnor ball about the type $\mathbb{A}_{17}$ singular point of $C$. Then the homomorphism $\pi_1(U \setminus C) \to \pi_1(\mathbb{P}^2 \setminus C)$ induced by the inclusion $U \hookrightarrow \mathbb{P}^2$ is surjective.

Proof. In the coordinates $\tilde{y} = y/x$, $\tilde{z} = 1/x$, the curve is given by the equation

$$(\tilde{y}^3 + \tilde{y}^2 \tilde{z} + \tilde{z})\left(\tilde{y}^3 + \tilde{y}^2 \tilde{z} + \tilde{z} - \frac{4}{27}\tilde{z}^3\right) = 0,$$

the type $\mathbb{A}_{17}$ singular point is at the origin, and each component is inflection tangent to the line $\{ \tilde{z} = 0 \}$ at this point. To compute the group, apply Zariski-van Kampen theorem [9] to the vertical pencil $\{ \tilde{z} = \text{const} \}$, choosing for the reference a generic fiber $F = \{ \tilde{z} = \epsilon \}$ close to the origin. On the one hand, one has an epimorphism $\pi_1(F \setminus C) \to \pi_1(\mathbb{P}^2 \setminus C)$. On the other hand, the intersection $C \cap \{ \tilde{z} = 0 \}$ consists of a single 6-fold point; hence, if $\epsilon$ is small enough, all six points of the intersection $C \cap F$ belong to $U$ and the generators of $\pi_1(F \setminus C)$ can be chosen inside $U$. □

3.3. Corollary. Let $C'$ be a perturbation of the curve $C$ as in 3.1 with the set of singularities $\mathbb{A}_{14} \oplus \mathbb{A}_2 \oplus 2\mathbb{A}_1$. Then $\pi_1(\mathbb{P}^2 \setminus C') \cong \mathbb{B}_3/(\sigma_1\sigma_2)^3$.

Proof. Let $U$ be as in Lemma 3.2. Then $\pi_1(U \setminus C') = \mathbb{B}_3$ and, due to the lemma, there is an epimorphism $\mathbb{B}_3 \to \pi_1(\mathbb{P}^2 \setminus C')$. Since $C'$ is necessarily irreducible and of torus type (so that the abelianization of $\pi_1(\mathbb{P}^2 \setminus C')$ is $\mathbb{Z}_6$ and $\pi_1(\mathbb{P}^2 \setminus C')$ factors to $\mathbb{B}_3/(\sigma_1\sigma_2)^3$), the latter epimorphism factors through an isomorphism $\mathbb{B}_3/(\sigma_1\sigma_2)^3 \cong \pi_1(\mathbb{P}^2 \setminus C')$. □

3.4. Remark. The other irreducible perturbations of $C$ that are of torus type are considered elsewhere, see [5]. Their groups are also $\mathbb{B}_3/(\sigma_1\sigma_2)^3$.

4. Proof of Theorem 1.1

4.1. The groups of all but one sextics of torus type occurring in classical Zariski pairs are known, see [5] for a ‘map’ and further references; all groups are $\mathbb{B}_3/(\sigma_1\sigma_2)^3$. The only missing curve has the set of singularities $\mathbb{A}_{14} \oplus \mathbb{A}_2 \oplus 2\mathbb{A}_1$. Such a curve can be obtained by a perturbation from a reducible sextic of torus type with the set of singularities $\mathbb{A}_{17} \oplus 2\mathbb{A}_1$ (see Proposition 5.1.1 in [4]), and its group is given by Corollary 3.3.
4.2. The fundamental groups of most non-abundant sextics appearing in classical Zariski pairs are computed in [5], [6], and [7], with a major contribution from [8]. According to [7], unknown are the groups of the curves with the sets of singularities $A_{17} \oplus A_1$, $A_{14} \oplus A_2 \oplus 2A_1$, $2A_8 \oplus 2A_1$, $2A_8 \oplus A_1$.

The first curve can be obtained by a perturbation from a sextic with a single type $A_{19}$ singular point. According to [1], its group is abelian. The three other curves are perturbations of the curve $C$ constructed in [2] and their groups are abelian due to Corollary 2.5 (Note that the perturbations exist due to Proposition 5.1.1 in [4], and the resulting curves are unique up to equisingular deformation due to [11].) □

4.3. Remark. A curve $C$ as in [2] can also be perturbed to a sextic with the set of singularities $A_{17} \oplus A_1$, but the result is reducible.

References


Department of Mathematics, Bilkent University, 06800 Ankara, Turkey
E-mail address: degt@fen.bilkent.edu.tr