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On the moduli space of real Enriques surfaces

Alexander DEGTYAREV and Viatcheslav KHARLAMOV

A. D.: Steklov Mathematical Institute, St. Petersburg branch, 191011, St. Petersburg, Russia
and Bilkent University, 06533 Bilkent–Ankara, Turkey.
E-mail: degt @ math.ucr.edu, degt @ pdmi.ras.ru;

V. K.: Institut de Recherche Mathématique Avancée, Université Louis Pasteur
et CNRS, 7 rue René-Descartes, 67084, Strasbourg, France.
E-mail: kharlam @ math.u-strasbg.fr.

Abstract. We introduce a new invariant, Pontrjagin-Viro form, of real algebraic surfaces. We evaluate it for real Enriques surfaces with non-negative minimal Euler characteristic of the components of the real part and prove that, when combined with the known topological invariants, it distinguishes the deformation types of such surfaces.

Sur l'espace de modules des surfaces d'Enriques réelles

Résumé. *On introduit un invariant des surfaces algébriques réelles, la forme de Pontryagin-Viro. Le résultat principal affirme qu'une surface d'Enriques réelle est déterminée à déformation près par cette forme combinée avec d'autres invariants topologiques lorsqu'une au moins des composantes de la partie réelle possède une caractéristique d'Euler négative ou nulle.*

Version française abrégée

Une *surface d'Enriques réelle* est une surface d'Enriques complexe E munie d'une involution antiholomorphe notée conj. On dit que deux surfaces d'Enriques réelles ont le même *type de déformation* si elles peuvent être reliées par une famille continue à un paramètre de surfaces d'Enriques réelles. Le type topologique de $E_{\mathbb{R}} = \operatorname{Fix} \operatorname{conj}$, appelé *partie réelle* de E, est invariant par déformation. On définit un autre invariant de déformation, la *décomposition fondamentale* de $E_{\mathbb{R}}$ en deux *moitiés* : deux composantes connexes C_a et C_b appartiennent à la même moitié si un, et donc tout, lacet de E, composé d'un chemin de C_a à C_b et de son conjugué, est contractible. Ainsi, $E_{\mathbb{R}}$ se scinde en deux $E_{\mathbb{R}}^{(1)}$ et $E_{\mathbb{R}}^{(2)}$, appelées moitiés. La classification à homéomorphisme près des surfaces $E_{\mathbb{R}}$ et de leurs décompositions $(E_{\mathbb{R}}; E_{\mathbb{R}}^{(1)}, E_{\mathbb{R}}^{(2)})$ a été commencée par Nikulin [9] et achevée dans [3], [4].

Nous divisons les surfaces d'Enriques réelles en trois groupes : E est dite de *type hyperbolique*, *parabolique* ou *elliptique*, si la caractéristique d'Euler minimale des composantes de $E_{\mathbb{R}}$ est négative, nulle ou positive. Dans cette Note nous présentons la classification à déformation près des surfaces hyperboliques et paraboliques. Elle est obtenue grâce à une version équivariante d'une astuce de

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Donaldson (basée sur une construction de Hitchin) : *La construction équivariante de Donaldson établit une bijection entre les types de déformation des surfaces d'Enriques réelles ayant une moitié marquée non vide et ceux des couples* $(\tilde Y, \tilde B)$, *où* $\tilde Y$ *est une surface rationnelle réelle et* $\tilde B \subset \tilde Y$ *est une courbe lisse réelle telle que* : (1) $\tilde B \in |-2K_{\tilde Y}|$, *(*$K$ *est la classe canonique), (2) la partie réelle de* $\tilde B$ *est vide et* (3) $\tilde B$ *et la partie réelle de* $\tilde Y$ *ne sont pas entrelacées* (au sens du §1).

Pour classer les surfaces maximales, on introduit au § 2 un nouvel invariant appelé *forme de Pontrjagin-Viro*. Rappelons que l'homomorphisme de Viro envoie le terme $\mathcal{F}^2 \subset H_*(E_{\mathbb R};\mathbb Z/2)$ de la filtration de Kalinin dans $H_2(E;\mathbb Z/2)$ (voir [4]). La forme de Pontryagin-Viro, notée P, est obtenue en composant cet homomorphisme avec le carré de Pontryagin $\mathcal P\colon H_2(E;\mathbb Z/2)\to\mathbb Z/4$. L'espace $\mathcal F^2$ contient les classes $\langle C_a - C_b\rangle \in H_0(E_{\mathbb R};\mathbb Z/2)$ réalisées par deux composantes C_a et C_b appartenant à la même moitié de $E_{\mathbb R}$. On définit la *séparation complexe* de $E_{\mathbb R}^{(i)}$, $i=1$ ou 2, en deux *quarts* en disant que deux composantes C_a, C_b sont dans le même quart si $P\langle C_a - C_b\rangle = 0$. On note $E_{\mathbb R}^{(i)} = (1^{er}\text{ quart}) \sqcup (2^e\text{ quart})$.

Une *simplification de Morse* (topologique) de $E_{\mathbb R}$ est une chirurgie élémentaire qui réduit le nombre $\beta_*(E_{\mathbb R})$. Une simplification de Morse est dite *algébrique* si elle se produit dans une famille continue de surfaces d'Enriques réelles. Nous posons $S=S^2$, $S_g = \#_g(S^1\times S^1)$ et $V_q = \#_q\mathbb{R}\mathrm{p}^2$.

Résultat principal. *Toute surface d'Enriques réelle* E *de type hyperbolique ou parabolique est obtenue par une suite de simplifications de Morse algébriques à partir d'une des surfaces dont la liste est donnée dans les Tableaux* I, II *et* III. *Le type de déformation de* E *est déterminé par la topologie de ses moitiés à trois exceptions suivantes près :*

1. $E_{\mathbb R} = V_{10}$: *il y a deux types de déformation selon que* E/conj *est* Spin *ou pas*;
2. $E_{\mathbb R} = 2V_2 \sqcup 4S$ *ou* $E_{\mathbb R} = V_2 \sqcup 2V_1 \sqcup 3S$: *leurs types de déformation sont distingués par leur forme de Pontrjagin-Viro*; *celle-ci est déterminée par la séparation complexe et la valeur* p *de* P *sur la classe caractéristique des composantes* V_2; *les valeurs possibles sont décrites dans le Tableau* II*;*
3. *surfaces paraboliques avec* $E_{\mathbb R}^{(2)} = 4S$: *chaque type topologique correspond à deux types de déformation distingués par la valeur—nulle ou non—de* $[E_{\mathbb R}^{(2)}]$ *dans* $H_2(X';\mathbb Z/2)$, *où* X' *est le revêtement double de* E/conj *ramifié le long de* $E_{\mathbb R}^{(2)}$.

Réciproquement, tout couple $E_{\mathbb R}^{(1)} \sqcup E_{\mathbb R}^{(2)}$ *obtenu par une suite de simplifications de Morse topologiques à partir des couples présentés dans les Tableaux* I–III *peut être réalisé par une surface d'Enriques réelle, à l'exception de deux familles :* $\{\dots\}\sqcup\{5S\}$ *et* $\{3S\sqcup\dots\}\sqcup\{2S\sqcup\dots\}$.

Introduction

A *real Enriques surface* is a complex Enriques surface E equipped with an antiholomorphic involution conj, called *complex conjugation*; the fixed point set $E_{\mathbb R} = \mathrm{Fix}\,\mathrm{conj}$ is called the *real part* of the surface. As is known, the study of a real Enriques surface is equivalent to the study of a real $K3$-surface X (the double covering of E) equipped with a fixed point free holomorphic involution τ (called the *Enriques involution*) commuting with the real structure (see, e.g., [3]).

Two real Enriques surfaces are said to have the same *deformation type* if they can be included into a continuous one-parameter family of real Enriques surfaces, or, equivalently, if they belong to the same connected component of the *moduli space of real Enriques surfaces.* Clearly, the topological type of the real part of a surface is preserved under deformation. (The real part is a closed 2-manifold with finitely many components, each component being either $S=S^2$, or $S_g = \#_g(S^1\times S^1)$, or $V_q = \#_q\mathbb{R}\mathrm{p}^2$.) Another immediate deformation invariant, which we suggest to call the *sign decomposition*, comes from the covering $K3$-surface X: since conj lifts to two real structures $t^{(1)}$, $t^{(2)}$ on X whose real parts $X_{\mathbb R}^{(1)}$, $X_{\mathbb R}^{(2)}$ are disjoint, $E_{\mathbb R}$ naturally splits into disjoint union of two *halves* $E_{\mathbb{R}}^{(i)} = X_{\mathbb{R}}^{(i)}/\tau$, each half being a union of whole components of $E_{\mathbb{R}}$.

The classification of the real parts $E_{\mathbb{R}}$ and decompositions $(E_{\mathbb{R}}; E_{\mathbb{R}}^{(1)}, E_{\mathbb{R}}^{(2)})$ up to homeomorphism was started by Nikulin [9] and completed by us in [3], [4]. For most topological types the decomposition $E_{\mathbb{R}}^{(1)} \sqcup E_{\mathbb{R}}^{(2)}$ is subject to the only restriction that the orientation double covering of each half must be the real part of a real $K3$-surface. However, for the topological types $E_{\mathbb{R}} = kS$ and $E_{\mathbb{R}} = V_{2q} \sqcup kS$ there are additional congruence type prohibitions, see [4].

It turns natural to divide the real Enriques surfaces in three groups: we say that a real Enriques surface E is of *hyperbolic*, *parabolic*, or *elliptic* type if the minimal Euler characteristic of the components of $E_{\mathbb{R}}$ is negative, zero, or positive, respectively. In this paper we classify up to deformation equivalence the surfaces of hyperbolic and parabolic types. In addition to the sign decomposition and its homological properties (i.e., whether the fundamental classes $[E_{\mathbb{R}}]$ and/or $[E_{\mathbb{R}}^{(i)}]$ vanish in E or some auxiliary manifolds) a new invariant, so called *Pontrjagin-Viro form* (see §2) is necessary (and sufficient) to distinguish the M-surfaces, i.e., those with the maximal total $\mathbb{Z}/2$-Betti number $\beta_*(E_{\mathbb{R}}) = 16$. The precise statement is found in §3. (A similar statement is also valid for the surfaces of elliptic type; it will be published elsewhere, cf. Remark in §3.)

It fact, the approach which we use gives as well some more or less explicit geometrical models of real Enriques surfaces and identifies the components of their moduli space with those of some other, more classical and easier to handle, objects (such as real plane quartics or order 6 curves on quadrics in P^3 with conditions on the singularities). It is based on an equivariant version of a trick by Donaldson [2], which transforms a real Enriques surface to a real rational surface with a certain curve on it (see §1; the original construction of [2] is applied to real $K3$-surfaces).

We are grateful to S. Finashin whose question attracted our attention to the Donaldson construction for real $K3$-surfaces.

1. Donaldson's trick

A $K3$-surface X with a Kähler metric has a canonical quaternionic Kähler structure (in [5] this result is deduced from the Calabi conjecture). Moreover, if Ω is the fundamental Kähler form and $\Re\omega$ and $\Im\omega$ are, respectively, the real and imaginary parts of a holomorphic 2-form $\omega \neq 0$, then any nonzero $\mathbb{R}$-linear combination of Ω, $\Re\omega$, and $\Im\omega$ defines a complex structure on X. We normalize Ω by $\Omega^2 = (\Im\omega)^2 (= (\Re\omega)^2)$ and consider the complex structure defined by $\Re\omega$: it is given by $\varsigma = \Omega - \sqrt{-1}\Im\omega$. The surface X equipped with this new structure will be denoted by $\tilde{X}$.

Let c be a real structure on X. Both the Kähler metric and holomorphic form ω can be chosen real (the latter in the sense that $c^*\omega = \overline{\omega}$). Then $c^*\varsigma = -\varsigma$, i.e., c is *holomorphic* on $\tilde{X}$ and $\operatorname{Fix} c$ is a holomorphic curve in $\tilde{X}$. If, further, τ is an Enriques involution on X commuting with c (i.e., c is one of the two lifts $t^{(1)}, t^{(2)}$ of a real structure on X/τ) and the Kähler metric is τ-equivariant, then $\tau^*\varsigma = \overline{\varsigma}$ (as necessary $\tau^*\omega = -\omega$); hence, τ is a real structure on $\tilde{X}$.

In what follows we always assume that $E_{\mathbb{R}}^{(1)} \neq \varnothing$ and $c = t^{(1)}$. Then $\tilde{Y} = \tilde{X}/t^{(1)}$ is a real rational surface (the real structure being induced by τ or $t^{(2)}$) and the projection $\tilde{X} \to \tilde{Y}$ is a real double covering ramified over a nonsingular real curve $\tilde{B} = \operatorname{Fix} t^{(1)} \subset \tilde{Y}$. Clearly, $E_{\mathbb{R}}^{(2)} = \tilde{Y}_{\mathbb{R}}$ and $E_{\mathbb{R}}^{(1)} = \tilde{B}/t^{(2)}$. A real curve $\tilde{B} \subset \tilde{Y}$ with $\tilde{B}_{\mathbb{R}} = \varnothing$ *is not linked* with $\tilde{Y}_{\mathbb{R}}$ if for any path $\gamma\colon [0,1] \to \tilde{Y} \smallsetminus \tilde{B}$ with $\gamma(0), \gamma(1) \in \tilde{Y}_{\mathbb{R}}$ the loop $\gamma^{-1} \cdot \operatorname{conj}_{\tilde{Y}} \gamma$ is $\mathbb{Z}/2$-homologous to zero in $\tilde{Y} \smallsetminus \tilde{B}$.

1.1. Theorem. *The Donaldson construction establishes a one-to-one correspondence between the deformation classes of real Enriques surfaces with distinguished nonempty half (i.e., pairs $(E, E_{\mathbb{R}}^{(1)})$ with $E_{\mathbb{R}}^{(1)} \neq \varnothing$) and deformation classes of pairs $(\tilde{Y}, \tilde{B})$, where $\tilde{Y}$ is a real rational surface and $\tilde{B} \subset \tilde{Y}$ is a nonsingular real curve such that that* (1) $\tilde{B} \in |-2K_{\tilde{Y}}|$ *(K being the canonical class),* (2) *the real point set of $\tilde{B}$ is empty, and* (3) *$\tilde{B}$ is not linked with the real point set $\tilde{Y}_{\mathbb{R}}$ of $\tilde{Y}$.*

(In this statement the first condition on $\tilde{B}$ ensures that the double covering $\tilde{X}$ of $\tilde{Y}$ ramified along $\tilde{B}$ is a $K3$-surface; the two others are equivalent to the requirement that one of the two lifts of the real structure to $\tilde{X}$ have empty fixed point set.)

Remark. Note that the fact that for $K3$-surfaces the classification of real structures coincides with that of holomorphic involutions reversing ω was first observed by Nikulin in [7] (without a geometrical explanation of this phenomenon; the two problems just lead to the same arithmetical question). In [7], [8] Nikulin also started the investigation of the resulting rational surfaces, which generalize to an extent the notion of Del-Pezzo surface. In particular, he showed that the minimal model of such a surface is either $\mathbb{C}P^2$ or the scroll F_n with $n = 0$, 2, 3, or 4.

Our study is based on Theorem 1.1 and the next result, mainly due to Comessatti (see [6]):

1.2. Theorem. *The following is the complete list of minimal over* $\mathbb{R}$ *real rational surfaces*: (1) $\mathbb{R}P^2$; (2) *the four real structures on* $\mathbb{C}P^1 \times \mathbb{C}P^1$; (3) *the scrolls* F_n *with* $n \geqslant 2$ (*there are two real structures for* n *even and one real structure for* n *odd*); (4) *conic bundles with* $2n \geqslant 8$ *singular fibers which all consist of pairs of conjugate imaginary lines*; (5) *Del-Pezzo surfaces of degree* 1 *with the real point set* $V_1 \sqcup 4S$; (6) *Del-Pezzo surfaces of degree* 2 *with the real point set* $3S$ *or* $4S$.

In order to study the real structures of pairs $(\tilde{Y}, \tilde{B})$ we use the anti-canonical (or anti-bicanonical) real model of $\tilde{Y}$. If $\tilde{B}$ has a component of positive genus (the case covered by this report), such a model exists, possibly, after several real blow-downs which can be controlled.

2. Pontrjagin-Viro form on M-surfaces

Let E be an M-surface. Then its Kalinin's spectral sequence degenerates; hence, the Viro homomorphism bv_2 maps the middle term $\mathcal{F}^2$ of Kalinin's filtration onto $H_2(E;\mathbb{Z}/2)$, and one can combine it with the Pontrjagin square $\mathcal{P}\colon H_2(E;\mathbb{Z}/2) \to \mathbb{Z}/4$ and define a quadratic refinement $P = \mathcal{P} \circ \mathrm{bv}_2\colon \mathcal{F}^2 \to \mathbb{Z}/4$ of Kalinin's intersection form. (Details on Kalinin's sequence are found in [4]. Recall that $\mathcal{P}([F]) = [F] \circ [F] + 2\chi(F) \bmod 4$ for an embedded surface F.)

If E is an Enriques surface, $\mathcal{F}^2$ is spanned by the fundamental classes $[C_i]$ of the components of $E_{\mathbb{R}}$ and classes of the form $\alpha + \langle C_a - C_b\rangle$, where $\alpha \in H_1(E_{\mathbb{R}};\mathbb{Z}/2)$, $\langle C_a - C_b\rangle$ is the 0-dimensional class realized by two components C_a, C_b, and either $\alpha^2 = 0$ and C_a, C_b belong to the same half of $E_{\mathbb{R}}$, or $\alpha^2 = 1$ and C_a, C_b belong to distinct halves. The restriction of P to $\{\sum\langle C_a - C_b\rangle \mid C_a, C_b \in E_{\mathbb{R}}^{(i)}\}$ is a homomorphism (with values in $\mathbb{Z}/2 \overset{\cdot 2}{\hookrightarrow} \mathbb{Z}/4$); hence, it divides $E_{\mathbb{R}}^{(i)}$ into two 'quarters': two components C_a, C_b belong to the same quarter iff $P\langle C_a - C_b\rangle = 0$. Following G. Mikhalkin we call this subdivision the *complex separation* of $E_{\mathbb{R}}$. To indicate the splitting of a half into quarters we use the notation $E_{\mathbb{R}}^{(i)} = (\text{quarter } 1) \sqcup (\text{quarter } 2)$.

Generalizations of this construction and its relation to other structures known in topology of real algebraic surfaces (such as Mikhalkin's complex separation, Guillou-Marin-Rokhlin form on characteristic surfaces in E/conj, and quadratic forms constructed by Viro) are discussed in [1].

3. Main result

A (topological) *Morse simplification* of the real part of a real Enriques surface E is a Morse surgery reducing the total $\mathbb{Z}/2$-Betti number $\beta_*(E_{\mathbb{R}})$. A simplification is called *algebraic* if it may occur in a continuous family of real Enriques surfaces. A surface E (or a half $E_{\mathbb{R}}^{(i)}$) is said to be of *type* I if $[E_{\mathbb{R}}]$ (respectively, $[E_{\mathbb{R}}^{(i)}]$) equals 0 or $w_2(E)$ in $H_2(E;\mathbb{Z}/2)$.

3.1. Main Theorem. *Any real Enriques surface of hyperbolic or parabolic type is obtained by a series of algebraic Morse simplifications from one of the extremal surfaces* (*which are all of type* I) *listed in Tables* I, II, *and* III (*where a* * *indicates that both the halves are of type* I.) *With few exceptions the deformation type of a surface is determined by its sign decomposition. The exceptions are*:

(1) $E_{\mathbb{R}} = V_{10}$: *there are two deformation types which differ by whether* E/conj *is* Spin *or not*;

(2) $E_{\mathbb{R}} = 2V_2 \sqcup 4S$ *or* $E_{\mathbb{R}} = V_2 \sqcup 2V_1 \sqcup 3S$: *the deformation types are distinguished by the Pontrjagin-Viro form; the latter can be recovered from the complex separation and the value* $p = P(\alpha)$ *on the characteristic class of the components* V_2, *see Table* II (*if there are two such components, the values of* $P(\alpha)$ *on them coincide*);

(3) *surfaces of parabolic type with* $E_{\mathbb{R}}^{(2)} = 4S$: *each topological type corresponds to two deformation types which differ by whether* $[E_{\mathbb{R}}^{(2)}]$ *vanishes in* $H_2(X/t^{(1)}; \mathbb{Z}/2)$ *or not.*

Conversely, with the exception of $\{\ldots\} \sqcup \{5S\}$ *and* $\{3S \sqcup \ldots\} \sqcup \{2S \sqcup \ldots\}$ *any pair* $E_{\mathbb{R}}^{(1)} \sqcup E_{\mathbb{R}}^{(2)}$ *obtained by a series of topological Morse simplifications from one of those listed in Tables* I–III *can be realized as the real part of a real Enriques surface.*

Remark. Mention another approach to the problem (cf. [10]): the assignment $\mathrm{conj} \mapsto (t_*^{(1)}, t_*^{(2)})$ establishes a one-to-one correspondence between the deformation classes of real Enriques surfaces and isomorphism classes of commuting pairs of involutions on the lattice $H_2(X) \cong 2E_8 \oplus 3U$ such that the eigenspaces $L_{\epsilon^{(1)},\epsilon^{(2)}} = \{x \in L \,|\, t_*^{(i)}x = \epsilon^{(i)}x\}$ have the inertia indices $\sigma_+ L_{++} = 0$, $\sigma_+ L_{+-} = \sigma_+ L_{-+} = \sigma_+ L_{--} = 1$ and the eigenlattice $\{x \in L \,|\, t_*^{(1)} t_*^{(2)} x = x\}$ is the double of $E_8 \oplus U$.

This approach covers all real Enriques surfaces (and we did use it to classify the surfaces of elliptic type). However, we do not know how to deduce Theorem 3.1 from this calculation: the relation between the topological invariants and the lattice action is still not always clear.

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Table I

Extremal hyperbolic types

$\beta_* = 16,\ \chi = -8$	$\beta_* = 16,\ \chi = 8$	$\beta_* = 12,\ \chi = 0$	
$^*\{V_{11} \sqcup V_1\} \sqcup \{\varnothing\}$	$^*\{V_4 \sqcup S\} \sqcup \{4S\}$	$\{V_6\} \sqcup \{2S\}$	$^*\{V_4 \sqcup S\} \sqcup \{S_1\}$
$^*\{V_{10}\} \sqcup \{S_1\}$	$^*\{V_3 \sqcup V_1 \sqcup 4S\} \sqcup \{\varnothing\}$	$^*\{V_5 \sqcup V_1 \sqcup S\} \sqcup \{\varnothing\}$	$\{V_4 \sqcup S\} \sqcup \{V_2\}$
$\{V_{11}\} \sqcup \{V_1\}$	$\{V_3 \sqcup V_1 \sqcup 3S\} \sqcup \{S\}$	$\{V_5 \sqcup V_1\} \sqcup \{S\}$	$\{V_4\} \sqcup \{V_2 \sqcup S\}$
$\{V_{10}\} \sqcup \{V_2\}$	$\{V_3 \sqcup V_1 \sqcup 2S\} \sqcup \{2S\}$	$\{V_5 \sqcup S\} \sqcup \{V_1\}$	$\{V_3 \sqcup V_1\} \sqcup \{V_2\}$
$\{V_9\} \sqcup \{V_3\}$	$\{V_3 \sqcup V_1 \sqcup S\} \sqcup \{3S\}$	$\{V_5\} \sqcup \{V_1 \sqcup S\}$	$\{V_3\} \sqcup \{V_2 \sqcup V_1\}$
$\{V_8\} \sqcup \{V_4\}$	$\{V_3 \sqcup V_1\} \sqcup \{4S\}$	$^*\{V_4 \sqcup 2V_1\} \sqcup \{\varnothing\}$	$\{V_3 \sqcup S\} \sqcup \{V_3\}$
$\{V_7\} \sqcup \{V_5\}$	$\{V_3 \sqcup 4S\} \sqcup \{V_1\}$	$\{V_4 \sqcup V_1\} \sqcup \{V_1\}$	
$\{V_6\} \sqcup \{V_6\}$	$\{V_3 \sqcup 3S\} \sqcup \{V_1 \sqcup S\}$	$\{V_4\} \sqcup \{2V_1\}$	
	$\{V_3 \sqcup 2S\} \sqcup \{V_1 \sqcup 2S\}$		
	$\{V_3 \sqcup S\} \sqcup \{V_1 \sqcup 3S\}$		
	$\{V_3\} \sqcup \{V_1 \sqcup 4S\}$		

Table II

Extremal parabolic types with $\beta_* = 16$, $\chi = 8$

$E_{\mathbb{R}}^{(1)}$	$E_{\mathbb{R}}^{(2)}$	p
Case $E_{\mathbb{R}} = S_1 \sqcup V_2 \sqcup 4S$		
$^*\ (V_2 \sqcup 2S) \sqcup (2S)$	$(S_1) \sqcup (\varnothing)$	0
Case $E_{\mathbb{R}} = 2V_2 \sqcup 4S$		
$^*\ (V_2) \sqcup (V_2)$	$(2S) \sqcup (2S)$	0
$^*\ (V_2) \sqcup (V_2)$	$(3S) \sqcup (S)$	2
$^*\ (2V_2) \sqcup (\varnothing)$	$(2S) \sqcup (2S)$	0 or 2
$(V_2 \sqcup 2S) \sqcup (2S)$	$(V_2) \sqcup (\varnothing)$	0
$(V_2 \sqcup S) \sqcup (2S)$	$(V_2 \sqcup S) \sqcup (\varnothing)$	2
$(V_2 \sqcup 2S) \sqcup (S)$	$(V_2) \sqcup (S)$	2
$(V_2 \sqcup S) \sqcup (S)$	$(V_2 \sqcup S) \sqcup (S)$	0
Case $E_{\mathbb{R}} = V_2 \sqcup 2V_1 \sqcup 3S$		
$^*\ (V_2 \sqcup 2S) \sqcup (2V_1 \sqcup S)$	$\varnothing$	0
$^*\ (V_2 \sqcup 2V_1 \sqcup S) \sqcup (2S)$	$\varnothing$	0 or 2
$(V_2 \sqcup V_1 \sqcup S) \sqcup (V_1 \sqcup S)$	$(S) \sqcup (\varnothing)$	0 or 2
$(V_2 \sqcup S) \sqcup (2V_1)$	$(S) \sqcup (S)$	0
$(V_2 \sqcup S) \sqcup (2V_1)$	$(2S) \sqcup (\varnothing)$	2
$(V_2 \sqcup 2V_1) \sqcup (S)$	$(S) \sqcup (S)$	0 or 2
$(V_2 \sqcup V_1) \sqcup (V_1)$	$(2S) \sqcup (S)$	0 or 2
$(V_2 \sqcup 2S) \sqcup (V_1 \sqcup S)$	$(V_1) \sqcup (\varnothing)$	0
$(V_2 \sqcup V_1 \sqcup S) \sqcup (2S)$	$(V_1) \sqcup (\varnothing)$	0 or 2
Case $E_{\mathbb{R}} = V_2 \sqcup 2V_1 \sqcup 3S$ (continued)		
$(V_2 \sqcup S) \sqcup (V_1 \sqcup S)$	$(V_1) \sqcup (S)$	0
$(V_2 \sqcup S) \sqcup (V_1 \sqcup S)$	$(V_1 \sqcup S) \sqcup (\varnothing)$	2
$(V_2 \sqcup V_1 \sqcup S) \sqcup (S)$	$(V_1) \sqcup (S)$	0 or 2
$(V_2 \sqcup S) \sqcup (V_1)$	$(V_1 \sqcup S) \sqcup (S)$	0
$(V_2 \sqcup S) \sqcup (V_1)$	$(V_1) \sqcup (2S)$	2
$(V_2 \sqcup V_1) \sqcup (S)$	$(V_1 \sqcup S) \sqcup (S)$	0 or 2
$(V_2) \sqcup (V_1)$	$(V_1 \sqcup S) \sqcup (2S)$	0
$(V_2) \sqcup (V_1)$	$(V_1 \sqcup 2S) \sqcup (S)$	2
$(V_2 \sqcup V_1) \sqcup (\varnothing)$	$(V_1 \sqcup S) \sqcup (2S)$	0 or 2
$(V_2 \sqcup S) \sqcup (2S)$	$(V_1) \sqcup (V_1)$	0
$(V_2 \sqcup S) \sqcup (2S)$	$(2V_1) \sqcup (\varnothing)$	2
$(V_2 \sqcup 2S) \sqcup (S)$	$(V_1) \sqcup (V_1)$	0
$(V_2 \sqcup S) \sqcup (S)$	$(2V_1) \sqcup (S)$	0
$(V_2 \sqcup S) \sqcup (S)$	$(V_1 \sqcup S) \sqcup (V_1)$	2
$(V_2) \sqcup (S)$	$(V_1 \sqcup S) \sqcup (V_1 \sqcup S)$	0
$(V_2) \sqcup (S)$	$(2V_1 \sqcup S) \sqcup (S)$	2
$(V_2 \sqcup S) \sqcup (\varnothing)$	$(V_1 \sqcup S) \sqcup (V_1 \sqcup S)$	0
$(V_2 \sqcup S) \sqcup (\varnothing)$	$(2V_1) \sqcup (2S)$	2
$(V_2) \sqcup (\varnothing)$	$(2V_1 \sqcup S) \sqcup (2S)$	0
$(V_2) \sqcup (\varnothing)$	$(V_1 \sqcup 2S) \sqcup (V_1 \sqcup S)$	2

Table III

Other special parabolic types

Extremal

$^*\{S_1\} \sqcup \{2V_2\}$	$\beta_* = 12,\ \chi = 0$	
$^*\{S_1\} \sqcup \{S_1\}$	$\beta_* = 8,\ \chi = 0$	
$^*\{S_1\} \sqcup \{4S\}$	$\beta_* = 12,\ \chi = 8$	2 types

Nonextremal

$\{V_2 \sqcup S\} \sqcup \{4S\}$	$\beta_* = 14,\ \chi = 10$	2 types
$\{V_2\} \sqcup \{4S\}$	$\beta_* = 12,\ \chi = 8$	2 types