LINES IN SUPERSINGULAR QUARTICS

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Abstract. We show that the number of lines contained in a supersingular quartic surface is 40 or at most 32, if the characteristic of the field equals 2, and it is 112, 58, or at most 52, if the characteristic equals 3. If the quartic is not supersingular, the number of lines is at most 60 in both cases. We also give a complete classification of large configurations of lines.

1. Introduction

Throughout the paper, unless specified otherwise, X stands for a nonsingular quartic surface in the projective space $\mathbb{P}^3$ over an algebraically closed field $\Bbbk$.

1.1. **Motivation.** A simple dimension count shows that, unlike quadrics or cubics, a generic quartic surface $X \subset \mathbb{P}^3$ contains no straight lines. On the other hand, it has been known since F. Schur [20] that there exists a quartic X_{64} containing 64 lines. B. Segre [21] proved that the number 64 is maximal possible. After a period of oblivion, S. Rams and M. Schütt [16] bridged a gap in Segre's arguments and extended his (correct) bound 64 to any algebraically closed field of characteristic $\operatorname{char}\Bbbk \neq 2, 3$. Since Schur's quartic X_{64} has a nonsingular reduction over such fields, the bound is sharp. If $\operatorname{char}\Bbbk = 3$, the maximal number of lines is 112, see [15]; if $\operatorname{char}\Bbbk = 2$, the maximal number is 60, see Theorem 1.3 below.

At the same time, in recent paper [6], we suggested an alternative approach to Segre's theorem over $\mathbb{C}$, using the theory of $K3$-surfaces and Nikulin's theory of discriminant forms [13]. We reestablished Segre's bound 64, proved that Schur's quartic X_{64} is the only one containing 64 lines (see Corollary 8.9 below for a similar statement over an arbitrary field), and gave a complete classification of all large configurations: up to projective equivalence, there are but ten quartics containing more than 52 lines. (Other results of [6] are the sharp bound 56 for the number of *real* lines in a *real* quartic and the bound 52 for the number of lines defined over $\mathbb{Q}$ in a quartic defined over $\mathbb{Q}$.)

In the present paper, we obtain similar refined results for the cases $\operatorname{char}\Bbbk = 2$ or 3. According to [11], if a quartic X is *not* supersingular, it is subject to the same lattice theoretical restrictions as quartics defined over $\mathbb{C}$. Hence, the list of large configurations found in [6] applies to such quartics as a "bound", with some entries missing over some fields. (An example of such missing entries is Theorem 1.3, which rules out the Schur configuration $\mathbf{X}_{64}$ in characteristics 2 and 3.) Therefore, we concentrate on supersingular surfaces; our principal results, Theorems 1.1 and 1.2,

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show that the configurations of lines realized by such surfaces do differ dramatically from the eight configurations found in [6].

Characteristics 2 and 3 are naturally special for quartics: these primes divide the degrees of the defining polynomial and its derivatives, and it is these (and only these) characteristics where pencils of curves of arithmetic genus 1 —one of the principal tools commonly used in the theory— may become quasi-elliptic. Note though that there also are interesting supersingular quartics over other fields: thus, the quartic in characteristic 7 discussed in Remark 5.2 beats by 10 all other known examples of the so-called triangle free configurations. Phenomena specific to fields of other characteristics will be the subject of a forthcoming paper.

1.2. **Principal results.** The set of lines in a quartic X is denoted by $\operatorname{Fn} X$, and the sublattice spanned by the classes of the lines and plane section is denoted by $\mathcal{F}(X) \subset NS(X)$. We denote by $\sigma := \sigma(X)$ the Artin invariant of a supersingular $K3$-surface X (see Theorem 3.1). An important easily comparable combinatorial invariant of a configuration of lines is its *pencil structure* $\mathfrak{p}$, *i.e.*, the list of the types (p, q) of all pencils $\mathcal{P}(l)$, $l \in \operatorname{Fn} X$ (see §4.2). We use the partition notation, a "factor" $(p, q)^m$ standing for m copies of the type (p, q).

In the statements, we identify "interesting" quartics by the triple $(\mathfrak{p}, \sigma, \operatorname{rk} \mathcal{F})$: these triples suffice to distinguish all examples found in the paper. More details, such as the Gram matrix of $\mathcal{F}(X)$ and coordinates of the lines in the Néron–Severi lattice $NS(X)$, are available from the author in electronic form.

The principal results of the paper are Theorem 1.1 (supersingular quartics in characteristic 2) and Theorem 1.2 (supersingular quartics in characteristic 3). In Theorem 1.3, we reduce Segre's bound for quartics that are not supersingular.

Theorem 1.1 (see §7.5)**.** *Assume that* $\operatorname{char} \Bbbk = 2$ *and* X *is supersingular. Then either* $|\operatorname{Fn} X| = 40$*, and there are at most five configurations*:

(1) $\mathfrak{p} = (2,6)^{40}$, $\sigma(X) = 3$, $\operatorname{rk} \mathcal{F}(X) = 22$,
(2) $\mathfrak{p} = (2,6)^{40}$, $\sigma(X) = 3$, $\operatorname{rk} \mathcal{F}(X) = 21$,
(3) $\mathfrak{p} = (4,0)^4(2,6)^{36}$, $\sigma(X) = 3$, $\operatorname{rk} \mathcal{F}(X) = 22$,
(4) $\mathfrak{p} = (4,0)^8(2,6)^{32}$, $\sigma(X) = 3$, $\operatorname{rk} \mathcal{F}(X) = 20$,
(5) $\mathfrak{p} = (4,0)^{40}$, $\sigma(X) = 3$, $\operatorname{rk} \mathcal{F}(X) = 16$,

or $|\operatorname{Fn} X| \leqslant 32$.

Due to the lack of the existence statement in Theorem 3.3 (*cf.* Remark 3.4 below), we do not assert that the five configurations listed in Theorem 1.1 are realizable: one would have to find explicit defining equations for these surfaces, but this task is beyond the scope of the present paper. Conjecturally, the last bound $|\operatorname{Fn} X| \leqslant 32$ is sharp: there are examples of configurations with 32 lines, but their realizability by smooth supersingular quartics is also open.

Theorem 1.2 (see §6.8)**.** *Assume that* $\operatorname{char} \Bbbk = 3$ *and* X *is supersingular. Then either* $|\operatorname{Fn} X| = 112$*, and* X *is the Fermat quartic*:

(1) $\mathfrak{p} = (10,0)^{112}$, $\sigma(X) = 1$, $\operatorname{rk} \mathcal{F}(X) = 22$,

or $|\operatorname{Fn} X| = 58$*, and there are three configurations*:

(2) $\mathfrak{p} = (10,0)^2(1,9)^{54}(1,0)^2$, $\sigma(X) = 2$, $\operatorname{rk} \mathcal{F}(X) = 22$,
(3) $\mathfrak{p} = (10,0)^1(4,6)^{27}(4,0)^{12}(1,9)^{18}$, $\sigma(X) = 2$, $\operatorname{rk} \mathcal{F}(X) = 22$,
(4) $\mathfrak{p} = (7,0)^2(4,6)^{18}(3,6)^{36}(1,9)^2$, $\sigma(X) = 2$, $\operatorname{rk} \mathcal{F}(X) = 21$,

or $|\operatorname{Fn} X| \leqslant 52$*, and this bound is sharp.*

D. Veniani (private communication) has found explicit defining equations of the three quartics with 58 lines. Alternatively, quartics as in Theorem 1.2(2) and (3) are described in Propositions 8.15 and 8.14, respectively.

In Theorem 1.1(5), the configuration $\operatorname{Fn} X$ constitutes (in the sense described in §2.5 below) the so-called generalized quadrangle $W(3)$. In Theorem 1.2(1), $\operatorname{Fn} X$ constitutes the only generalized quadrangle $\mathrm{GQ}(3,9) \cong Q(5,3)$.

If X is not supersingular, the situation also differs from that in characteristic 0, as some quartics defined over algebraic number fields become singular and/or acquire extra lines when reduced to positive characteristics. We have the following bound; its sharpness is discussed in Remark 8.10.

Theorem 1.3 (see §8.3)**.** *Assume that* $\operatorname{char} \Bbbk = 2$ *or* 3 *and* X *is not supersingular. Then* $|\operatorname{Fn} X| \leqslant 60$*.*

According to [6], there are considerable gaps in the set of values taken by the number of lines in a nonsingular quartic defined over $\mathbb{C}$. We conjecture similar gaps for supersingular quartics in characteristics 2 and 3.

Conjecture 1.4 (see Remark 7.7)**.** Under the assumptions of Theorem 1.1, the number $|\operatorname{Fn} X|$ takes values in the set $\{0, 1, \ldots, 17, 18, 20, 22, 24, 28, 32, 40\}$.

Conjecture 1.5 (see Remark 6.12)**.** Under the assumptions of Theorem 1.2, one has $|\operatorname{Fn} X| = 1 \bmod 3$ whenever $|\operatorname{Fn} X| \geqslant 40$.

1.3. **Contents of the paper.** Sections 2 and 3 are preliminary: we summarize the necessary facts concerning integral lattices, discriminant forms, $K3$-surfaces, and (quasi-)elliptic pencils. In §4, we summarize and extend some intermediate results of [6], introducing the principal technical tools—configurations and pencils. Then, in §5, we treat the so-called triangle free configurations, also following [6]. We prove a (rather week) characteristic independent bound and a few intermediate lemmas that are used later. The principal results of the paper, *viz.* Theorems 1.2 and 1.1, are proved in §6 and §7, where we study in detail pencils in supersingular quartics over fields of characteristic 3 and 2, respectively. Finally, in §8, intuitive geometric arguments are used to rule out Schur's configuration $\mathbf{X}_{64}$ in characteristics 3 and 2 and prove Theorem 1.3; we conclude this section with explicit defining equations of several supersingular quartics in characteristic 3.

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2. Lattices

In this introductory section we recall briefly a few elementary facts concerning integral lattices and their discriminant forms. The principal reference is [13].

2.1. **Finite quadratic forms** (see [12, 13])**.** A *finite quadratic form* is a finite abelian group $\mathcal{L}$ equipped with a map $q\colon \mathcal{L} \to \mathbb{Q}/2\mathbb{Z}$ quadratic in the sense that

$$q(x+y) = q(x) + q(y) + 2b(x,y), \quad q(nx) = n^2 q(x), \quad x, y \in \mathcal{L},\ n \in \mathbb{Z},$$

where $b\colon \mathcal{L} \otimes \mathcal{L} \to \mathbb{Q}/\mathbb{Z}$ is a symmetric bilinear form and $2\colon \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/2\mathbb{Z}$ is the natural isomorphism. We often abbreviate $x^2 := q(x)$ and $x \cdot y := b(x,y)$. Clearly, b is determined by q; the converse holds if and only if $|\mathcal{L}|$ is prime to 2.

There is a direct sum decomposition $\mathcal{L} = \bigoplus_p \mathcal{L}_p$, where $\mathcal{L}_p := \mathcal{L} \otimes \mathbb{Z}_p$ is the p-primary part of $\mathcal{L}$ and p runs over all primes. The *length* $\ell(\mathcal{L})$ is the minimal number of generators of $\mathcal{L}$; we abbreviate $\ell_p(\mathcal{L}) := \ell(\mathcal{L}_p)$. The form on $\mathcal{L}_2$ is called *even* if $x^2 = 0 \bmod \mathbb{Z}$ for each element $x \in \mathcal{L}_2$ of order 2; otherwise, it is called *odd*. There is a unique vector $c \in \mathcal{L}_2/2\mathcal{L}_2$ with the property $x^2 = x \cdot c \bmod \mathbb{Z}$ for each element $x \in \mathcal{L}_2$ or order 2; it is called the *characteristic vector*. The form on $\mathcal{L}_2$ is even if and only if $c = 0$.

A finite quadratic/bilinear form is *nondegenerate* if the *associated map*

$$\mathcal{L} \longrightarrow \operatorname{Hom}(\mathcal{L}, \mathbb{Q}/\mathbb{Z}), \quad x \longmapsto (y \mapsto x \cdot y)$$

is an isomorphism. A nondegenerate finite quadratic form splits into an orthogonal direct sum of cyclic forms $\langle \frac{m}{n} \rangle$, g.c.d.$(m,n) = 1$, $mn = 0 \bmod 2$ (defined on the cyclic group $\mathbb{Z}/m$) and length 2 blocks (on the group $(\mathbb{Z}/n)^2$)

$$\mathcal{U}_n := \begin{pmatrix} 0 & 1/n \\ 1/n & 0 \end{pmatrix}, \quad \mathcal{V}_n := \begin{pmatrix} 2/n & 1/n \\ 1/n & 2/n \end{pmatrix}, \quad \text{where } n = 2^k,\ k \geqslant 1.$$

Given a prime p, the *determinant* $\det_p \mathcal{L}$ of a nondegenerate finite quadratic form is the determinant of the matrix of the form on the p-group $\mathcal{L}_p$ in any minimal basis. According to [12], one has $\det_p \mathcal{L} = u|\mathcal{L}_p|^{-1}$, where $u \in \mathbb{Z}_p^\times$; the unit u is well defined modulo $(\mathbb{Z}_p^\times)^2$ unless $p = 2$ and $\mathcal{L}_2$ is odd; in the latter case, $\det_2 \mathcal{L}$ is well defined modulo the subgroup generated by $(\mathbb{Z}_2^\times)^2$ and 5.

The *Brown invariant* of a nondegenerate finite quadratic form $\mathcal{L}$ is the residue $\operatorname{Br} q = \operatorname{Br} \mathcal{L} \in \mathbb{Z}/8$ defined by the Gauss sum

$$\exp\bigl(\tfrac{1}{4} i\pi \operatorname{Br} \mathcal{L}\bigr) = |\mathcal{L}|^{-\frac{1}{2}} \sum_{x \in \mathcal{L}} \exp\bigl(i\pi x^2\bigr).$$

The Brown invariant is additive: $\operatorname{Br}(\mathcal{L}' \oplus \mathcal{L}'') = \operatorname{Br} \mathcal{L}' + \operatorname{Br} \mathcal{L}''$.

A finite quadratic form q (respectively, bilinear form b) on $\mathcal{L}$ is *null-cobordant* if there exists a q-isotropic (respectively, b-isotropic) subgroup $\mathcal{K} \subset \mathcal{L}$ of maximal order, *i.e.*, such that $|\mathcal{L}| = |\mathcal{K}|^2$ or, equivalently, $\mathcal{K} = \mathcal{K}^\perp$. If a quadratic form q is null-cobordant, then $\operatorname{Br} q = 0$; if q is defined on a 2- or 3-elementary group, the converse also holds. More generally, for any q-isotropic subgroup $\mathcal{K} \subset \mathcal{L}$ one has the identity $\operatorname{Br}(\mathcal{K}^\perp/\mathcal{K}) = \operatorname{Br} \mathcal{L}$ (*cf.* Theorem 2.3 below).

2.2. **Integral lattices** (see [13])**.** An (*integral*) *lattice* is a finitely generated free abelian group L equipped with a symmetric bilinear form $b\colon L \otimes L \to \mathbb{Z}$; usually, we abbreviate $x^2 := b(x,x)$ and $x \cdot y := b(x,y)$. A lattice L is *even* if $x^2 = 0 \bmod 2$ for all $x \in L$; otherwise, L is *odd*. The *determinant* $\det L \in \mathbb{Z}$ is the determinant of the matrix of b in any integral basis. The lattice L is called *nondegenerate* if $\det L \neq 0$; it is called *unimodular* if $\det L = \pm 1$. Equivalently, L is nondegenerate if and only if its *kernel*

$$\ker L = L^\perp := \bigl\{ x \in L \bigm| x \cdot y = 0 \text{ for all } y \in L \bigr\}$$

is trivial. A *characteristic vector* of a unimodular lattice L is a vector $u \in L$ such that $x^2 = x \cdot u \bmod 2$ for all $x \in L$. Such a vector exists and is unique $\bmod 2L$.

The inertia indices $\sigma_\pm$ and signature $\sigma := \sigma_+ - \sigma_-$ of a lattice L are defined as those of $L \otimes \mathbb{Q}$; a nondegenerate lattice L is called *hyperbolic* if $\sigma_+ L = 1$.

Let L be a nondegenerate lattice. Then, we have a canonical inclusion

$$L \subset L^\vee := \operatorname{Hom}(L, \mathbb{Z}) = \bigl\{x \in L \otimes \mathbb{Q} \bigm| x \cdot y \in \mathbb{Z} \text{ for all } y \in L\bigr\}. \tag{2.1}$$

The finite group $\operatorname{discr} L := L^\vee/L$ of order $|\det L|$ is called the *discriminant* (*group*) of L. This group inherits from $L \otimes \mathbb{Q}$ the symmetric bilinear *discriminant form*

$$b\colon (x \bmod L) \otimes (y \bmod L) \mapsto (x \cdot y) \bmod \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$$

and, if L is even, its quadratic extension

$$q\colon (x \bmod L) \mapsto x^2 \bmod 2\mathbb{Z} \in \mathbb{Q}/2\mathbb{Z}.$$

These forms are taken into account whenever we speak about (anti-)isometries of discriminant groups. We abbreviate $\operatorname{discr}_p L := (\operatorname{discr} L) \otimes \mathbb{Z}_p$. A lattice L is said to be *p-elementary* if $\operatorname{discr} L$ is a p-elementary group.

To avoid confusion, we fix the notation:

- L^n, $n \in \mathbb{N}$, is the orthogonal direct sum of n copies of L;
- $L(q)$, $q \in \mathbb{Q}$, is the abelian group L equipped with the symmetric bilinear form $x \otimes y \mapsto q(x \cdot y)$, provided that it is still a lattice;
- $qL \subset L \otimes \mathbb{Q}$, $q \in \mathbb{Q}$, is the subgroup $\{qx \,|\, x \in L\}$, also equipped with the restricted bilinear form; as an abstract lattice, $qL \cong L(q^2)$.

The same notation applies to discriminant forms whenever it makes sense. Note that, if p is a prime, $L(\frac{1}{p})$ is a lattice if and only if $\ell(\operatorname{discr}_p L) = \operatorname{rk} L$; this lattice is even if and only if L is even and either $p \neq 2$ or $\operatorname{discr}_2 L$ is even.

Usually, we do *not* assume isometries bijective; for an isometry $\psi\colon L \to S$, one has $\operatorname{Ker} \psi \subset \ker L$. The group of bijective autoisometries of a lattice L is denoted by $O(L)$. There is a canonical homomorphism $O(L) \to \operatorname{Aut} \operatorname{discr} L$.

A 4-*polarization* of a lattice L is a distinguished vector $h \in L$ of square 4; this vector is usually assumed but not present in the notation. The group of polarized autoisometries is denoted by $O_h(L)$. A *line* in a 4-polarized hyperbolic lattice L is an element of the set

$$\operatorname{Fn} L := \bigl\{a \in L \bigm| a^2 = -2,\ a \cdot h = 1\bigr\}.$$

The set $\operatorname{Fn} L$ is finite; it admits a natural action of $O_h(L)$.

Two lattices L', L'' are said to be in the same *genus* if $L' \otimes \mathbb{R} \cong L'' \otimes \mathbb{R}$ and $L' \otimes \mathbb{Q}_p \cong L'' \otimes \mathbb{Q}_p$ for each prime p. Each genus contains finitely many isomorphism classes. According to [13], the genus of an *even* nondegenerate lattice is determined by its rank, signature, and discriminant form. A realizability criterion is given by the following theorem.

Theorem 2.2 (see [13, Theorem 1.10.1])**.** *A nondegenerate even lattice L with given inertia indices (σ_+, σ_-) and discriminant form $\mathcal{L}$ exists if and only if*

(1) $\ell(\mathcal{L}) \leqslant r := \operatorname{rk} L = \sigma_+ + \sigma_-$,
(2) $\operatorname{Br} \mathcal{L} = \sigma_+ - \sigma_- \bmod 8$ (*van der Blij formula* [24]),

and the following conditions are satisfied:

- $|\mathcal{L}| \det_p \mathcal{L} = (-1)^{\sigma_-} \bmod (\mathbb{Z}_p^\times)^2$ *for any prime $p > 2$ for which $\ell_p(\mathcal{L}) = r$;*
- *either $\ell_2(\mathcal{L}) < r$, or $\mathcal{L}_2$ is odd, or $|\mathcal{L}| \det_2 \mathcal{L} = \pm 1 \bmod (\mathbb{Z}_2^\times)^2$.*

We fix the following notation for a few special lattices:

- $\mathbf{H}_n := \bigoplus_{i=1}^n \mathbb{Z}\mathrm{e}_i$, $\mathrm{e}_i^2 = -1$; once the basis is fixed, we have a distinguished characteristic vector $\bar{\mathrm{e}} := \mathrm{e}_1 + \ldots + \mathrm{e}_n \in \mathbf{H}_n$;
- $\mathbf{U} := \mathbb{Z}u_1 + \mathbb{Z}u_2$, $u_1^2 = u_2^2 = 0$, $u_1 \cdot u_2 = 1$, is the *hyperbolic plane*;
- $\mathbf{A}_n$, $\mathbf{D}_n$, $\mathbf{E}_n$ are the *negative* definite lattices generated by the root systems of the same name, see [3];
- $\mathbf{L} := H_2(X) \cong \mathbf{E}_8^2 \oplus \mathbf{U}^3$ is the intersection form of a $K3$-surface X over $\mathbb{C}$;
- $\mathbf{S}_{p,\sigma} := NS(X)$ is the Néron–Severi lattice of a supersingular $K3$-surface X over a field of characteristic p with Artin invariant $\sigma = 1, \ldots, 10$.

Recall that $\mathbf{A}_n$ can be interpreted as the orthogonal complement $\bar{\mathrm{e}}^\perp \subset \mathbf{H}_{n+1}$ and $\mathbf{D}_n$ is the maximal even sublattice in $\mathbf{H}_n$. The nondegenerate even lattice $\mathbf{S}_{p,\sigma}$ is uniquely determined by the properties $\sigma_+\mathbf{S}_{p,\sigma} = 1$, $\sigma_-\mathbf{S}_{p,\sigma} = 21$, and $\operatorname{discr} \mathbf{S}_{p,\sigma}$ is a p-elementary group of length 2σ, even if $p = 2$. Similarly, $\mathbf{L}$ is the only even unimodular lattice with $\sigma_+\mathbf{L} = 3$ and $\sigma_-\mathbf{L} = 19$.

We also use freely the classification of definite unimodular lattices of small rank found in [4], explaining the extra notation L^+ on the fly: usually, it stands for the only "interesting" unimodular extension of L.

2.3. **Lattice extensions** (see [13]). From now on, unless specified otherwise, all lattices considered are even and nondegenerate. Respectively, q-isotropic subgroups of a finite quadratic form $\mathcal{L}$ are called just isotropic.

An *extension* of a lattice S is any overlattice $L \supset S$. Two extensions $L', L'' \supset S$ are *isomorphic* if there is a bijective isometry $L' \to L''$ identical on S. One can also fix a subgroup $G \subset O(S)$ and speak about G-*isomorphisms* of extensions, *i.e.*, bijective isometries whose restriction to S is in G.

Let $L \supset S$ be a finite index extension. Then we have natural inclusions

$$S \subset L \subset L^\vee \subset S^\vee,$$

cf. (2.1), and, hence, a well defined subgroup $\mathcal{K} := L/S \subset \operatorname{discr} S = S^\vee/S$. This subgroup is isotropic (since L is an even integral lattice); it is called the *kernel* of the extension $L \supset S$. Conversely, if $\mathcal{K} \subset \operatorname{discr} S$ is isotropic, the lattice

$$L := \left\{ x \in S \otimes \mathbb{Q} \;\middle|\; x \bmod L \in \mathcal{K} \right\}$$

is an extension of S. (If $\mathcal{K}$ is b-, but not q-isotropic, then L is odd.) We say that L is the extension of S by $\mathcal{K}$ (or by any collection of vectors $a_1, a_2, \ldots \in S \otimes \mathbb{Q}$ such that $a_i \bmod S$ generate $\mathcal{K}$). Thus, we have the following statement.

Theorem 2.3 (see [13]). *Given a subgroup $G \subset O(S)$, the map*

$$(L \supset S) \longmapsto \mathcal{K} := L/S \subset \operatorname{discr} S$$

is a one-to-one correspondence between the set of G-isomorphism classes of finite index extensions $L \supset S$ and the set of G-orbits of isotropic subgroups $\mathcal{K} \subset \operatorname{discr} S$. Under this correspondence, one has $\operatorname{discr} L = \mathcal{K}^\perp/\mathcal{K}$.

In general, an extension $L \supset S$ can be described by fixing a finite index sublattice $T \subset S_L^\perp$: then L is a finite index extension of $S \oplus T$ and, as such, is determined by an isotropic subgroup

$$\mathcal{K} \subset \operatorname{discr}(S \oplus T) = \operatorname{discr} S \oplus \operatorname{discr} T.$$

This subgroup $\mathcal{K}$ can be regarded as the graph of an anti-isometric additive relation (also known as partially defined multi-valued homomorphism)

$$\psi\colon \operatorname{discr} S \dashrightarrow \operatorname{discr} T;$$

denoting by pr_S, pr_T the projections to the two summands, we have

$$\begin{aligned} \operatorname{Domain}\psi &= \operatorname{pr}_S(\mathcal{K}), & \operatorname{Ker}\psi &= \mathcal{K}\cap\operatorname{discr} S,\\ \operatorname{Im}\psi &= \operatorname{pr}_T(\mathcal{K}), & \operatorname{Indet}\psi &= \mathcal{K}\cap\operatorname{discr} T. \end{aligned}$$

Hence, if $T = S^\perp_L$ is primitive in L, then $\psi\colon \operatorname{pr}_S(\mathcal{K}) \to \operatorname{discr} T$ is a conventional anti-isometry; if S is also primitive, then ψ is injective. With T fixed and $G \subset O(S)$ as above, the G-isomorphism classes of extensions are enumerated by the orbits of the two-sided action of $G \times O(T)$ on the set of anti-isometric additive relations ψ. Note, though, that if we do not insist that T should be primitive, distinct pairs (T, ψ) may give rise to isomorphic extensions.

An important consequence is the following restriction on the genus of T.

Proposition 2.4 (see [13])**.** *If both S and T are primitive in L and $\operatorname{discr}_p L = 0$ for some prime p, then $\psi_p\colon \operatorname{discr}_p S \to \operatorname{discr}_p T$ is a bijective anti-isometry.*

2.4. **Lemmas on discriminant forms.** In this section, we state a few lemmas which would help us identify negative definite lattices.

Lemma 2.5. *Let T be a lattice with 2-elementary group $\operatorname{discr}_2 T$. Then there is a finite index sublattice $T' \subset T$ such that $\operatorname{discr}_p T' = \operatorname{discr}_p T$ for all primes $p \neq 2$ and $\operatorname{discr}_2 T'$ is a 2-elementary group of maximal length: $\ell(\operatorname{discr}_2 T') = \operatorname{rk} T'$.*

Proof. We start with the sublattice $2T$ and extend it to $T_0 \subset T$ *via* the obviously isotropic subgroup $4\operatorname{discr}_2(2T)$; the new discriminant $\mathcal{T} := \operatorname{discr}_2 T_0$ has only 2- and 4-torsion. Such discriminant forms have been studied in [5]. There is a well-defined nondegenerate symmetric bilinear form

$$\circ\colon 2\mathcal{T}\otimes 2\mathcal{T} \longrightarrow \mathbb{Q}/\mathbb{Z}, \quad 2x\otimes 2y \longmapsto 2(x\cdot y),$$

and, given an isotropic subgroup $\mathcal{K} \subset \mathcal{T}$, one has (see [5, 4.2.2])

$$2(\mathcal{K}^\perp/\mathcal{K}) = (\mathcal{K}\cap 2\mathcal{T})^\perp_\circ \big/ \big(\mathcal{K}\cap(\mathcal{K}\cap 2\mathcal{T})^\perp_\circ\big);$$

in particular, $\mathcal{K}^\perp/\mathcal{K}$ is 2-elementary if and only if $(\mathcal{K}\cap 2\mathcal{T})^\perp_\circ$ is $\circ$-isotropic. The kernel of the extension $T \supset T_0$ is $2\mathcal{T}$; hence, a lattice T' as in the statement is obtained by extending T_0 by any maximal $\circ$-isotropic subgroup $\mathcal{K}' \subset 2\mathcal{T}$. Note that we always have the congruence $\ell(2\mathcal{T}) = \operatorname{rk} T - \ell(\operatorname{discr}_2 T) = 0 \bmod 2$ and any nondegenerate symmetric bilinear form on an $\mathbb{F}_2$-vector space of even dimension is null-cobordant (as follows from the classification of such forms). □

The arguments of [5] can easily be extended to finite forms with 3- and 9-torsion only. Given such a form $\mathcal{T}$, we have a well-defined nondegenerate quadratic form

$$q_\circ\colon 3\mathcal{T} \longrightarrow \mathbb{Q}/2\mathbb{Z}, \quad 3x \longmapsto 3x^2.$$

The following statement is immediate.

Lemma 2.6. *Given a quadratic form $\mathcal{T}$ as above and an isotropic subgroup $\mathcal{K} \subset \mathcal{T}$, one has $\ell(\mathcal{K}^\perp/\mathcal{K}) = \ell(\mathcal{T})$ if and only if $\mathcal{K} \subset 3\mathcal{T}$ and $\mathcal{K}$ is $q_\circ$-isotropic.*

Furthermore, for a 3-elementary quadratic form $\mathcal{T}$, using the obvious additivity, one can easily check the congruences

$$\delta := \operatorname{Br}\mathcal{T} - 2\ell(\mathcal{T}) = 0 \bmod 4, \qquad \det\mathcal{T} = (-1)^{\delta/4}\cdot|\mathcal{T}|^{-1}.$$

These congruences apply to $q_\circ$; combining, for a quadratic form $\mathcal{T}$ with 3- and 9-torsion only, we have

$$\delta := \operatorname{Br}\mathcal{T} + \operatorname{Br}q_\circ - 2\ell(\mathcal{T}) = 0 \bmod 4, \qquad \det\mathcal{T} = (-1)^{\delta/4}\cdot|\mathcal{T}|^{-1}. \tag{2.7}$$

(Recall that $\operatorname{Br}q = 0$ for any form q on $\mathbb{Z}/9$.)

Lemma 2.8. *Let T be a negative definite lattice with the following properties*:

- $\operatorname{discr}_2 T$ *is odd* 2*-elementary and* $\operatorname{Br}(\operatorname{discr}_2 T) = \operatorname{rk}T \bmod 8$*,*
- $\operatorname{discr}_3 T$ *is* 3*-elementary, and*
- $\operatorname{discr}_p T = 0$ *for all primes* $p > 3$*.*

Then T contains a finite index sublattice $T' \cong \bar{T}(6)$, where $\bar{T}$ is odd unimodular.

Proof. As in the proof of Lemma 2.5, we extend $3T$ *via* $9\operatorname{discr}_3(3T)$ to obtain a sublattice $T_0 \subset T$ whose discriminant $\operatorname{discr}_3 T_0$ has only 3- and 9-torsion. Using Theorem 2.2(2) and (2.7), we obtain $\operatorname{Br}q_\circ = 0$. Hence, $q_\circ$ is null-cobordant and T contains a sublattice T_3 with 3-elementary group $\operatorname{discr}_3 T_3$ of maximal length.

There remains to apply Lemma 2.5 to T_3 to produce a sublattice $T' \subset T$ with both 2- and 3-discriminants elementary and of maximal length. Then $\bar{T} := T'(\frac{1}{6})$ is integral and unimodular; it is odd since so is $\operatorname{discr}_2 T'$. □

The next lemma describes a maximal 3-elementary finite index sublattice.

Lemma 2.9. *Any* 3*-elementary lattice T has a finite index sublattice $T' = \bar{T}^\vee(3)$, where $\operatorname{discr}\bar{T} = \langle\frac{2}{3}r\rangle^m$ and $r = \pm1$, $m \leqslant 2$ are such that $2mr = \sigma(T) \bmod 8$.*

Proof. Consider the extension $T_0 \supset 3T$ by $9\operatorname{discr}_3(3T)$, followed by the extension $T_1 \supset T_0$ by any maximal isotropic subgroup of $(3\operatorname{discr}T_0, q_\circ)$. Then, $T_1 = \bar{T}(3)$, where $\bar{T}$ is as in the statement, and $T' \supset T_1$ is the extension by the isotropic subgroup $3\operatorname{discr}T_1$. □

The following well-known lemma is easily proved by induction.

Lemma 2.10. *Let $\mathcal{A}$ be an affine subspace in a quadratic $\mathbb{F}_3$-vector space, and let $n_r(\mathcal{A})$ be the number of vector in $\mathcal{A}$ of square $\frac{2}{3}r$, $r \in \mathbb{F}_3$. Then, for each $r \in \mathbb{F}_3$, one has either* $\dim\mathcal{A} \leqslant 2$ *and* $n_r(\mathcal{A}) \leqslant 5$ *or* $n_r(\mathcal{A}) = 0 \bmod 3$*.*

Finally, consider a finite quadratic form $\mathcal{S}_n$ generated by n orthogonal elements α_i, each of order 2 and square $\frac{1}{2}$. We have an obvious inclusion $\mathbb{S}_n \subset \operatorname{Aut}\mathcal{S}_n$, the symmetric group acting *via* permutations of the generators. The *reflection* against an element $\alpha \in \mathcal{S}_n$, $\alpha^2 = 1$, is the autoisometry $t_\alpha\colon x \mapsto x - 2(x\cdot\alpha)\alpha$. The group $\operatorname{Aut}\mathcal{S}_n$ is generated by all reflections, whereas $\mathbb{S}_n$ is generated by the reflections against vectors of Hamming norm 2. Denote by $\mathfrak{t}_i \in \operatorname{Aut}\mathcal{S}_n$ the reflection against $\alpha_i + \ldots + \alpha_{i+5}$ (assuming that $n \geqslant i+5$), and let $\mathfrak{t} \in \operatorname{Aut}\mathcal{S}_n$ be the reflection against $\alpha_1 + \ldots + \alpha_{10}$ (assuming that $n \geqslant 10$).

Lemma 2.11 (D. Pasechnik, private communication)**.** *For $n \leqslant 9$, a complete list of representatives of the double cosets $\mathfrak{S}_n := \mathbb{S}_n\backslash\operatorname{Aut}\mathcal{S}_n/\mathbb{S}_n$ is as follows*:

$$\text{identity}, \quad \mathfrak{t}_1\ (\textit{if } n \geqslant 6), \quad \mathfrak{t}_1\mathfrak{t}_3\ (\textit{if } n \geqslant 8), \quad \mathfrak{t}_1\mathfrak{t}_4\ (\textit{if } n \geqslant 9).$$

The set $\mathfrak{S}_{10}$ is represented by $\{\mathfrak{t}_1\mathfrak{t}_3\mathfrak{t}_5\} \cup \{u, \mathfrak{t}u \mid u \in \mathfrak{S}_9\}$; one has $|\mathfrak{S}_{10}| = 9$.

In practice, we use Lemma 2.11 to classify the bijective anti-isometries between two copies, $\mathcal{S}_n$ and $-\mathcal{S}_n$, each equipped with a basis canonical up to order, up to the two-sided action of the group $\mathbb{S}_n \times \mathbb{S}_n$, *cf.* §2.3. To do so, we identify the two groups by means of some bijection of their bases and, with a certain abuse of the language, speak about the anti-isometries $\mathfrak{t}_i$, $\mathfrak{t}$, *etc.*

2.5. **Generalized quadrangles** (see [14]). The intersection of a quartic X and a plane in $\mathbb{P}^3$ is a curve of degree 4. It may happen that this curve is completely reducible, *i.e.*, splits into four lines $l_1, \ldots, l_4$. (Note that these lines must be pairwise distinct, as otherwise X would have a singular point.) If this is the case, we say that the lines $l_1, \ldots, l_4$ constitute a *plane* $\alpha \subset \operatorname{Fn} X$, *cf.* Lemma 4.1 below and the definition thereafter. By definition, each plane consists of four lines. Occasionally, we consider subconfigurations $\mathcal{F} \subset \operatorname{Fn} X$ with the following properties:

- each line is contained in a certain fixed number $p \geqslant 2$ of planes;
- if two lines $l_1, l_2 \in \mathcal{F}$ intersect, they are contained in a plane $\alpha \subset \mathcal{F}$.

In this case, renaming (lines, planes) to (points, lines) and taking the inclusion for the incidence relation, we obtain a combinatorial structure known as a *generalized quadrangle* of order $(3, p-1)$, or $\operatorname{GQ}(3, p-1)$. Specifically, a $\operatorname{GQ}(s,t)$ consists of two sets, $\mathcal{P}$ (points) and $\mathcal{B}$ (lines), and an incidence relation $|$, so that

- each point is incident with $1 + t \geqslant 2$ lines,
- each line is incident with $1 + s \geqslant 2$ points, and
- for a point l and line $\alpha \nmid l$, there is a unique pair α', l' such that $l \mid \alpha' \mid l' \mid \alpha$,

see [14] for the precise definition and further details. (For the last axiom, one uses the obvious fact that a line $l \in \operatorname{Fn} X$ not contained in a plane α intersects exactly one line $l' \in \alpha$, *cf.* Remark 4.5 below; then, l, l' are contained in a plane α'.)

According to [14, §6.2], a generalized quadrangle $\operatorname{GQ}(3,t)$ exists if and only if $t = 1, 3, 5, 9$, and, unless $t = 3$, a quadrangle is unique up to isomorphism. In the exceptional case $t = 3$, there are two quadrangles: $Q(4,3)$ and its dual $W(3)$. Here, $Q(d,q)$, $d = 3, 4, 5$, can be described as the collection of points and lines in a fixed nonsingular quadric of index 2 in the projective space $\mathbb{P}^d$ over $\mathbb{F}_q$, whereas $W(q)$ is the collection of points in the projective space $\mathbb{P}^3$ over $\mathbb{F}_q$ and lines Lagrangian with respect to any fixed symplectic form.

By our definition, two lines in a generalized quadrangle $\mathcal{F} \subset \operatorname{Fn} X$ intersect if and only if they are contained in a plane $\alpha \subset \mathcal{F}$. Hence, the adjacency graph of the lines is uniquely determined by the combinatorics, and we denote by $\mathbf{Q}_{16}$, $\mathbf{Q}'_{40}$, $\mathbf{Q}''_{40}$, $\mathbf{Q}_{64}$, $\mathbf{Q}_{112}$ the corresponding lattices modulo kernel, with the 4-polarization defined as the sum of the four lines constituting any plane (*cf.* Lemma 4.1 below). (By convention, the lattices $\mathbf{Q}'_{40}$ and $\mathbf{Q}''_{40}$ correspond to the generalized quadrangles $Q(4,3)$ and $W(3)$, respectively.) All five lattices are hyperbolic; we have

- $\operatorname{rk} \mathbf{Q}_{112} = 22$ and $|\operatorname{Fn} \mathbf{Q}_{112}| = 112$ (see Remark 6.15 and §7.6.1),
- $\operatorname{rk} \mathbf{Q}_{64} = 19$ and $|\operatorname{Fn} \mathbf{Q}_{64}| = 64$ (see §7.6.2),
- $\operatorname{rk} \mathbf{Q}^*_{40} = 16$ and $|\operatorname{Fn} \mathbf{Q}^*_{40}| = 40$ (see §7.5 and §7.6.3),
- $\operatorname{rk} \mathbf{Q}_{16} = 10$ and $|\operatorname{Fn} \mathbf{Q}_{16}| = 16$ (see §7.6.4).

(References indicate parts of the paper where the realizability of the generalized quadrangles by configurations of lines in nonsingular quartics is discussed.)

3. $K3$-SURFACES

Here, we give a brief account of the theory of $K3$-surfaces; for more details and further references, we address the reader to [9].

3.1. **$K3$-surfaces.** An (*algebraic*) *$K3$-surface* over an algebraically closed field $\Bbbk$ is a complete nonsingular variety X over $\Bbbk$ of dimension two such that

$$\Omega^2_X \cong \mathcal{O}_X, \quad H^1(X;\mathcal{O}_X)=0.$$

If X is a $K3$-surface, the canonical epimorphism $\operatorname{Pic} X \twoheadrightarrow NS(X)$ is an isomorphism; furthermore, the lattice $NS(X)$ is even and hyperbolic and $\operatorname{rk} NS(X) \leqslant 22$.

If $\Bbbk = \mathbb{C}$, one also considers *analytic $K3$-surfaces*, which are simply connected compact complex surfaces with the trivial canonical bundle. All $K3$-surfaces are Kähler. In general, $\sigma_+ NS(X) \leqslant 1$, and X is algebraic if and only if $\sigma_+ NS(X) = 1$; in this case, $NS(X)$ is nondegenerate. We have a primitive embedding

$$NS(X) \subset H_2(X;\mathbb{Z}) \cong \mathbf{L} = \mathbf{E}_8^2 \oplus \mathbf{U}^3,$$

see §2.2; hence, $\operatorname{rk} NS(X) \leqslant 20$. These statements on $NS(X)$ extend to $K3$-surfaces over any algebraically closed field $\Bbbk$ of characteristic 0.

A $K3$-surface X is called (*Shioda*) *supersingular* if $\operatorname{rk} NS(X) = 22$.

Theorem 3.1 (see [1])**.** *Assume that a $K3$-surface X over an algebraically closed field $\Bbbk$ is supersingular. Then $p := \operatorname{char}\Bbbk > 0$ and $NS(X) \cong \mathbf{S}_{p,\sigma}$ (see §2.2) for some integer $\sigma = 1,\ldots,10$, called the* Artin invariant *of X.*

If X is not supersingular, then $\operatorname{rk} NS(X) \leqslant 20$. Furthermore, according to the next theorem, in this case we have (al least) the same restrictions on $NS(X)$ as in the case of characteristic 0.

Theorem 3.2 (see [7, 11])**.** *If a $K3$-surface X is not supersingular, there exists a $K3$-surface X_0 over a field $\Bbbk_0$, $\operatorname{char}\Bbbk_0 = 0$, with the property that $NS(X_0) \cong NS(X)$. In particular, there exists a primitive extension $\mathbf{L} \supset NS(X)$.*

3.2. **Quartics.** Any nonsingular quartic $X \subset \mathbb{P}^3$ is a $K3$-surface. This surface is equipped with a canonical 4-polarization $h \in NS(X)$, *viz.* the hyperplane sections; this polarization is always assumed when we speak about quartics.

Theorem 3.3 (see [19])**.** *The 4-polarization $h \in NS(X)$ of a nonsingular quartic $X \in \mathbb{P}^3$ has the following property: there is no vector $e \in NS(X)$ such that either*

(1) *$e^2 = -2$ and $e \cdot h = 0$* (exceptional divisor) *or*
(2) *$e^2 = 0$ and $e \cdot h = 2$.*

Conversely, given a $K3$-surface X over an algebraically closed field $\Bbbk$, $\operatorname{char}\Bbbk \neq 2$, a 4-polarization $h \in NS(X)$ contained in the positive cone of $NS(X)$ and satisfying the two conditions above embeds X into $\mathbb{P}^3$ as a nonsingular quartic.

Remark 3.4. Strictly speaking, [19] makes a global assumption that $\operatorname{char}\Bbbk \neq 2$. Nevertheless, the arguments leading to the necessity of conditions (1), (2) work in any characteristic, as essentially they rely upon the Riemann–Roch theorem only. It would not be a surprise if the conditions were also sufficient. Unfortunately, I could not find in the literature an appropriate reference covering all the details of the proof; that is why I refrain from stating the realizability of the found configurations in Theorem 1.1, leaving the latter as an "upper bound."

A 4-polarized hyperbolic lattice S satisfying the necessary conditions (1), (2) in Theorem 3.3 is called *admissible*.

A *geometric realization* of an admissible lattice S is a lattice extension $\mathbf{L} \supset S$ or $\mathbf{S}_{p,\sigma} \supset S$ (see §2.2), where $1 \leqslant \sigma \leqslant 10$ and p is a prime; we also require that

- the primitive hull $\tilde{S} := (S \otimes \mathbb{Q}) \cap \mathbf{L}$ (in the former case) or
- the 4-polarized lattice $(\mathbf{S}_{p,\sigma}, h)$ (in the latter case)

should still be admissible. In view of Theorems 3.1 and 3.2, the following simple consequence of Theorem 2.2 (applied to the orthogonal complement of $T := S^\perp$ in $\mathbf{L}$ or $\mathbf{S}_{p,\sigma}$) and Proposition 2.4 gives us a necessary condition for the existence of a primitive geometric realization of a given hyperbolic lattice S.

Theorem 3.5. *Consider a primitive hyperbolic sublattice $S \subset NS(X)$ and denote $\delta := 22 - \operatorname{rk} S$ and $\mathcal{S} := \operatorname{discr} S$. If X is supersingular, let $p := \operatorname{char} \Bbbk$; otherwise, let $p := 0$. Then we have $\ell_q(\mathcal{S}) \leqslant \delta$ for each prime $q \neq p$ and*

- *$|\mathcal{S}| \det_q \mathcal{S} = 1 \bmod (\mathbb{Z}_q^\times)^2$ for any prime $q \neq 2$ or p for which $\ell_q(\mathcal{S}) = \delta$;*
- *either $p = 2$, or $\ell_2(\mathcal{S}) < \delta$, or $\mathcal{S}_2$ is odd, or $|\mathcal{S}| \det_2 \mathcal{S} = \pm 1 \bmod (\mathbb{Z}_2^\times)^2$.*

(Note that, by the additivity of both Brown invariant Br and signature $\sigma_+ - \sigma_-$, condition (2) in Theorem 2.2 holds automatically.) An important observation is the fact that the restriction imposed by Theorem 3.5 at a prime $q \neq \operatorname{char} \Bbbk$ does not depend on $\operatorname{char} \Bbbk$.

If $\operatorname{char} \Bbbk = 0$, the condition given by Theorem 3.5 is also sufficient, as follows from the surjectivity of the period map [10] (see also [6] for the details on the moduli space). If X is supersingular, a sufficient condition is that at least one of the (finite set of) extensions $\mathbf{S}_{p,\sigma} \supset S$ obtained by Nikulin's construction (see §2.3) should be admissible. Note, though, that in this case $\operatorname{Fn} S$ may be a proper subset of the set $\operatorname{Fn} X = \operatorname{Fn} \mathbf{S}_{p,\sigma}$; these phenomena are discussed in §6 and §7.

Let $X \subset \mathbb{P}^3$ be a nonsingular quartic. Then, sending a line $l \subset X$ to its class $[l] \in NS(X)$, we obtain a map $\operatorname{Fn} X \to NS(X)$.

Lemma 3.6 (*cf.* [6]). *The map $l \mapsto [l]$ establishes a bijection $\operatorname{Fn} X = \operatorname{Fn} NS(X)$.*

Proof. The map is obviously well defined and injective (since each line has negative self-intersection), and its image is in $\operatorname{Fn} NS(X)$. By the Riemann–Roch theorem, any element $a \in \operatorname{Fn} NS(X)$ is realized by a unique (-2) curve C. Assume that C is reducible, $C = C_1 + \ldots + C_k$, where all components C_i are also (-2)-curves. Then, since $1 = C \cdot h = \sum_i C_i \cdot h$, all but one components of C are exceptional divisors, contradicting Theorem 3.3(1). Thus, C is irreducible; since also C has projective degree $1 = C \cdot h$, we conclude that C is a line. □

3.3. **(Quasi-)elliptic pencils.** Let $\pi \colon X \to \mathbb{P}^1$ be a pencil of curves of arithmetic genus 1. If a generic fiber of π is a smooth elliptic curve, the pencil π is *elliptic*; otherwise (generic fiber is singular), π is *quasi-elliptic*.

All fibers of a (quasi-)elliptic pencil $\pi \colon X \to \mathbb{P}^1$ are linearly equivalent and, for each fiber F, one has $F^2 = 0$. Conversely, if X is a $K3$-surface, then each primitive, effective, and numerically effective divisor $F \subset X$ such that $F^2 = 0$ is a fiber of a unique (quasi-)elliptic pencil $\pi \colon X \to \mathbb{P}^1$.

Let $F = \sum_i r_i C_i$ be a reducible fiber. Each reduced component C_i is a smooth rational (-2)-curve, and the dual intersection graph of F is a certain affine Dynkin diagram $\tilde{D}$, see [3]. Denoting by $\mathbb{Z}\tilde{D}$ the intersection lattice freely generated by the vertices $C_i \in \tilde{D}$, the kernel $\ker \mathbb{Z}\tilde{D}$ has rank 1 and is generated by a unique positive linear combination $\sum_i r_i C_i$; this generator of $\ker \mathbb{Z}\tilde{D}$ is F.

We define the *Milnor number* of an (affine or elliptic) Dynkin diagram D as the rank $\mu(D) := \operatorname{rk}(\mathbb{Z}D/\ker)$. Thus, $\mu(D)$ is the number of vertices of D in the elliptic case and the number of vertices minus 1 in the affine case.

Theorem 3.7 (see [18]). *Let $\pi\colon X \to \mathbb{P}^2$ be a pencil of curves of arithmetic genus 1, and denote by $\tilde{D}_1, \ldots$ the dual intersection graphs of the components of the reducible fibers of π. Then, π is quasi-elliptic if and only if*

1. $p := \operatorname{char} \Bbbk = 2$ *or* 3;
2. *each lattice* $\mathbb{Z}\tilde{D}_i/\ker$ *is* p*-elementary*;
3. *one has* $\sum_i \mu(\tilde{D}_i) = b_2(X) - 2$.

If X is a $K3$-surface, then $b_2(X) = 22$ and $e(X) = 12\chi(X) = 24$ (the étale Euler characteristic). Recall also that the affine Dynkin diagrams $\tilde{D}$ with p-elementary lattice $\mathbb{Z}\tilde{D}/\ker$ are:

- $\tilde{\mathbf{A}}_1$, $\tilde{\mathbf{E}}_7$, $\tilde{\mathbf{E}}_8$, $\tilde{\mathbf{D}}_{2k}$, if $p = 2$, and
- $\tilde{\mathbf{A}}_2$, $\tilde{\mathbf{E}}_6$, $\tilde{\mathbf{E}}_8$, if $p = 3$.

If a pencil $\pi\colon X \to \mathbb{P}^2$ is elliptic, instead of Theorem 3.7(3) we have the identity

$$\sum \bigl(e(F_i) + d(F_i)\bigr) = e(X), \tag{3.8}$$

the summation running over all singular fibers F_i of π. Here, $d(F_i) \geqslant 0$ is the wild ramification index and

$$e(F_i) \geqslant |\text{irreducible components of } F_i| \geqslant 1.$$

Let $X \subset \mathbb{P}^3$ be a nonsingular quartic, and let $\pi\colon X \to \mathbb{P}^1$ be a (quasi-)elliptic pencil. A fiber of π consisting entirely of lines is called *parabolic*; any other singular fiber is called *elliptic*. Each parabolic fiber is an affine Dynkin diagram $\tilde{D} \subset \operatorname{Fn} X$, whereas the configuration of lines contained in an elliptic fiber is a Dynkin diagram, possibly empty or disconnected, $D \subset \operatorname{Fn} X$; the type of D is referred to as the *linear type* of the elliptic fiber. Regarded as spatial curves, all fibers of π have the same degree. This fact limits the types of parabolic fibers and linear types of elliptic fibers appearing in the same pencil.

We denote by $\ln(\pi)$ the number of lines contained in the fibers of π. The following statement is an immediate consequence of Theorem 3.7(3) and (3.8).

Corollary 3.9. *If a pencil $\pi\colon X \to \mathbb{P}^1$ is elliptic, then*

$$\ln(\pi) \leqslant |\text{components in the singular fibers of } \pi| \leqslant 24;$$

If π is quasi-elliptic, then

$$\ln(\pi) \leqslant 20 + |\text{parabolic fibers of } \pi|.$$

When applying the first inequality, we often use the fact that the upper bound 24 is reduced by at least 1 by each elliptic fiber of π, as such a fiber contains at least one component that is not a line.

4. Configurations and pencils

In this section, we discuss simplest arithmetical properties of configurations of lines. Most results here either are contained in [6] or can be regarded as immediate extensions/generalizations thereof.

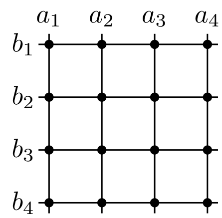

FIGURE 1. The configuration in Lemma 4.2 (the graph $K_{4,4}$)

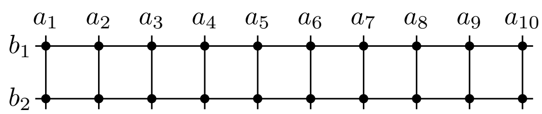

FIGURE 2. The configuration in Lemma 4.3 (the graph $K_{10,2}$)

4.1. **Configurations of lines.** Let S be a 4-polarized hyperbolic lattice; we will always assume that S is admissible, *i.e.*, satisfies the conditions in Theorem 3.3. Recall (see §2.2) that the *configuration of lines* in S is the set

$$\operatorname{Fn} S := \bigl\{a \in S \bigm| a^2 = -2,\ a \cdot h = 1\bigr\}.$$

Elements of $\operatorname{Fn} S$ are called *lines*. The hyperbolicity of S and Theorem 3.3 imply that, for any pair $a_1, a_2 \in \operatorname{Fn} S$, one has $a_1 \cdot a_2 = 0$ or 1 (*cf.* [6]); respectively, we say that the two lines a_1, a_2 are *disjoint* or *intersect*. The set $\operatorname{Fn} X$ is often treated as a graph, with two lines connected by an edge if and only if they intersect.

The next few lemmas are based on the following simple observation, *cf.* [6]: each sublattice $S' \subset S$ containing h and at least one line is hyperbolic; hence, $\ker S' = 0$ and any numeric relation $u = 0 \bmod (S')^\vee$ implies a true relation $u = 0$.

Lemma 4.1. *Assume that four lines $a_1, \ldots, a_4 \in \operatorname{Fn} S$ intersect each other,* i.e.*,*

$$a_i \cdot a_j = 1 \textit{ for } 1 \leqslant i < j \leqslant 4.$$

(In other words, the lines constitute the complete graph K_4.) Then

$$a_1 + a_2 + a_3 + a_4 = h.$$

If $S = NS(X)$, then the lines $a_1, \ldots, a_4$ are cut on X by a plane.

A quadruple $a_1, \ldots, a_4 \in \operatorname{Fn} S$ as in Lemma 4.1 is called a *plane* in S.

The *valency* $\operatorname{val} l$ of a line $l \in \operatorname{Fn} X$ is the number of lines $a \in \operatorname{Fn} X$ intersecting l (alternatively, this is the valency of l as a vertex of the graph $\operatorname{Fn} X$), whereas the *multiplicity* $\operatorname{mult} l$ is the number of planes $\alpha \subset \operatorname{Fn} X$ containing l.

Lemma 4.2. *Assume that eight lines $a_1, \ldots, a_4, b_1, \ldots, b_4 \in \operatorname{Fn} S$ intersect as shown in Figure 1,* i.e.*,*

$$a_i \cdot a_j = b_i \cdot b_j = 0 \textit{ for } 1 \leqslant i < j \leqslant 4, \quad a_i \cdot b_j = 1 \textit{ for all } i, j = 1, \ldots, 4.$$

(In other words, the lines constitute the complete bipartite graph $K_{4,4}$.) Then

$$a_1 + a_2 + a_3 + a_4 + b_1 + b_2 + b_3 + b_4 = 2h.$$

If $S = NS(X)$, then the lines $a_1, \ldots, b_4$ are cut on X by a quadric.

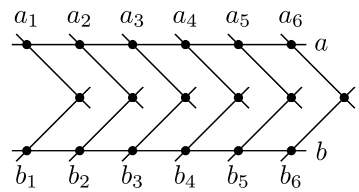

FIGURE 3. The configuration in Lemma 4.4

Lemma 4.3. *Assume that twelve lines* $a_1, \ldots, a_{10}, b_1, b_2 \in \operatorname{Fn} S$ *intersect as shown in Figure 2,* i.e.*,*

$$a_i \cdot a_j = b_1 \cdot b_2 = 0 \text{ for } 1 \leqslant i < j \leqslant 10, \quad a_i \cdot b_j = 1 \text{ for } i = 1, \ldots, 10,\ j = 1, 2.$$

(*In other words, the lines constitute the complete bipartite graph* $K_{10,2}$*.*) *Then*

$$a_1 + a_2 + \ldots + a_{10} + 3b_1 + 3b_2 = 4h.$$

In Proposition 8.6 below we show that a configuration as in Lemma 4.3 cannot exist if $\operatorname{char} \Bbbk = 2$.

Lemma 4.4. *Assume that fourteen lines* $a, b, a_1, \ldots, a_6, b_1, \ldots, b_6 \in \operatorname{Fn} S$ *intersect as shown in Figure 3,* i.e.*,*

$$a \cdot b = a \cdot b_i = a_i \cdot b = a_i \cdot a_j = b_i \cdot b_j = 0,\ a \cdot a_i = b \cdot b_i = a_i \cdot b_i = 1$$

for all pairs $i, j = 1, \ldots, 6$ *such that* $i \neq j$*. Then*

$$3a + a_1 + \ldots + a_6 = 3b + b_1 \ldots + b_6.$$

Remark 4.5. The relations in Lemmas 4.1–4.4 can be used to assert the existence and uniqueness of lines with certain properties. If all but one lines of a configuration as in one of the lemmas are known, there is at most one "missing" line completing the configuration; if the corresponding coefficient in the relation equals ± 1, such a line does exist (as it is a linear combination of the known ones). In addition, we can use the lemmas to control the intersection of a line l other than the given ones with the lines constituting the configuration. Thus,

- in Lemma 4.1, l intersects exactly one of $a_1, \ldots, a_4$, and
- in Lemma 4.2, l intersects exactly two of $a_1, \ldots, b_4$.

(In the other two lemmas, one should weight the intersections with the coefficients present in the relations.)

4.2. **Pencils.** The pencil of planes in $\mathbb{P}^3$ passing through a fixed line $l \subset X$ defines a (quasi-)elliptic pencil $\pi := \pi[l] \colon X \to \mathbb{P}^1$: the fibers of π are the *residual cubics* obtained by removing the common component l from the quartic curve cut on X by a plane. Clearly, the lines that are in the fibers of π are precisely those intersecting l. From this point of view, there are two kinds of fibers: 3-*fibers*, split into three lines (constituting, together with l, a plane in $\operatorname{Fn} X$), and 1-*fibers*, consisting of a single line (and an irreducible residual conic). Each line in X that is disjoint from l intersects each fiber of π at a single point; hence, it is a section of π.

Motivated by this construction, given a line $l \in \operatorname{Fn} S$, we define the *maximal pencil* with the *axis* l as the set

$$\mathcal{P}(l) := \bigl\{ a \in \operatorname{Fn} S \bigm| a \cdot l = 1 \bigr\}.$$

By Lemma 4.1, the lines constituting $\mathcal{P}(l)$ split into pairwise disjoint groups, each consisting of three or one line; they are called 3- and 1-fibers of $\mathcal{P}(l)$, respectively. The *type* of a pencil is the pair (p, q) representing the numbers of its 3- and 1-fibers. The *pencil structure* of a configuration $\operatorname{Fn} S$ is the list of types of all pencils $\mathcal{P}(l)$, $l \in \operatorname{Fn} S$; it is usually represented in the partition notation and can be used as an easily comparable combinatorial invariant.

Sometimes, we consider pencils $\mathcal{P} \subset \mathcal{P}(l)$ that are not necessarily maximal; however, we always insist that, whenever $\mathcal{P}$ contains two lines a_1, a_2 that intersect, it also contains the third line $h - l - a_1 - a_2$ of the same 3-fiber. In other words, $\mathcal{P}$ is a maximal pencil in the sublattice $S' \subset S$ spanned by h, l, and *some* of the lines $a \in \operatorname{Fn} S$ that intersect l.

A *section* of a pencil $\mathcal{P} \subset \mathcal{P}(l)$ is any line $a \in \operatorname{Fn} S$ disjoint from l. The number of sections of $\mathcal{P}$ is denoted by $\operatorname{s}(\mathcal{P})$.

Two pencils $\mathcal{P}(l_1)$, $\mathcal{P}(l_2)$ are called *obverse* if their axes l_1, l_2 are disjoint.

4.3. **The lattice $P_{p,q}$ (see [6]).** Fix a pencil $\mathcal{P} \subset \operatorname{Fn} S$ (not necessarily maximal) of type (p, q) and denote by $P_{p,q} \subset S$ the sublattice spanned by the polarization h, axis l, and the members of $\mathcal{P}$.

Let $\operatorname{fb}_3 \mathcal{P} := \{1, \dots, p\}$ and $\operatorname{fb}_1 \mathcal{P} := \{1, \dots, q\}$ be the sets of the 3- and 1-fibers of $\mathcal{P}$, respectively. (We emphasize that we regard fb_3 and fb_1 as two *disjoint* sets.) The lattice $P_{p,q}$ is generated by the classes h, l, $m_{i,\pm}$, $i \in \operatorname{fb}_3 \mathcal{P}$ (two lines from each 3-fiber), and n_k, $k \in \operatorname{fb}_1 \mathcal{P}$ (the lines constituting the 1-fibers). The third line in the i-th 3-fiber is $m_{i,0} = h - l - (m_{i,+} + m_{i,-})$, see Lemma 4.1. When speaking about dual vectors h^*, l^*, *etc.*, we always refer to this distinguished basis (which involves some choice for each 3-fiber).

Observation 4.6. The 3-primary part $\operatorname{discr}_3 P_{p,q}$ contains the classes represented by the following mutually orthogonal vectors:

- $\lambda := \frac{1}{3}(l - h)$: one has $\lambda^2 = 0$ and $\lambda \cdot h = \lambda \cdot l = -1$;
- $\mu_i := \frac{1}{3}(m_{i,+} - m_{i,-})$, $i \in \operatorname{fb}_3 \mathcal{P}$: one has $\mu_i^2 = -\frac{2}{3}$ and $\mu_i \cdot h = 0$.

If $r := p + q - 1 \neq 0 \bmod 3$, then $\operatorname{discr}_3 P_{p,q}$ is generated by μ_i, $i \in \operatorname{fb}_3 \mathcal{P}$, and the order 9 class of the vector

- $\upsilon := \frac{1}{3}\bigl(l - r\lambda + \sum_{i=1}^{p}(m_{i,+} + m_{i,-}) - \sum_{k=1}^{q} n_k\bigr)$;

note that $3\upsilon = -r\lambda \neq 0 \bmod P_{p,q}$. Hence, in this case, the subgroup of elements of order 3 is generated by λ and μ_i. If $p + q = 1 \bmod 3$, then $\operatorname{discr}_3 P_{p,q}$ is generated by λ, μ_i, and the order 3 class of

- $\omega := \frac{1}{3}\bigl(l + \sum_{i=1}^{p}(m_{i,+} + m_{i,-}) - \sum_{k=1}^{q} n_k\bigr)$.

The $O_h(P_{p,q})$-orbits of order 3 elements in $\operatorname{discr}_3 P_{p,q}$ are $r\Lambda := \{r\lambda\}$, $r = \pm$, and

$$\mathrm{M}_k := \bigl\{r\lambda + \textstyle\sum_{i \in I} \pm\mu_i \bigm| r \in \mathbb{F}_3,\ I \subset \operatorname{fb}_3 \mathcal{P},\ |I| = k\bigr\}, \quad k = 0, \dots, p;$$

if $p + q = 1 \bmod 3$, there also are at most six orbits

$$r\Omega_s := \bigl\{\alpha \in \operatorname{discr}_3 P_{p,q} \bigm| \alpha \cdot \lambda = -\tfrac{1}{3}r,\ \alpha^2 = \tfrac{2}{3}s\bigr\}, \quad r = \pm,\ s \in \mathbb{F}_3.$$

Note also that any element of $\operatorname{Ker}\bigl[\operatorname{Aut}\lambda^\perp \to \operatorname{Aut}(\lambda^\perp/\lambda)\bigr]$ lifts to $O_h(P_{p,q})$.

The 2-primary part $\operatorname{discr}_2 P_{p,q}$ is generated by the classes of $3\nu_k$, where

- $\nu_k := n_k^* = -\frac{1}{2}(\lambda + n_k)$, $k \in \operatorname{fb}_1 \mathcal{P}$: one has $\nu_k^2 = -\frac{1}{2}$ and $\nu_k \cdot h = 0$.

TABLE 1. The bounds on (p, q) and $\operatorname{val} l$, see Proposition 4.8

$p =$	6	5	4	3	2	1	0
$q \leqslant$	2	3	6	7	9	10	12
$\operatorname{val} l \leqslant$	20	18	18	16	15	13	12

With respect to this basis, the image of $O_h(P_{p,q})$ in $\operatorname{Aut} \operatorname{discr}_2 P_{p,q}$ is the subgroup $\mathbb{S}_q$ permuting the generators, and the orbits are

$$\mathrm{N}_k := \bigl\{\textstyle\sum_{i\in I} \nu_i \bigm| I \subset \mathrm{fb}_1 \mathcal{P},\ |I| = k\bigr\}, \quad k = 0, \ldots, q.$$

Note also that $O_h(P_{p,q})$ acts on the 2- and 3-torsion independently, *i.e.*, the image of $O_h(P_{p,q})$ in $\operatorname{Aut} \operatorname{discr} P_{p,q} = \operatorname{Aut} \operatorname{discr}_2 P_{p,q} \times \operatorname{Aut} \operatorname{discr}_3 P_{p,q}$ is the product of its images in the two factors.

We are interested in the possible finite index extensions $P_{p,q} \subset \tilde{P} \subset NS(X)$; recall that each such extension is described by its kernel $\tilde{P}/P_{p,q}$, which is an isotropic subgroup of $\operatorname{discr} P_{p,q}$, see Theorem 2.3. A pencil $\mathcal{P}$ is called *primitive* (in a given lattice S) if $\tilde{P}/P_{p,q} = 0$; otherwise, it is called *imprimitive*. Due to Observation 4.6, the isotropic vectors are those in the orbits $\pm\Lambda$, $\pm\Omega_0$, M_{3i}, $i > 0$, and N_{4k}, $k > 0$.

Lemma 4.7. *The kernel $\tilde{P}/P_{p.q}$ is disjoint from the orbits $\pm\Lambda$, M_3, and N_4. If $p + q = 1$, 4, or 7, it is also disjoint from $\pm\Omega_0$.*

Proof. If $\lambda \in NS(X)$, then so is $e := -2\lambda$, which contradicts Theorem 3.3(2).

The orbits M_3 and N_4 are represented by the exceptional divisors $\mu_1 + \mu_2 + \mu_3$ and $\frac{1}{2}(n_1 - n_2 + n_3 - n_4)$, respectively, see Theorem 3.3(1). For the last statement, it suffices to consider the case $p = 0$; then, the vectors $2l^*$, $l^* + \nu_1 - \nu_2 - \nu_3 - \nu_4$, and $l^* - \sum_{k=1}^{7} \nu_k$, respectively, are exceptional divisors. □

4.4. **Types of pencils.** In the next statement, we show that, with one exception, a pencil of the form $\pi[l]$, see §4.2, is elliptic.

Proposition 4.8. *With one exception,* viz. *the case where*

- $\operatorname{char} \Bbbk = 3$ *and $\mathcal{P}(l)$ is of type $(10, 0)$, hence $\pi[l]$ is quasi-elliptic,*

the type (p, q) of a pencil $\mathcal{P} \subset NS(X)$ satisfies Euler's inequality $3p + 2q \leqslant 24$.

Proof. The fibers of $\pi[l]$ are all of type $\tilde{\mathbf{A}}_2$, $\tilde{\mathbf{A}}_2^*$ (p copies) or $\tilde{\mathbf{A}}_1$, $\tilde{\mathbf{A}}_1^*$ (q copies). By Theorem 3.7, such a pencil can be quasi-elliptic only if either

- $\operatorname{char} \Bbbk = 2$ and $(p, q) = (0, 20)$, or
- $\operatorname{char} \Bbbk = 3$ and $(p, q) = (10, 0)$.

The former possibility is eliminated by Theorem 3.5 and Lemma 4.7, as the only isotropic orbits in $\operatorname{discr}_3 P_{0,20}$ are $\pm\Lambda$, see Observation 4.6. If $\pi[l]$ is elliptic, the inequality $3p + 2q \leqslant 24$ follows from Corollary 3.9. □

Proposition 4.9 (see [6])**.** *Assume that* $\operatorname{char} \Bbbk \neq 2, 3$ *or X is not supersingular. Then the type (p, q) of a pencil $\mathcal{P}(l) \in \operatorname{Fn} X$ takes values listed in Table 1. Besides, if $p = 6$ and $q > 0$ or $(p, q) = (4, 6)$, the pencil is necessary imprimitive.*

For a pencil $\mathcal{P}$ of type $(4, 6)$, the imprimitivity implies the existence of a section intersecting all ten fibers: by Lemma 4.7, the kernel $\tilde{P}/P_{4,6}$ is generated by ω, and the section is given by Lemma 4.3. Pencils in supersingular quartics over fields of characteristic 2 or 3 are considered in subsequent sections; the restrictions are listed in Proposition 7.5 and Proposition 6.13, respectively.

Proof of Proposition 4.9. Since $3p+2q \leqslant 24$, we only need to eliminate the values $p=7$ and $(p,q)=(6,3)$ or $(5,4)$. To do so, assume that $P_{p,q} \subset \tilde{P} \subset NS(X)$, where $\tilde{P}$ is a finite index extension, and use Lemma 4.7 to bound the size of the group $\operatorname{discr}_3 \tilde{P}$ from below. Then, Theorem 3.5 implies that $\tilde{P}$ does not admit a primitive geometric realization. The imprimitivity is proved similarly, using the group $\operatorname{discr}_3 P_{p,q}$ instead of $\operatorname{discr}_3 \tilde{P}$. For more details, see [6]. □

Proposition 4.10 (see [6])**.** *Assume that* $\operatorname{char} \Bbbk \neq 2$ *or* X *is not supersingular. Then any pencil of type* (p,q)*,* $p+q \geqslant 11$*, is imprimitive; up to reordering the* 1*-fibers, the kernel* $\tilde{P}/P_{p,q} \subset \operatorname{discr}_2 P_{p,q}$ *is generated by*

- $3(\nu_1+\ldots+\nu_8)$ *and* $3(\nu_5+\ldots+\nu_{12})$ *for* $(p,q)=(0,12)$*, or*
- $3(\nu_1+\ldots+\nu_8)$ *for* $p+q=11$*.*

This statement is proved similar to Proposition 4.9. As a consequence, in the case $p+q=11$, each section intersects an even number of 1-fibers $n_1,\ldots,n_8$. If $q=12$, the fibers split into three groups, $n_s,\ldots,n_{s+3}$, $s=1,5,9$, and the intersections of each section with these groups are either all even or all odd.

Formally, Proposition 4.10 still holds if $\operatorname{char} \Bbbk = 2$ and X is supersingular, but the statement becomes void as $p+q \leqslant 8$ in this case (see Proposition 7.3 below).

Corollary 4.11. *Assume that* $\operatorname{char} \Bbbk \neq 2$ *or* X *is not supersingular, and let* $\mathcal{P}$ *be a pencil of type* $(0,q)$*. If* $q \geqslant 11$*, a section* s *of* $\mathcal{P}$ *can intersect at most seven fibers of* $\mathcal{P}$*; if* $q=12$*, a section can intersect at most six fibers.*

Proof. A section cannot intersect more than eight fibers by Lemma 4.3. If $q=11$ and s intersects eight fibers, then, up to reordering, these fibers are either $n_1,\ldots,n_8$ or $n_1,\ldots,n_6,n_9,n_{10}$, see Proposition 4.10. In the former case, $\tilde{P}$ contains a vector as in Theorem 3.3(2), *viz.*

$$e := -h + l + \tfrac{1}{2}(n_1+\ldots+n_8) + s;$$

in the latter case, it contains an exceptional divisor $e-n_7-n_8$. Similarly, if $q=12$ and s intersects seven fibers, these fibers can be chosen $n_1,n_2,n_3,n_5,n_6,n_7,n_9$; then, the vector

$$-h + l + \tfrac{1}{2}(n_1+n_2+n_3-n_4+n_5+n_6+n_7-n_8) + s$$

is an exceptional divisor. □

5. Triangle free configurations

A configuration S is said to be *triangle free* if the graph $\operatorname{Fn} S$ has no cycles of length 3. According to Lemma 4.1, this condition is equivalent to the requirement that S should contain no plane.

5.1. **Statement and setup.** The principal result of this section is the following theorem. The proof follows that found in [6]. In fact, in the present paper we are mainly interested in a few intermediate lemmas.

Theorem 5.1 (see §5.5)**.** *If the configuration* $\operatorname{Fn} X$ *is triangle free and* $\operatorname{char} \Bbbk \neq 2$*, then* $|\operatorname{Fn} X| \leqslant 61$*.*

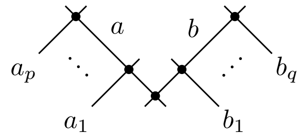

FIGURE 4. The configuration in Lemma 5.5

Remark 5.2. Probably, the bound given by Theorem 5.1 is not sharp; the best known example is a supersingular quartic X in characteristic 7 with 47 lines:

- $\mathfrak{p} = (0,12)^{20}(0,10)^{24}(0,8)^3$, $\sigma(X) = 1$, and $\operatorname{rk}\mathcal{F}(X) = 21$.

The best (known to me) examples over other fields are 33 lines if $\Bbbk = \mathbb{C}$ and 37 lines if $\operatorname{char}\Bbbk = 3$ (see Proposition 6.11 below).

Throughout this section, we use the following simple observation. Let $\tilde{D} \subset \operatorname{Fn} X$ be an affine Dynkin diagram, and let $\sum_i r_i l_i$, $l_i \in \tilde{D}$, be the positive generator of $\ker \mathbb{Z}\tilde{D}$. This generator, regarded as a divisor, is obviously primitive, effective, and numerically effective; hence, the lines in $\tilde{D}$ constitute a reducible fiber of a (quasi-) elliptic pencil $\pi := \pi_{\tilde{D}} \colon X \to \mathbb{P}^1$, see §3.3. Any other line $l \in \operatorname{Fn} X$ either is in a reducible singular fiber of π or intersects each fiber of π. Thus,

$$|\operatorname{Fn} X| = \ln(\pi) + |\text{lines adjacent to a vertex of } \tilde{D}|. \tag{5.3}$$

Another fact used freely without further references is that, in a triangle free configuration $\operatorname{Fn} X$, a pencil of the form $\mathcal{P}(l)$, $l \in \operatorname{Fn} X$, cannot have 3-fibers. Hence, all pencils are of type $(0, q)$, $q \leqslant 12$, and $\operatorname{val} l \leqslant 12$ for any line $l \in \operatorname{Fn} X$.

5.2. **Locally elliptic configurations.** A configuration S or graph $\operatorname{Fn} S$ are called *locally elliptic* if $\operatorname{val} l \leqslant 3$ for each line $l \in \operatorname{Fn} S$.

Lemma 5.4. *If the graph* $\operatorname{Fn} X$ *is locally elliptic, then* $|\operatorname{Fn} X| \leqslant 31$.

Proof. If $\operatorname{Fn} X$ is elliptic, then $|\operatorname{Fn} X| \leqslant \operatorname{rk} NS(X) \leqslant 22$. If $\operatorname{Fn} X$ contains a plane $a_1, \ldots, a_4$, then $|\operatorname{Fn} X| = 4$, as any other line would increase the valency of one of a_i, see Lemma 4.1. Thus, assume that $\operatorname{Fn} X$ is triangle free and contains an affine Dynkin diagram, which cannot be of type $\tilde{\mathbf{D}}_4$. Choose a diagram $\tilde{D} \subset \operatorname{Fn} X$ such that $\mu(\tilde{D})$ is minimal possible and let $\pi := \pi_{\tilde{D}}$, see §5.1.

Analyzing affine Dynkin diagrams one-by-one and using the minimality of $\tilde{D}$, one can easily see that, unless $\tilde{D} = \tilde{\mathbf{D}}_5$, the last term in (5.3) is bounded by 6. If the pencil π is elliptic, then $\ln(\pi) \leqslant 24$ by Corollary 3.9. If π is quasi-elliptic, Theorem 3.7 implies that $\operatorname{char}\Bbbk = 2$ and the parabolic fibers F_i of π are of type $\tilde{\mathbf{D}}_{2k}$, $k \geqslant 3$, $\tilde{\mathbf{E}}_7$, or $\tilde{\mathbf{E}}_8$, with $\sum \mu(F_i) \leqslant 20$. The number of parabolic fibers is at most 4 and $\ln(\pi) \leqslant 24$, see Corollary 3.9. In any case, $|\operatorname{Fn} X| \leqslant 30$.

If $\tilde{D} = \tilde{\mathbf{D}}_5$, two vertices can be attached to each of the four monovalent vertices of $\tilde{D}$. However, if a line $s \in \operatorname{Fn} X$ is a section of π, then π has at most three parabolic fibers, as otherwise $\operatorname{val} s \geqslant 4$. As the types of the parabolic fibers are $\tilde{\mathbf{D}}_5$ or $\tilde{\mathbf{A}}_7$, with at least one $\tilde{\mathbf{D}}_5$, it follows that at least one fiber is elliptic; hence, we have $\ln(\pi) \leqslant 23$, see (3.8), and $|\operatorname{Fn} X| \leqslant 31$. □

5.3. **Quadrangle-free configurations.** A configuration S (or graph $\operatorname{Fn} S$) is said to be *quadrangle free* if $\operatorname{Fn} S$ has no cycles of length 3 or 4.

Lemma 5.5. *Assume that lines $a, b, a_1, \ldots, a_p, b_1, \ldots, b_q \in \operatorname{Fn} X$ intersect as shown in Figure 4,* i.e.,

$$a \cdot b = a \cdot a_i = b \cdot b_i = 1, \ a \cdot b_i = a_i \cdot b = a_i \cdot b_j = 0$$

for all $i = 1, \ldots, p$ and $j = 1, \ldots, q$. Then, up to reordering the pair (p, q), one has either $p, q \leqslant 6$, or $p = 4$ and $q \leqslant 8$, or $p \leqslant 3$ and $q \leqslant 11$.

Furthermore, in the case $p = q = 6$, if the configuration $\operatorname{Fn} X$ is quadrangle free and $\operatorname{char} \Bbbk \neq 2$, one also has $\operatorname{val} a_i \leqslant 5$, $\operatorname{val} b_i \leqslant 5$ for all $i = 1, \ldots, 6$.

Proof. If $p = q = 6$, we have a relation

$$2a + a_1 + \ldots + a_6 = 2b + b_1 + \ldots + b_6. \tag{5.6}$$

Hence, $p \geqslant 5$ implies $q \leqslant 6$, *cf.* Remark 4.5. If $p = 4$ and $q \geqslant 9$, then

$$e := h + 2a + 2a_1 + 2a_2 + a_3 + a_4 - 3b - b_1 - \ldots - b_9$$

is an exceptional divisor, see Theorem 3.3(1). Finally, one always has $q \leqslant 11$, as the pencil $\mathcal{P}(b)$ cannot have more than twelve fibers, see Proposition 4.8.

For the last statement, due to relation (5.6), it suffices to show that $\operatorname{val} a_1 < 7$. By the same relation, any line $c \in \mathcal{P}(a_1) \smallsetminus a$ intersects exactly one of b_i and is disjoint from all other lines. Assume that there are six such lines $c_1, \ldots, c_6$, so that $c_i \cdot b_j = \delta_{ij}$, and consider the sublattice $S \subset NS(X)$ generated by h and all lines a, b, a_i, b_i, c_i. We have $\operatorname{rk} S = 19$ and $\operatorname{discr}_2 S$ is the $\mathbb{F}_2$-vector space generated by $\frac{1}{2}(h + a + b + a_1 + \ldots + a_6)$ and $\frac{1}{2}(a_i + a_{i+1})$, $i = 2, \ldots, 5$. The isotropic vectors are those of the form $\frac{1}{2}(a_2 + \ldots + a_6 - a_i)$, $i = 2, \ldots, 6$, and they are all represented by exceptional divisors, see Theorem 3.3(1). Hence, S has no admissible extensions of index 2 and, by Theorem 3.5, S does not admit a geometric realization. □

Lemma 5.7. *If $\operatorname{char} \Bbbk \neq 2$ and $\operatorname{Fn} X$ is quadrangle free, then $|\operatorname{Fn} X| \leqslant 44$.*

Proof. Let $l_0 \in \operatorname{Fn} X$ be a line of maximal valency. In view of Lemma 5.4, we can assume that $\operatorname{val} l_0 \geqslant 4$. Choose four lines $l_1, \ldots, l_4 \in \operatorname{Fn} X$ adjacent to l_0; together with l_0, they constitute a type $\tilde{\mathbf{D}}_4$ affine Dynkin diagram contained in $\operatorname{Fn} X$. Applying (5.3) to the pencil $\pi := \pi_{\tilde{D}} \colon X \to \mathbb{P}^1$, we have

$$|\operatorname{Fn} X| = \ln(\pi) + \operatorname{val} l_0 + \operatorname{val} l_1 + \ldots + \operatorname{val} l_4 - 8.$$

The valencies are estimated by applying Lemma 5.5 to $a = l_i$, $i = 1, \ldots, 4$, and $b = l_0$ (recall that we assume $\operatorname{val} l_0 \geqslant \operatorname{val} l_i$). In the worst case, where $\operatorname{val} l_i = 6$, $i = 0, \ldots, 4$, we have $\sum_i \operatorname{val} l_i = 30$; otherwise, $\sum_i \operatorname{val} l_i \leqslant 29$.

The parabolic fibers of π are of types $\tilde{\mathbf{D}}_4$ or $\tilde{\mathbf{A}}_5$, with at least one $\tilde{\mathbf{D}}_4$, and its elliptic fibers are of linear types $\mathbf{A}_p$, $p \leqslant 4$, or $\mathbf{A}_1^2$. Thus, we have $\ln(\pi) \leqslant 23$, see Corollary 3.9. We assert that, if $\operatorname{val} l_0 = 6$, the pencil has at most three parabolic fibers and, hence, $\ln(\pi) \leqslant 22$; these inequalities imply $|\operatorname{Fn} X| \leqslant 44$ in the statement. For the last assertion, let $n_1, n_2 \neq l_1, \ldots, l_4$ be the two extra lines adjacent to l_0. They are bisections of π. If F is another type $\tilde{\mathbf{D}}_4$ fiber, then either one of n_1, n_2 intersects two monovalent vertices of F or both lines intersect the central vertex; in either case, we obtain a quadrangle. If F_1, F_2, F_3 are three type $\tilde{\mathbf{A}}_5$ fibers, then, each n_i intersecting two lines in each F_j, we obtain $\operatorname{val} n_i \geqslant 7$, which contradicts to our choice of l_0. □

5.4. **Configurations with a quadrangle.** Assume that a triangle free configuration $\operatorname{Fn} X$ contains a quadrangle $Q := \{l_1, l_2, l_3, l_4\}$. (The lines constituting a quadrangle are always listed according to their cyclic order in the affine Dynkin diagram). Let $\pi_Q\colon X \to \mathbb{P}^1$ be the corresponding elliptic pencil. It has a certain number s_3 of parabolic fibers of type $\tilde{\mathbf{A}}_3$ and, for $p = 1, 2$, a certain number s_p of elliptic fibers of linear type $\mathbf{A}_p$. By Corollary 3.9, we have

$$\ln(\pi_Q) = 4s_3 + 2s_2 + s_1, \qquad 4s_3 + 3s_2 + 2s_1 \leqslant 24. \tag{5.8}$$

Identity (5.3) becomes

$$|\operatorname{Fn} X| = \ln(\pi_Q) + \operatorname{val} l_1 + \ldots + \operatorname{val} l_4 - \mathrm{d}(l_1, l_3) - \mathrm{d}(l_2, l_4) - 8, \tag{5.9}$$

where the correction terms $\mathrm{d}(l_i, l_j) := |\mathcal{P}(l_i) \cap \mathcal{P}(l_j)| - 2$ are nonnegative.

Lemma 5.10 (see [6])**.** *If* $\operatorname{char} \Bbbk \neq 2$ *and* $\operatorname{Fn} X$ *is triangle free, then* $\ln(\pi_Q) \leqslant 21$.

Proof. In view of (5.8), we only need to eliminate the triples $(s_3, s_2, s_1) = (6, 0, 0)$, $(5, 1, 0)$, and $(5, 0, 2)$, the former being a consequence of the two latter. To this end, we consider the lattice S generated by h and the lines in the fibers, use an analog of Lemma 4.7 to estimate from below the 2-torsion of the discriminants of admissible finite index extensions $\tilde{S} \supset S$ such that $\operatorname{Fn} \tilde{S}$ is still triangle free, and apply Theorem 3.5 (the part related to $p = 2$) to show that $\tilde{S}$ does not admit a primitive geometric realization. Details are left to the reader. □

Now, the following lemma is an immediate consequence of (5.9), Lemma 5.10, and the bound $\operatorname{val} l \leqslant 12$ for each line $l \in \operatorname{Fn} X$.

Lemma 5.11. *If* $\operatorname{char} \Bbbk \neq 2$ *and* $\operatorname{Fn} X$ *is triangle free and contains a quadrangle, then* $|\operatorname{Fn} X| \leqslant 61$.

5.5. **Proof of Theorem 5.1.** If X is not supersingular, we can use Theorem 3.2 and assume that $\operatorname{char} \Bbbk = 0$. Hence, depending on the type of the configuration $\operatorname{Fn} X$, the statement of the theorem follows from Lemma 5.11 ($\operatorname{Fn} X$ contains a quadrangle), Lemma 5.7 ($\operatorname{Fn} X$ is quadrangle free and has a line of valency at least 4) and Lemma 5.4 ($\operatorname{val} l \leqslant 3$ for all $l \in \operatorname{Fn} X$). □

6. Exotic pencils

Exotic are pencils contained in a supersingular quartic over an algebraically closed field of characteristic 3. The most interesting feature of such pencils is the fact that the existence of a pencil of a certain type does not guarantee the existence of a maximal pencil of any smaller type, see Proposition 6.13 below.

6.1. **Quasi-elliptic pencils.** According to Proposition 4.8, a pencil of the form $\pi[l]$, $l \in \operatorname{Fn} X$, is quasi-elliptic if and only if $\operatorname{char} \Bbbk = 3$ and $\mathcal{P}(l)$ is of type $(10, 0)$. Since $\operatorname{rk} P_{10,0} = 22$, the lattice $NS(X)$ is a finite index extension of $P_{10,0}$; its kernel is denoted by $\mathcal{X} := NS(X)/P_{10,0} \subset \operatorname{discr} P_{10,0}$. Note that $\mathcal{X}$ is an $\mathbb{F}_3$-vector space, as so is $\operatorname{discr} P_{10,0}$, see Observation 4.6.

Proposition 6.1. *The map* $s \mapsto (s \bmod P_{10,0}) \in \mathcal{X}$ *establishes a bijection between the set of sections of* $\mathcal{P}(l)$ *and the set* $\{\alpha \in \mathcal{X} \mid \alpha \cdot \lambda = \frac{2}{3} \bmod \mathbb{Z}\} = \mathcal{X} \cap \Omega_0$.

Proof. Since $NS(X) \subset P_{10,0}^\vee$, each section represents an element $\alpha \in \mathcal{X}$ as in the statement. Distinct sections represent distinct elements by Lemma 4.3 and, by the same lemma, any element in the $O_h(P_{10,0})$-orbit of $\omega + 6\lambda$ is a section. □

Remark 6.2. If $s_1 \neq s_2$ are two sections, then $s_1 - s_2 \bmod P_{10,0}$ is an isotropic vector orthogonal to λ. By Observation 4.6 and Lemma 4.7, this difference is in $\mathrm{M}_6 \cup \mathrm{M}_9$. Each section intersects all ten fibers of $\mathcal{P}(l)$, see Lemma 4.1; hence, applying Lemma 4.3 (see also Remark 4.5) to l, s_1, and the ten common lines, we conclude that $s_1 \cdot s_2 = 1$ if and only if $s_1 - s_2 \in \mathrm{M}_9$.

Corollary 6.3. *Let $\mathcal{P} \subset \operatorname{Fn} X$ be a pencil of type $(10, 0)$. Then*

- $\mathrm{s}(\mathcal{P}) = 0$ *or* 3^r, $0 \leqslant r \leqslant 4$; *respectively,* $|\operatorname{Fn} X| = 31$ *or* $31 + 3^r$ *and, in the latter case, one has* $\sigma(X) = 5 - r$.

There is a unique configuration of size 112 *and two configurations of size* 58; *they are as in Theorem 1.2(1) and (2), (3), respectively.*

Proof. According to Proposition 6.1, the sections of $\mathcal{P}$ constitute the affine subspace $\mathcal{X} \cap \Omega_0$; it is either empty or has dimension $r := \dim \mathcal{X} - 1 \leqslant 4$. In the latter case, up to automorphism, we have $\mathcal{X} = \mathbb{F}_3\omega \oplus \mathcal{X}'$, where $\mathcal{X}' := \mathcal{X} \cap \omega^\perp \cong (\mathcal{X} \cap \lambda^\perp)/\lambda$ is a ternary code of length 10 with all Hamming norms 6 or 9 (see Lemma 4.7). The number of such codes of dimension 1, 2, 3, 4 is, respectively, 2, 3, 2, 1. (The statement on codes of dimension 4, *i.e.*, the uniqueness of the Fermat quartic, can also be established by other means, see Remark 6.15 below.) □

We refer to §8.4 for a geometric description of quasi-elliptic pencils.

6.2. **The lattice $NS(X)$.** We often need to establish the (non-)uniqueness in the genus of the orthogonal complement of $P_{p,q}$ in $NS(X)$. To do so, we use the results of §2.4 and start with a "minimal" lattice T, *i.e.*, the one with largest discriminant. More precisely, Proposition 2.4 and Observation 4.6 imply that

$$\operatorname{discr}_2 T \cong \langle \tfrac{1}{2} \rangle^q, \qquad \operatorname{discr}_r T = 0 \quad \text{for } r > 3.$$

The 9-torsion is also determined by (p, q): in the terminology of §2.4, we have

$$(3 \operatorname{discr}_3 T, q_\circ) \cong (3 \operatorname{discr}_3 P_{p,q}, -q_\circ).$$

However, the 3-elementary part of $\operatorname{discr}_3 T$ is not fixed, and we make it as large as possible. Then, $NS(X)$ is a finite index extension of $N := P_{p,q} \oplus T$, and it this extension that we try to describe by means of its kernel

$$\mathcal{X} := NS(X)/N \subset \operatorname{discr} N = \operatorname{discr} P_{p,q} \oplus \operatorname{discr} T.$$

We reserve this notation till the end of the section. Clearly, $\mathcal{X} = \bigoplus_r \mathcal{X}_r$, where we let $\mathcal{X}_r := \mathcal{X} \cap \operatorname{discr}_r N$ for a prime r. Then, $\mathcal{X}_r = 0$ for $r > 3$ and $\mathcal{X}_2$ is the graph of a bijective anti-isometry

$$\psi_2 \colon \operatorname{discr}_2 P_{p,q} \longrightarrow \operatorname{discr}_2 T.$$

(The case where the quotient $NS(X)/P_{p,q}$ has 2-torsion is discussed separately in §6.6.) Our principal concern is the natural map

$$\mathrm{sec}_3 \colon \{\text{sections of } \mathcal{P}\} \longrightarrow \mathcal{X}_3$$

sending a section s to the projection of $(s \bmod N)$ to $\mathcal{X}_3$. All nonempty fibers of the composition of sec_3 and the projection to $\operatorname{discr}_3 P_{p,q}$ are over the affine space

$$\Omega_* := \bigl\{\alpha \in \operatorname{discr}_3 P_{p,q} \bigm| \alpha \cdot \lambda = -\tfrac{1}{3} r\bigr\} = \Omega_0 \cup \Omega_+ \cup \Omega_-.$$

Since we do not always assume a pencil $\mathcal{P} \subset \mathcal{P}(l)$ maximal, there is a similar map

$$\mathrm{fib}_3 \colon \mathcal{P}(l) \smallsetminus \mathcal{P} \longrightarrow \mathcal{X}_3$$

with all nonempty fibers over $-\Lambda \cup \mathrm{M}_0$.

We make a few general observations. Fix an anti-isometry ψ_2 as above; it defines an extension $N_3 \supset N$ with $\operatorname{discr}_r N_3 = 0$ for $r \neq 3$. Moreover, if $p + q = 1 \bmod 3$ (which we usually assume), the lattice N_3 is 3-elementary. Recall that the image of $O_h(P_{p,q})$ in $\operatorname{Aut}\operatorname{discr}_2 P_{p,q}$ is the symmetric group $\mathbb{S}_q$, see Observation 4.6. Let

$$O(T,\psi_2) := \bigl\{ g \in O(T) \bigm| \psi_2^{-1} \circ \bar{g} \circ \psi_2 \in \mathbb{S}_q \bigr\},$$

where $\bar{g}$ is the image of g in $\operatorname{Aut}\operatorname{discr}_2 T$. Then, since the actions of $O_h(P_{p,q})$ on $\operatorname{discr}_2 P_{p,q}$ and $\operatorname{discr}_3 P_{p,q}$ are independent, see Observation 4.6, the action of the group $O_h(N_3)$ on $\operatorname{discr} N_3$ reduces to the product action of $O_h(P_{p,q}) \times O(T,\psi_2)$, so that the orbits of elements of order 3 are products of the form

$$\mathrm{M}_k \times \mathrm{O},\ k = 0,\ldots,p, \quad \pm\Lambda \times \mathrm{O}, \quad \pm\Omega_s \times \mathrm{O},\ s \in \mathbb{F}_3,$$

where $\mathrm{O} \subset \operatorname{discr}_3 T$ is an $O(T,\psi_2)$-orbit. When describing the fibers of the maps sec_3 and fib_3 and discussing the admissibility of the extension (the non-existence of exceptional divisors), it suffices to check a single representative of each orbit.

Another observation concerns the choice of the anti-isometry ψ_2. We have the following obvious lemma.

Lemma 6.4. *Assume that a generator $3\nu_k \in \operatorname{discr}_2 P_{p,q}$, $k = 1,\ldots,q$, is such that the image $\psi_2(3\nu_k)$ is represented by $\frac{1}{2}a$, where $a \in T$, $a^2 = -6$. Then, the vectors*

$$n_k + 3\nu_k \pm \tfrac{1}{2}a = -\lambda + \nu_k \pm \tfrac{1}{2}a \in NS(X)$$

are lines that belong to the maximal pencil $\mathcal{P}(l)$ containing $\mathcal{P}$. As a consequence, the k-th 1-fiber of $\mathcal{P}$ becomes a 3-fiber of $\mathcal{P}(l)$ and $\mathcal{P}$ is not maximal.

6.3. **Pencils of type** $(7,0)$**.** By Proposition 4.9, any pencil of type $(p,q) = (7,0)$ is exotic, and the minimal lattice T is given by Lemma 2.9 as $\bar{\mathbf{E}}_6 := \mathbf{E}_6^\vee(3)$. The homomorphism $O(\bar{\mathbf{E}}_6,\psi_2) = O(\bar{\mathbf{E}}_6) = O(\mathbf{E}_6) \to \operatorname{Aut}\operatorname{discr}\bar{\mathbf{E}}_6$ is an isomorphism; hence, the $O(\bar{\mathbf{E}}_6)$-orbits in $\operatorname{discr}\bar{\mathbf{E}}_6$ are $\bar{\mathrm{E}}_0 := \{0\}$ and

$$\bar{\mathrm{E}}_r := \bigl\{ \alpha \in \operatorname{discr}\bar{\mathbf{E}}_6 \bigm| \alpha \neq 0,\ \alpha^2 = -\tfrac{2}{3}r \bmod 2\mathbb{Z} \bigr\}, \quad r = 1,2,3,$$

and each $\alpha \in \bar{\mathrm{E}}_r$ has a representative of the form $\frac{1}{3}a$, where $a \in \bar{\mathbf{E}}_6$, $a^2 = -6r$. The following statement is immediate, *cf.* Lemma 4.7 or Proposition 6.1: one can easily check all orbits one by one.

Proposition 6.5. *Let $\mathcal{P} \subset \operatorname{Fn} X$ be a pencil of type $(7,0)$. Then the kernel $\mathcal{X}$ is disjoint from the $O_h(N)$-orbits*

$$\mathrm{M}_0 \times \bar{\mathrm{E}}_3, \quad \mathrm{M}_1 \times \bar{\mathrm{E}}_2, \quad \mathrm{M}_2 \times \bar{\mathrm{E}}_1, \quad \mathrm{M}_3 \times \bar{\mathrm{E}}_0, \quad \pm\Lambda \times \bar{\mathrm{E}}_0, \quad \pm\Omega_0 \times \bar{\mathrm{E}}_0.$$

The map sec_3 is a bijection onto $\mathcal{X} \cap (\Omega_1 \times \bar{\mathrm{E}}_1)$. The pencil $\mathcal{P}$ is not maximal if and only if it extends to a quasi-elliptic pencil if and only if $\mathcal{X} \cap (\Lambda \times \bar{\mathrm{E}}_3) \neq \varnothing$.

Corollary 6.6. *Let $\mathcal{P} \subset \operatorname{Fn} X$ be a maximal pencil of type $(7,0)$. Then:*

- $\mathrm{s}(\mathcal{P}) = 36$ *or* $\mathrm{s}(\mathcal{P}) \leqslant 27$; *respectively,* $|\operatorname{Fn} X| = 58$ *or* $|\operatorname{Fn} X| \leqslant 49$;
- *if* $\mathrm{s}(\mathcal{P}) > 5$*, then* $\mathrm{s}(\mathcal{P}) = 0 \bmod 3$.

These bounds are sharp, and there is a unique configuration $\operatorname{Fn} X$ of size 58; it is as in Theorem 1.2(4).

Proof. By Proposition 6.5, we have $\mathcal{X} \cap \operatorname{discr}\bar{\mathbf{E}}_6 = 0$; hence, $\mathcal{X}$ is the graph of a certain anti-isometry $\psi\colon \mathcal{D} \to \operatorname{discr}\bar{\mathbf{E}}_6$, where the domain $\mathcal{D} \subset \operatorname{discr} P_{7,0}$ is disjoint from Λ, M_1, M_2. The projection $\mathcal{D} \cap \lambda^\perp \to \lambda^\perp/\lambda$ is injective and its image is a ternary code of length 7 and minimal Hamming distance 3; it has dimension at most 4. Hence, $\dim\mathcal{D}\leqslant 5$. The sections of $\mathcal{P}$ are in a one-to-one correspondence with the vectors $\alpha\in\mathcal{D}':=\mathcal{D}\cap\Omega_*$ such that $\psi(\alpha)\in\bar{\mathrm{E}}_1$, *i.e.*, $\alpha^2=\frac{2}{3}$. Thus, the congruence in the statement follows from Lemma 2.10.

If $\dim\mathcal{D}\leqslant 4$, then $\mathrm{s}(\mathcal{P})\leqslant|\mathcal{D}'|\leqslant 27$; henceforth, we assume that $\dim\mathcal{D}=5$.

If $\operatorname{Ker}\psi=0$, then ψ restricts to an injective map from $\mathcal{D}'$ to a proper affine subspace $\mathcal{E}'\subset\operatorname{discr}\bar{\mathbf{E}}_6$ disjoint from 0. For any such space, $|\mathcal{E}'\cap\bar{\mathrm{E}}_1|\leqslant 27$.

Assume that $\operatorname{Ker}\psi\neq 0$. By Proposition 6.5 (or Lemma 4.7), $\dim\operatorname{Ker}\psi=1$ and $\operatorname{Ker}\psi$ is generated by an element of M_6. In this case, ψ restricts to a three-to-one map $\mathcal{D}'\twoheadrightarrow\mathcal{E}'$, where $\mathcal{E}'\subset\operatorname{discr}\bar{\mathbf{E}}_6$ is an affine subspace disjoint from 0 and $\dim\mathcal{E}'=3$. With one exception, one has $|\mathcal{E}'\cap\bar{\mathrm{E}}_1|\leqslant 9$. In the exceptional case, $|\mathcal{E}'\cap\bar{\mathrm{E}}_1|=12$ (hence $\mathrm{s}(\mathcal{P})=36$), both spaces $\mathcal{E}_0:=\psi(\lambda^\perp)\subset\mathcal{E}:=\operatorname{Im}\psi$ are nondegenerate, $\operatorname{Br}\mathcal{E}_0=2$, and $\operatorname{Br}\mathcal{E}=0$; in other words,

$$\mathcal{E}_0\cong\left\langle\tfrac{4}{3}\right\rangle^3\subset\mathcal{E}=\mathcal{E}_0\oplus\left\langle\tfrac{4}{3}\right\rangle\subset\operatorname{discr}\bar{\mathbf{E}}_6=\mathcal{E}\oplus\left\langle\tfrac{2}{3}\right\rangle.$$

On the other hand, in the space $\operatorname{discr}_3 P_{7,0}$, there is a unique pair $(\mathcal{D},\mathcal{D}\cap\lambda^\perp)$ satisfying all the assumptions above and anti-isomorphic (modulo kernel) to $(\mathcal{E},\mathcal{E}_0)$. Since the stabilizer of $(\mathcal{E},\mathcal{E}_0)$ restricts to the full group $\operatorname{Aut}\mathcal{E}_0$, the anti-isometry $\psi\colon\mathcal{D}\to\mathcal{E}$ is also unique; it gives rise to a quartic as in the statement. ☐

6.4. **Exotic pencils of type** $(4,6)$**.** Consider a pencil $\mathcal{P}$ of type $(4,6)$ and *assume* that it is exotic. By Lemma 2.8, the minimal orthogonal complement T of $P_{4,6}$ is $\mathbf{H}_6(6)$, which is unique in its genus. The $O(T)$-orbits in $\operatorname{discr}_3 T$ are

$$\mathrm{H}_r:=\left\{\alpha\in\operatorname{discr}_3 T\;\middle|\;\|\alpha\|=r\right\},\quad r=0,\ldots,6,$$

where $\|\cdot\|$ is the Hamming norm in the obvious basis.

By Lemma 2.11, there are two essentially distinct choices for the anti-isometry $\psi_2\colon\operatorname{discr}_2 P_{4,6}\to\operatorname{discr}_2 T$, *viz.* the identity and $\mathfrak{t}_1$. In the former case, $\mathcal{P}$ extends to a quasi-elliptic pencil, see Lemma 6.4. Thus, from now on we assume that $\psi_2=\mathfrak{t}_1$; such a pencil $\mathcal{P}\subset NS(X)$ is called *non-trivially* exotic. We have $O(T,\psi_2)=O(T)$, and, as above, the following statement is straightforward.

Proposition 6.7. *Let $\mathcal{P}\subset\operatorname{Fn}X$ be a non-trivially exotic pencil of type $(4,6)$. Then the kernel $\mathcal{X}$ is disjoint from the $O_h(N)$-orbits*

$$\mathrm{M}_0\times\mathrm{H}_3,\quad\mathrm{M}_1\times\mathrm{H}_*,\quad\mathrm{M}_2\times\mathrm{H}_1,\quad\mathrm{M}_3\times\mathrm{H}_0,\quad\pm\Lambda\times\mathrm{H}_*,\quad\pm\Omega_1\times\mathrm{H}_1$$

(*where $*$ stands for whichever index is appropriate*)*. Furthermore, the map $\sec_3$ is*

- *three-to-one over $\mathcal{X}\cap(\Omega_2\times\mathrm{H}_2)$ and*
- *one-to-one over $\mathcal{X}\cap(\Omega_0\times\mathrm{H}_*)$ and $\mathcal{X}\cap(\Omega_1\times\mathrm{H}_4)$;*

all other fibers are empty. The pencil $\mathcal{P}$ is not maximal if and only if it extends to a quasi-elliptic pencil if and only if $\mathcal{X}\cap(\mathrm{M}_0\times\mathrm{H}_6)\neq\varnothing$.

Corollary 6.8. *Let $\mathcal{P}\subset\operatorname{Fn}X$ be a maximal exotic pencil of type $(4,6)$. Then*:

- $\mathrm{s}(\mathcal{P})=39$ *or* $\mathrm{s}(\mathcal{P})\leqslant 33$; *respectively,* $|\operatorname{Fn}X|=58$ *or* $|\operatorname{Fn}X|\leqslant 52$;
- *if* $\mathrm{s}(\mathcal{P})>19$*, then* $\mathrm{s}(\mathcal{P})=0\bmod 3$.

These bounds are sharp. If $|\operatorname{Fn}X|=58$, then $\operatorname{Fn}X$ contains a pencil of type $(10,0)$ or $(7,0)$ and, hence, is as in Theorem 1.2(3) or (4).

Proof. Consider the projections $\mathcal{D}\subset\operatorname{discr}_3 P_{4,6}$ and $\mathcal{H}\subset\operatorname{discr}_3 T$ of the kernel $\mathcal{X}_3$ to the two summands of $\operatorname{discr}N$. Since $\mathcal{P}$ is assumed maximal, Proposition 6.7

implies that $\mathcal{X}_3 \cap \operatorname{discr}_3 T = 0$; hence, $\mathcal{X}_3$ is the graph of a certain anti-isometry $\psi\colon \mathcal{D} \twoheadrightarrow \mathcal{H}$. Conversely, an anti-isometry $\psi\colon \mathcal{D} \twoheadrightarrow \mathcal{H}$ gives rise to a nonsingular quartic if and only if

(1) $\mathcal{D}$ is disjoint from the orbits M_1 and $\pm\Lambda$,
(2) $\mathcal{H}$ is disjoint from the orbit H_1; in other words, $\mathcal{H} \subset \operatorname{discr}_3 T$ is a ternary code of minimal Hamming distance at least 2, and
(3) $\operatorname{Ker}\psi$ is disjoint from the orbit M_3.

The first condition implies that $\dim \mathcal{D} \leqslant 4$ and, up to the action of $O_h(P_{4,6})$, there are three subspaces of dimension 4, *viz.*

$$\mathcal{D}_r := \bigl[\mathbb{F}_3(\mu_1 + \ldots + \mu_4) \oplus \mathbb{F}_3(\omega + r\lambda)\bigr]^\perp, \quad r \in \mathbb{F}_3.$$

Let $\mathcal{D}' := \mathcal{D} \cap \Omega_*$. Since $\operatorname{Ker}\psi \cap \lambda^\perp = 0$, the restriction $\psi|_{\mathcal{D}'}$ is a one-to-one map onto an affine subspace $\mathcal{H}' \subset \mathcal{H}$. Denote $h_r := |\mathcal{H}' \cap \mathrm{H}_r|$, $r = 0, \ldots, 6$; we have $h_1 = 0$ and $\mathrm{s}(\mathcal{P})$ is given by Proposition 6.7 as

$$\mathrm{s}(\mathcal{P}) = h_0 + 3h_2 + h_3 + h_4 + h_6. \tag{6.9}$$

If $\dim \mathcal{D} \leqslant 3$, then $\dim \mathcal{D}' \leqslant 2$ and, even if each element of $\mathcal{D}'$ triples, we have $\mathrm{s}(\mathcal{P}) \leqslant 27$. Therefore, we consider the three spaces $\mathcal{D}_r$, $r \in \mathbb{F}_3$.

Let $\mathcal{D} = \mathcal{D}_r$ and $r \neq 0$. Then $\omega_r := r\omega - \lambda \in \mathcal{D}$ and $\mathcal{D}'$ is given intrinsically as

$$\mathcal{D}' = \bigl\{\alpha \in \mathcal{D} \bigm| \alpha \cdot \omega_r = -\tfrac{1}{3}\bigr\}.$$

Specializing r to ± 1, we have $\operatorname{Br}\mathcal{D}_r = 2 + 2r$ and $\omega_r^2 = \frac{2}{3}r$; since $\operatorname{Ker}\psi = 0$, these invariants determine the isomorphism type of $(\mathcal{H}, \beta) := \psi(\mathcal{D}_r, \omega_r)$. Then, for each $O(T)$-orbit of pairs $(\mathcal{H}, \beta)$, the number of sections $\mathrm{s}(\mathcal{P})$ is given by (6.9). With condition (2) taken into account, we have

- six orbits, if $r = 1$, and then $\mathrm{s}(\mathcal{P}) = 39, 33, 27, 27, 24, 21$, or
- three orbits, if $r = -1$, and then $\mathrm{s}(\mathcal{P}) = 24, 21, 18$.

Let $\mathcal{D} = \mathcal{D}_0 \cong (\mathcal{D}_0/\ker) \oplus \mathbb{F}_3\omega$, $\omega^2 = 0$. In this case, we have $\operatorname{Br}(\mathcal{D}/\ker) = 2$ and $\mathcal{D}' = (\mathcal{D}/\ker) + \omega$. If $\operatorname{Ker}\psi = 0$, the pair $(\mathcal{H}, \mathcal{H}')$ is anti-isomorphic to $(\mathcal{D}, \mathcal{D}')$; otherwise, ψ maps $\mathcal{D}'$ onto $\mathcal{H}' = \mathcal{H}$, which is anti-isomorphic to $\mathcal{D}/\ker$. Taking condition (2) into account and using (6.9), we obtain

- four orbits, if $\operatorname{Ker}\psi = 0$, and then $\mathrm{s}(\mathcal{P}) = 30, 27, 24, 21$, or
- four orbits, if $\operatorname{Ker}\psi \neq 0$, and then $\mathrm{s}(\mathcal{P}) = 39, 33, 27, 21$.

There are two cases where $\mathrm{s}(\mathcal{P}) = 39$. If $r \neq 0$, the stabilizer of $(\mathcal{D}_r, \omega_r)$ induces the full group $\operatorname{Aut}(\mathcal{D}_r, \omega_r)$. Similarly, the stabilizer of $\mathcal{D}_0$ induces the full group $\operatorname{Aut}(\mathcal{D}_0/\ker)$. Hence, in both cases, an anti-isometry $\psi\colon \mathcal{D} \twoheadrightarrow \mathcal{H}$ is unique up to the two-sided action of the group $O_h(P_{4,6}) \times O(\mathbf{H}_6)$. Choosing a representative and computing all lines, we conclude that the quartic is one of those considered above as it contains a pencil (other than $\mathcal{P}$) of type $(10, 0)$ or $(7, 0)$.

For the congruence, rewrite (6.9) in the form

$$\mathrm{s}(\mathcal{P}) = |\mathcal{H}'| + 3h_2 - (h_2 + h_5).$$

In the notation of Lemma 2.10, we have $h_2 + h_5 = n_2(\mathcal{H}')$. By the lemma, either $|\mathcal{H}'| \leqslant 9$ and $h_2 \leqslant h_2 + h_5 \leqslant 5$, and then $\mathrm{s}(\mathcal{P}) \leqslant 19$, or $\mathrm{s}(\mathcal{P}) = 0 \bmod 3$. □

6.5. **Exotic pencils of type** $(4,0)$**.** Consider a pencil $\mathcal{P}$ of type $(4,0)$ and assume that it is exotic. For the lattice $\bar{T}$ in Lemma 2.9, we have $\operatorname{discr}\bar{T} = \langle\frac{2}{3}\rangle^2$; hence, $\bar{T}$ is the orthogonal complement of a sublattice $\mathbf{A}_2^2 \subset U$, where U is even unimodular of rank 16, *i.e.*, $U = \mathbf{E}_8^2$ or $\mathbf{D}_{16}^+$ (the unique, up to isomorphism, even unimodular extension of $\mathbf{D}_{16}$). Then, the minimal orthogonal complement is $T = \bar{T}^\vee(3)$.

The sublattice $\mathbf{A}_2^2$ is embedded into the maximal root system contained in U; there are two embeddings to $\mathbf{E}_8^2$ and one embedding to $\mathbf{D}_{16}$.

If $U = \mathbf{E}_8^2$, we have $T = \bar{\mathbf{E}}_6^2$ or $\mathbf{A}_2^2 \oplus \mathbf{E}_8$ (as obviously $\mathbf{A}_2^\vee(3) \cong \mathbf{A}_2$), and the latter lattice contains (-2)-vectors, contradicting Theorem 3.3(1).

If $U = \mathbf{D}_{16}^+$ and $\mathbf{D}_{16}$ is represented as the maximal even sublattice in $\mathbf{H}_{16}$, so that $\mathbf{D}_{16}^+$ is the extension by $\frac{1}{2}\bar{\mathrm{e}}$, we can assume that the two copies of $\mathbf{A}_2 \subset \mathbf{D}_{16}$ are generated by the pairs of roots $\{\mathrm{e}_1 - \mathrm{e}_2, \mathrm{e}_2 - \mathrm{e}_3\}$ and $\{\mathrm{e}_4 - \mathrm{e}_5, \mathrm{e}_5 - \mathrm{e}_6\}$; then, $\frac{1}{3}(\mathrm{e}_1 + \ldots + \mathrm{e}_6) \in T$ is a (-2)-vector.

Thus, we are left with $T = \bar{\mathbf{E}}_6^2$. The orbits of the $O(T)$-action on $\operatorname{discr} T$ are described in §6.3; we will abbreviate $\bar{\mathrm{E}}_{r,s} := (\bar{\mathrm{E}}_r \times \bar{\mathrm{E}}_s) \cup (\bar{\mathrm{E}}_s \times \bar{\mathrm{E}}_r)$.

Proposition 6.10. *Let* $\mathcal{P} \subset \operatorname{Fn} X$ *be an exotic pencil of type* $(4,0)$*. Then*

- $\mathcal{X} \cap \operatorname{discr} P_{4,0} = 0$ *and* $\mathcal{X} \cap \operatorname{discr} T \subset \bar{\mathrm{E}}_{0,0} \cup \bar{\mathrm{E}}_{3,3}$;
- $\mathcal{X}$ *is disjoint from* $\mathrm{M}_1 \times \bar{\mathrm{E}}_{1,1}$, $\mathrm{M}_1 \times \bar{\mathrm{E}}_{0,2}$, $\mathrm{M}_2 \times \bar{\mathrm{E}}_{0,1}$, *and* $\Omega_1 \times \bar{\mathrm{E}}_{0,1}$.

Furthermore, the map $\sec_3$ *is*

- *three-to-one over* $\mathcal{X} \cap (\Omega_2 \times \bar{\mathrm{E}}_{0,2})$ *and*
- *one-to-one over* $\mathcal{X} \cap (\Omega_2 \times \bar{\mathrm{E}}_{1,1})$;

all other fibers are empty. The pencil $\mathcal{P}$ *is not maximal if and only if* $\mathcal{X}$ *intersects one of the following orbits*:

- $\Lambda \times \bar{\mathrm{E}}_{0,3}$, *and then* $\mathcal{P}$ *is contained in a pencil of type* $(7,0)$, *or*
- $\Lambda \times \bar{\mathrm{E}}_{1,2}$, *and then* $\mathcal{P}$ *is contained in a pencil of type* $(4,3)$.

As an immediate consequence, for a maximal exotic pencil of type (p,q), $p \geqslant 4$, we have either $p = 4$ and $q = 0$, 3, 6 or $(p,q) = (7,0)$ or $(10,0)$.

6.6. **Exotic pencils of type** $(0,q)$**.** Consider an exotic pencil $\mathcal{P}$ of type $(0,10)$. Here, the situation is much more diverse than in the previous sections.

First, by Lemma 2.8, the minimal orthogonal complement T has the form $\bar{T}(6)$, where $\bar{T}$ is an odd unimodular lattice of rank 10, *i.e.*, $\bar{T} = \mathbf{H}_{10}$ or $\mathbf{E}_8 \oplus \mathbf{H}_2$.

Second, the intersection $\mathcal{X}_2 \cap \operatorname{discr}_2 P_{0,10}$ may be nontrivial: it may contain an element $\alpha \in \mathrm{N}_8 \subset \operatorname{discr}_2 P_{0,10}$ (see Lemma 4.7), and then also $\mathcal{X}_2 \cap \operatorname{discr}_2 T \neq 0$. If this is the case, we still choose an anti-isometry $\psi_2\colon \operatorname{discr}_2 P_{0,10} \to \operatorname{discr}_2 T$ and represent $\mathcal{X}_2$ as $\mathbb{F}_2\alpha \oplus \operatorname{graph}(\psi_2|_{\alpha^\perp})$. If ψ_2 is fixed, the orbit N_8 splits into the orbits of the subgroup

$$G := \bigl\{ g \in \mathbb{S}_q \bigm| \psi_2 \circ g \circ \psi_2^{-1} \text{ is in the image of } O(T) \bigr\},$$

and it suffices to consider for α a single representative of each suborbit. Then, with $\alpha \neq 0$ fixed, the group $O(T, \psi_2)$ in §6.2 can be extended to the larger subgroup

$$O(T, \psi_2, \alpha) := \bigl\{ g \in O(T) \bigm| (\psi_2^{-1} \circ \bar{g} \circ \psi_2)|_{\alpha^\perp} = s|_{\alpha^\perp} \text{ for some } s \in \mathbb{S}_q(\alpha) \bigr\},$$

where $\mathbb{S}_q(\alpha)$ is the stabilizer of α. Lemma 6.4 should be restricted to the generators $\nu_k \in \alpha^\perp$, leaving more choice for ψ_2. Other observations made in §6.2 apply literally.

Finally, there are several choices for ψ_2 itself: if $\bar{T} = \mathbf{H}_{10}$, the nine classes are given by Lemma 2.11, and if $\bar{T} = \mathbf{E}_8 \oplus \mathbf{H}_2$, the inverse ψ_2^{-1} is determined by the

Table 2. The types (p,q) of exotic pencils, see Proposition 6.13

$p:$	10	7	4	3	2	1	0
$q:$	0	0	$0,3,6$	$\leqslant 6$	$\leqslant 8$	$\leqslant 9$	$\leqslant 12$

Hamming norms of the images of the two generators $\frac{1}{2}\mathrm{e}_1, \frac{1}{2}\mathrm{e}_2 \in \operatorname{discr}_2 \mathbf{H}_2(6)$, which may be $(1,9)$ or $(5,5)$. (Recall that the characteristic vector $\frac{1}{2}(\mathrm{e}_1+\mathrm{e}_2)$ is mapped to the characteristic vector and the map $O(\mathbf{E}_8) \to \operatorname{Aut}\operatorname{discr}_2 \mathbf{E}_8(6)$ is surjective.)

Summarizing, we have two choices for $\bar{T}$, $(9+2)$ choices for ψ_2, and, for each ψ_2, up to four choices for $\alpha \in \mathrm{N}_0 \cup \mathrm{N}_8$. Furthermore, in most cases, there is no simple description of the group $O(T,\psi_2,\alpha)$ and its orbits on $\operatorname{discr}_3 T$. Hence, we use `GAP` [8] to enumerate the orbits and, afterwards, quartics. (We disregard the quartics in which a 1-fiber of $\mathcal{P}$ becomes a 3-fiber; however, we do allow extra 1- and 3-fibers.) The resulting statement is as follows.

Proposition 6.11. *Let $\mathcal{P} \subset \operatorname{Fn} X$ be a maximal exotic pencil of type $(0,q)$, $q \geqslant 10$.*

- *If $q = 10$, then $\mathrm{s}(\mathcal{P}) \leqslant 29$; hence, $|\operatorname{Fn} X| \leqslant 40$.*
- *If $q = 11$, then $\mathrm{s}(\mathcal{P}) \leqslant 22$; hence, $|\operatorname{Fn} X| \leqslant 34$.*
- *If $q = 12$, then $\mathrm{s}(\mathcal{P}) \leqslant 39$; hence, $|\operatorname{Fn} X| \leqslant 52$.*

If $\operatorname{Fn} X$ as above is triangle free, then $|\operatorname{Fn} X| \leqslant 37$; this bound is sharp and it is attained by a unique configuration:

- *$\mathfrak{p} = (0,12)^1(0,9)^{22}(0,6)^{14}$, $\sigma(X) = 2$, and $\operatorname{rk}\mathcal{F}(X) = 21$.*

Remark 6.12. The counts $|\operatorname{Fn} X|$ observed in the proof of Proposition 6.11 are

$$\{11, 12, \ldots, 33, 34, 36, 37, 40, 43, 46, 52\}.$$

Together with Corollaries 6.3, 6.6, and 6.8, this list substantiates Conjecture 1.5. Note that the value $|\operatorname{Fn} X| = 49$ is also taken, as follows from Proposition 6.5 or the proof of Proposition 6.7.

6.7. **Types of exotic pencils.** In conclusion, we list the possible types of exotic pencils. Comparison of Tables 1 and 2 explains the term “exotic.”

Proposition 6.13. *The type (p,q) of a maximal exotic pencil $\mathcal{P} \subset \operatorname{Fn} X$ takes only the values listed in Table 2.*

There are a few further restrictions, which we do not discuss. For example, a maximal exotic pencil of type $(3,q)$ exists if and only if $q \leqslant 4$ or $q = 6$. The upper bounds for q in Table 2 are sharp.

Proof of Proposition 6.13. In view of Propositions 4.8 and 6.10, there only remains to show that maximal pencils of types $(p,q) = (3,7)$, $(2,9)$, or $(1,10)$ do not exist.

For the first two types, we start with a subpencil $\mathcal{P}$ of type $(2,8)$. According to Lemma 4.7 (*cf.* also Proposition 4.10), this subpencil can be chosen so that the quotient group $NS(X)/P_{2,8}$ does not have 2-torsion. Then, the minimal orthogonal complement of $P_{2,8}$ is $T = \mathbf{H}_8(6)$ (see Lemma 2.8), and, by Lemma 2.11, there are three essentially distinct choices for the anti-isometry $\psi_2\colon \operatorname{discr}_2 P_{2,8} \to \operatorname{discr}_2 T$: the identity, $\mathfrak{t}_1$, and $\mathfrak{t}_1\mathfrak{t}_3$. The first two give rise to “immediate” extra fibers and embed $\mathcal{P}$ to a pencil of type $(10,0)$ or $(4,6)$, see Lemma 6.4; hence, we choose the last one. The resulting quartic is nonsingular; in particular, we conclude that maximal exotic pencils of type $(2,8)$ do exist.

The orbits of the $O(T,\psi_2)$-action on $\operatorname{discr}_3 T$ are characterized by the "triple" Hamming norm, *i.e.*, the sequence (u,v,w) of Hamming norms in the coordinates $\{1,2\}$, $\{3,4,5,6\}$, and $\{7,8\}$; we denote these orbits by $\mathrm{H}_{u,v,w}$. (In fact, $O(T,\psi_2)$ contains also the permutation $(1,3)(2,4)(5,7)(6,8)$ of the basis vectors, which we ignore to simplify the description of the orbits.) As in the previous sections, we check that the kernel $\mathcal{X}_3 \subset \operatorname{discr}_3 N$ defining the quartic is disjoint from $\pm\Lambda\times\mathrm{H}_*$ and the pencil $\mathcal{P}$ is not maximal if and only if $\mathcal{X}_3$ intersects any of

$$\mathrm{M}_0\times\mathrm{H}_{2,2,2},\quad \mathrm{M}_0\times\mathrm{H}_{1,4,1},\quad \mathrm{M}_0\times\mathrm{H}_{2,4,0},\quad \mathrm{M}_0\times\mathrm{H}_{0,4,2}.$$

(In fact, due to the presence of an extra permutation, this is a single orbit.) In each of these cases, there are four extra lines and $\mathcal{P}$ extends to a pencil of type $(4,6)$. Hence, pencils of type $(2,9)$ do not exist, and any pencil of type $(3,7)$ is contained in one of type $(4,6)$.

The value $(p,q)=(1,10)$ can be ruled out similarly, starting with a subpencil of type $(1,9)$; this time, there are two possibilities $T=\mathbf{H}_9(6)$ or $T=\mathbf{E}_8(6)\oplus\mathbf{H}_1(6)$. However, we merely refer to the computation leading to Proposition 6.11, where pencils of type $(1,10)$ are not excluded *a priori* but do not appear. □

6.8. **Proof of Theorem 1.2.** First, assume that $\operatorname{Fn}X$ is triangle free. If it is also quadrangle free, we have $|\operatorname{Fn}X|\leqslant 44$ by Lemma 5.7. If $\operatorname{Fn}X$ has a line of valency 10 or more, the bound $|\operatorname{Fn}X|\leqslant 37$ is given by Proposition 6.11. In the remaining case, where $\operatorname{Fn}X$ has a quadrangle and $\operatorname{val} l\leqslant 9$ for each line $l\in\operatorname{Fn}X$, we obtain $|\operatorname{Fn}X|\leqslant 49$ directly from (5.9) and Lemma 5.10.

Now, assume that $\operatorname{Fn}X$ contains a plane $\{a_1,a_2,a_3,a_4\}$. We use repeatedly the following special case of (5.3), due to B. Segre [21]: since any other line $l\in\operatorname{Fn}X$ intersects exactly one of a_i, $i=1,\ldots,4$ (see Lemma 4.1 and Remark 4.5), we have

$$|\operatorname{Fn}X| = \operatorname{val}a_1+\operatorname{val}a_2+\operatorname{val}a_3+\operatorname{val}a_4-8. \tag{6.14}$$

Configurations containing a pencil of type $(10,0)$, $(7,0)$, or $(4,6)$ are considered in Corollaries 6.3, 6.6, and 6.8, respectively. Otherwise, by Proposition 6.13, for any line $l\in\operatorname{Fn}X$ we have $\operatorname{val} l=|\mathcal{P}(l)|\leqslant 15$. Hence, $|\operatorname{Fn}X|\leqslant 52$ by (6.14). □

Remark 6.15. For an alternative proof of the uniqueness of the Fermat quartic, observe that, by (6.14) again, a configuration of size 112 constitutes a generalized quadrangle $\mathrm{GQ}(3,9)$, see §2.5, which is unique up to isomorphism. As $\operatorname{rk}\mathbf{Q}_{112}=22$ and $\operatorname{discr}\mathbf{Q}_{112}=\langle\frac{2}{3}\rangle^2$, this lattice admits no further extension.

7. Pencils in 2-supersingular quartics

In this section, we discuss pencils in supersingular quartics over an algebraically closed field $\Bbbk$ of characteristic 2 (for short, 2-*supersingular quartics*). Our principal observation is the fact that such quartics are related to indecomposable *odd* negative definite unimodular lattices.

7.1. **The lattice** $NS(X)$**.** Consider a pencil $\mathcal{P}$, not necessarily maximal, of type $(0,q)$, $q>0$. First, note that the 3-torsion of the quotient $NS(X)/P_{0,q}$ is trivial. Indeed, by Lemma 4.7, it could be nontrivial only if $q=10$. Then, $P^\perp$ would be a 2-elementary lattice, see Proposition 2.4, and, by Lemma 2.5, it would contain a sublattice of the form $\bar{T}(2)$, where $\bar{T}=\mathbf{H}_{10}$ or $\mathbf{E}_8\oplus\mathbf{H}_2$. Hence, $P^\perp$, and then also $NS(X)$, would contain a (-2)-vector, contradicting Theorem 3.3(1).

Thus, denoting by T the minimal orthogonal complement and representing the lattice $NS(X)$ as an extension of $N := P_{0,q} \oplus T$ with a certain kernel $\mathcal{X}$ (*cf.* §6.2), from Proposition 2.4 we conclude that the 3-primary part $\mathcal{X}_3$ is the graph of a bijective anti-isometry

$$\psi_3\colon \operatorname{discr}_3 T \longrightarrow \operatorname{discr}_3 P_{0,q}.$$

In particular, $|\operatorname{discr}_3 T| = 9$ and $\operatorname{discr}_3 T$ has an isotropic vector, *viz.* the pull-back of λ (see Observation 4.6); hence, T is an index 3 sublattice of a 2-elementary lattice. By Lemma 2.5 again, the latter contains a sublattice of the form $\bar{T}(2)$, where $\bar{T}$ is unimodular. Till the rest of this section, we will use the bilinear form in $\bar{T}$; in T, all products are doubled.

Summarizing, $NS(X)$ is a finite index extension of $N := P_{0,q} \oplus T$, where T can be described in terms of a unimodular lattice $\bar{T}$ of rank $20 - q$ *via*

$$T\bigl(\tfrac{1}{2}\bigr) = \bigl\{a \in \bar{T} \bigm| a \cdot u = 0 \bmod 3\bigr\}, \quad \text{where } u \in \bar{T} \text{ is fixed, } u^2 = q - 1 \bmod 3.$$

The kernel $\mathcal{X}_3$ is the graph of an anti-isometry

$$\psi_3\colon \operatorname{discr}_3 T \longrightarrow \operatorname{discr}_3 P_{0,q}, \qquad \psi_3\bigl(\bar{T}(2)/T\bigr) = \mathbb{F}_3\lambda. \tag{7.1}$$

This anti-isometry is unique up to the action of $O_h(P_{0,q})$; hence, with the pair $(\bar{T}, u)$ fixed, the quartic is determined by the 2-primary part $\mathcal{X}_2$.

There is a necessary condition for the realizability of a pair $(\bar{T}, u)$ by a quartic; we state it as a separate lemma.

Lemma 7.2. *For $(\bar{T}, u)$ as above, the class $u \bmod 3\bar{T}$ cannot be represented by a vector of square $(q-7)$ or a characteristic vector of square $(q-28)$.*

Proof. If $a = u \bmod 3\bar{T}$ is as in the statement, then one of the square (-2) vectors $l^* - \sum_{k=1}^{q} \nu_k \pm a$ (in the former case) or $l^* \pm \frac{1}{6}a$ (in the latter case) is in $NS(X)$, contradicting Theorem 3.3. □

Proposition 7.3. *Let $\mathcal{P}$ and $(\bar{T}, u)$ be as above, and assume that the maximal pencil containing $\mathcal{P}$ has exactly q fibers. Then*:

1. *the unimodular lattice $\bar{T}$ is odd and indecomposable*; *in particular, $q \leqslant 8$;*
2. *the pencil $\mathcal{P}$ is primitive.*

Proof. First, we show that $\bar{T}$ does not represent (-1). Indeed, if $a \in \bar{T}$, $a^2 = -1$, then $a \notin T$ by Theorem 3.3(1); hence, a represents a nontrivial class in $\bar{T}(2)/T$ and $\psi_3(a \bmod T) = \pm\lambda$, see (7.1). It follows that $\lambda \pm a \in NS(X)$ and, since we have $(a \pm \lambda) \cdot n_i = 0$ for $i = 1, \ldots, q$, the pencil has an extra fiber.

If $q = 12$ and $\bar{T} = \mathbf{E}_8$, we can assume that $u^2 = -4$; then the vector $-2a$ is characteristic, contradicting Lemma 7.2. With $\mathbf{E}_8$ eliminated, we have $q \leqslant 8$ and $\bar{T}$ is automatically odd unless $q = 4$. However, in the latter case, $\mathcal{P}$ is primitive, see Lemma 4.7, and $\operatorname{discr}_2 T$ must be odd; hence, $\bar{T}$ is also odd. Now, it follows that T is indecomposable, as the smallest decomposable odd unimodular lattice not representing (-1) is $\mathbf{E}_8 \oplus \mathbf{D}_{12}^+$ of rank 20 (corresponding to $q = 0$).

Similarly, by Lemma 4.7, $\mathcal{P}$ is primitive unless $q = 8$, in which case the only admissible extension $\tilde{P}$ has even discriminant $\operatorname{discr}_2 \tilde{P}$. Then $NS(X)$ contains $\frac{1}{2}a$, where $a \in \bar{T} = \mathbf{D}_{12}^+$ is a characteristic vector. One can check that, for any choice of $u \bmod 3\bar{T}$, there is a characteristic vector $a \in u^\perp$ of square (-4) (see §7.3.1 below for a detailed description of this lattice). □

7.2. **Orbits.** Due to Proposition 7.3(2), the kernel $\mathcal{X}_2 \subset \operatorname{discr}_2 N$ is the graph of an anti-isometry

$$\psi\colon \mathcal{H} \longrightarrow \operatorname{discr}_2 P_{0,q}, \qquad \mathcal{H} \subset \operatorname{discr}_2 T = \bar{T}/2\bar{T}.$$

Denote by $c_P \in \operatorname{discr}_2 P_{0,q}$ and $c_T \in \operatorname{discr}_2 T$ the characteristic vectors; they are both nontrivial. Since $\operatorname{discr}_2 NS(X)$ must be even, we have

$$c_T \in \mathcal{H} = \operatorname{Domain}\psi \quad \text{and} \quad \psi(c_T) = c_P.$$

Consider the group

$$O(\bar{T}, u) := \bigl\{ g \in O(\bar{T}) \bigm| g(u) = \pm u \bmod 3\bar{T} \bigr\}.$$

As in the case of characteristic 3, the orbits of the $O_h(N)$-action on $\operatorname{discr}_2 N$ split into products

$$\mathrm{N}_k \times \mathrm{O}, \quad \text{where } k = 0, \ldots, q \text{ and } \mathrm{O} \subset \bar{T}/2\bar{T} \text{ is an } O(\bar{T}, u)\text{-orbit}.$$

Furthermore, we only need to check the admissibility and compute the number of extra lines, *i.e.*, the fibers of the natural maps

$$\operatorname{sec}_2\colon \{\text{sections of } \mathcal{P}\} \longrightarrow \mathcal{X}_2, \qquad \operatorname{fib}_2\colon \mathcal{P}(l) \smallsetminus \mathcal{P} \longrightarrow \mathcal{X}_2,$$

for one representative of each orbit. Then, an isotropic subgroup $\mathcal{X}_2 \subset \operatorname{discr}_2 N$ is admissible if and only if so are all its elements, and the number of lines is additive. Note that $N_q = \{c_P\}$ and the orbit $\mathrm{N}_q \times \{c_T\}$ is always in $\mathcal{X}_2$.

Remark 7.4. In the case of characteristic 2, we have a better control over the geometry of the extra lines. Denote by

$$\operatorname{pr}\colon \operatorname{discr}_2 N \longrightarrow \operatorname{discr}_2 P_{0,q}$$

the projection. Then, for an extra line $s \in NS(X) \smallsetminus N$, we have

$$\operatorname{pr}(s \bmod N) = \sum_{k \in I} 3\nu_k, \quad \text{where } I := \{k \in \operatorname{fb}_1(\mathcal{P}) \,|\, s \cdot n_k = 1\}.$$

As a consequence, all nonempty fibers of the composition $\operatorname{pr} \circ \operatorname{fib}_2$ are over N_1: a generator $3\nu_k$ is in the image if and only if the k-th 1-fiber of $\mathcal{P}$ becomes a 3-fiber. Alternatively, all extra lines in $\mathcal{P}(l) \smallsetminus \mathcal{P}$ are of the form

$$n_k + 3\nu_k \pm \tfrac{1}{2}a = -\lambda + \nu_k \pm \tfrac{1}{2}a,$$

$a \in \bar{T}$, $a^2 = -3$, provided that these vectors are in $NS(X)$, *cf.* Lemma 6.4.

7.3. **The values $q = 5$, 6, and 8.** In this section, we describe the isomorphism classes of pairs $(\bar{T}, u)$ appearing in Proposition 7.3 for the large values $q = 5$, 6, and 8. (Note that $q \neq 7$, as there is no indecomposable lattice of rank 13.) There are eight classes, listed below. Afterwards, it is straightforward, although tedious, to use `GAP` [8] and enumerate all kernels $\mathcal{X}_2$; we merely state the final result in Proposition 7.5. An important experimental fact is that T is necessarily primitive in $NS(X)$; hence, instead of $\psi\colon \mathcal{H} \to \mathcal{D} \subset \operatorname{discr}_2 P_{0,q}$, we can consider its inverse $\psi^{-1}\colon \mathcal{D} \to \operatorname{discr}_2 T$, which is defined on a subspace $\mathcal{D} \subset -\mathcal{S}_q$ (see §2.4) containing the characteristic vector. Up to the action of $\mathbb{S}_q$ by permutations of the generators, which is induced from $O(P_{0,q})$, there are relatively few such subspaces.

7.3.1. *The case $q = 8$ and $\bar{T} = \mathbf{D}_{12}^{+}$.* We represent $\mathbf{D}_{12}$ as the maximal even sublattice in $\mathbf{H}_{12}$. This lattice has three unimodular extensions, all odd: one is the original lattice $\mathbf{H}_{12}$, and the two others are isomorphic. We denote by $\mathbf{D}_{12}^{+}$ the extension by $\frac{1}{2}\bar{\mathrm{e}}$; then, $O(\mathbf{D}_{12}^{+}) \subset O(\mathbf{D}_{12})$ is the index 2 subgroup generated by reflections against vectors of square (-2). Since $u^2 = 1 \bmod 3$, the $O(\mathbf{D}_{12}^{+})$-orbits of vectors $u \bmod 3\mathbf{D}_{12}^{+}$ are characterized by the Hamming norm: we have H_2, H_5, H_8, and H_{11}. (Note, in particular, that there always is a characteristic vector, *viz.* $2\mathrm{e}_{12} \in u^{\perp}$, of square (-4); this fact was used in the proof of Proposition 7.3.) The only orbit satisfying Lemma 7.2 is H_8.

7.3.2. *The case $q = 6$ and $\bar{T} = (\mathbf{E}_7^2)^{+}$.* This lattice is the only nontrivial extension of $\mathbf{E}_7^2$; its kernel is generated by the vector of square 1 in $\operatorname{discr} \mathbf{E}_7^2 = \langle \frac{1}{2} \rangle^2$. The orbits of the action of $O(\mathbf{E}_7)$ on $\mathbf{E}_7/3\mathbf{E}_7$ are almost distinguished by the length of the shortest representatives: we have

$$\mathrm{E}_r := \bigl\{ u \bmod 3\mathbf{E}_7 \bigm| u \in \mathbf{E}_7,\ u^2 = -r \bigr\}, \quad r = 0, 2, 4, 6, 8,$$

but the set E_6 splits into two orbits E_6', E_6'', where E_6'' is characterized by the fact that its shortest representatives vanish $\bmod 2\mathbf{E}_7^{\vee}$. Since $u^2 = 2 \bmod 3$ (and in view of the obvious symmetry), we have the $O(\bar{T})$-orbits

- $\mathrm{E}_2 \times \mathrm{E}_8$, $\mathrm{E}_4 \times \mathrm{E}_0$, $\mathrm{E}_4 \times \mathrm{E}_6''$, which contradict Lemma 7.2, and
- $\mathrm{E}_2 \times \mathrm{E}_2$, $\mathrm{E}_8 \times \mathrm{E}_8$, $\mathrm{E}_4 \times \mathrm{E}_6'$, which give rise to nonsingular quartics.

7.3.3. *The case $q = 5$ and $\bar{T} = \mathbf{A}_{15}^{+}$.* This is the only unimodular extension of $\mathbf{A}_{15}$, *viz.* the one by $4 \operatorname{discr} \mathbf{A}_{15}$. Represent $\mathbf{A}_{15}$ as $\bar{\mathrm{e}}^{\perp} \subset \mathbf{H}_{16}$. Then, $O(\mathbf{A}_{15}^{+}) = O(\mathbf{A}_{15})$ is the subgroup of $O(\mathbf{H}_{16})$ stabilizing the set $\{\pm\bar{\mathrm{e}}\}$. The orbits of the action of this group on the $\mathbb{F}_3$-vector space $\mathbf{A}_{15}/3\mathbf{A}_{15}$ are

$$\mathrm{H}_{r,s} \ni 2(\mathrm{e}_1 + \ldots + \mathrm{e}_r) + (\mathrm{e}_{r+1} + \ldots + \mathrm{e}_{r+s}) - (\mathrm{e}_{r+s+1} + \ldots + \mathrm{e}_{3r+2s});$$

we have $r, s \geqslant 0$, $3r + 2s \leqslant 16$, and $s = 1 \bmod 3$ (since $u^2 = 1 \bmod 3$). Five orbits contradict Lemma 7.2; the remaining valid orbits are $\mathrm{H}_{0,4}$, $\mathrm{H}_{0,7}$, $\mathrm{H}_{1,4}$, $\mathrm{H}_{3,1}$.

Proposition 7.5. *Let $X \subset \mathbb{P}^3$ be a 2-supersingular quartic, and let $\mathcal{P} \subset \operatorname{Fn} X$ be a maximal pencil of type (p, q). Then the number $p + q$ of fibers of $\mathcal{P}$ can take only one of the following values*:

- $p + q = 8$: *then* $p \leqslant 3$ *and* $|\operatorname{Fn} X| = 40$ *or* $|\operatorname{Fn} X| \leqslant 32$;
- $p + q = 6$: *then* $p \leqslant 5$ *and* $|\operatorname{Fn} X| \leqslant 32$;
- $p + q = 5$: *then* $|\operatorname{Fn} X| \leqslant 24$;
- $p + q \leqslant 4$.

If $p + q = 8$, there are at most four distinct configurations $\operatorname{Fn} X$ *of size* 40; *they are as in Theorem 1.1(1)–(5).*

Hypothetically, the bounds in Proposition 7.5 are sharp: they are realizable by homological configurations, *cf.* Remark 3.4. In the spirit of Propositions 4.9 and 6.13, the part of Proposition 7.5 concerning the types of the pencils can be summarized in the form of the table

$p =$	5	4	3	2	1	0
$q \leqslant$	1	2	5	6	7	8,

with the extra restriction that $p + q \neq 7$.

7.4. **The value** $q = 4$. The only indecomposable odd unimodular lattice of rank 16 is $\bar{T} = (\mathbf{D}_8^2)^+$. Recall that $\mathbf{D}_8$ is the maximal even sublattice in $\mathbf{H}_8$. Then, $(\mathbf{D}_8^2)^+$ can be described as the extension of $\mathbf{D}_8^2$ by the two $(\operatorname{mod} \mathbb{Z})$-orthogonal vectors $\frac{1}{2}\bar{\mathrm{e}} \oplus \mathrm{e}_8$, $\mathrm{e}_8 \oplus \frac{1}{2}\bar{\mathrm{e}}$ of square 1 mod $2\mathbb{Z}$. The $O(\mathbf{D}_8)$-orbits on $\mathbf{D}_8/3\mathbf{D}_8$ are almost characterized by the Hamming norm: we have H_0 through H_7, and H_8 splits into two orbits $\mathrm{H}_8^+ \ni \bar{\mathrm{e}}$, $\mathrm{H}_8^- \ni \bar{\mathrm{e}} - 2\mathrm{e}_8$. Hence, with the obvious symmetry taken into account, there are 17 orbits of vectors $u \bmod 3\bar{T}$ of square 0 mod 3. Nine of them contradict Lemma 7.2, and the remaining eight are

- $\mathrm{H}_4 \times \mathrm{H}_5$, $\mathrm{H}_4 \times \mathrm{H}_8^+$, $\mathrm{H}_7 \times \mathrm{H}_5$, $\mathrm{H}_3 \times \mathrm{H}_6$, $\mathrm{H}_6 \times \mathrm{H}_6$, and
- $\mathrm{H}_4 \times \mathrm{H}_2$, $\mathrm{H}_7 \times \mathrm{H}_8^-$, $\mathrm{H}_0 \times \mathrm{H}_3$.

The last three orbits are characterized by the fact that a generic quartic has at least one section; this section intersects all four fibers (in particular, the configuration of lines is not quadrangle free).

Unlike the three cases considered in the previous section, this time we have $\dim(\mathcal{X}_2 \cap \operatorname{discr}_2 T) \leqslant 3$. Together with the size of the spaces and groups involved, this fact makes the computation difficult. For this reason, we only consider two extremal cases: quadrangle free configurations and those where the maximal pencil containing $\mathcal{P}$ has at least three 3-fibers.

Proposition 7.6. *Let $X \subset \mathbb{P}^3$ be a 2-supersingular quartic, and let $\mathcal{P} \subset \operatorname{Fn} X$ be a maximal pencil of type (p, q), $p + q = 4$. Then*:

- *if $p \geqslant 3$, then $|\operatorname{Fn} X| = 40$ or $|\operatorname{Fn} X| \leqslant 32$;*
- *if $\operatorname{Fn} X$ is quadrangle free, then $|\operatorname{Fn} X| \leqslant 9$.*

Remark 7.7. The potential line counts $|\operatorname{Fn} X|$ observed in the course of the proof of Propositions 7.5 and 7.6 are

$$\{5, 6, \ldots, 17, 18, 20, 22, 24, 28, 32, 40\}.$$

This list is complete for the homological (*cf.* Remark 3.4) configurations containing a pencil with at least five fibers. In more detail, for a nonsingular quartic $X \subset \mathbb{P}^3$ we have $\sigma(X) \geqslant 3$ and the values of $|\operatorname{Fn} X|$ are distributed as follows:

- if $\sigma(X) = 3$, then $|\operatorname{Fn} X| = 40$ (five quartics);
- if $\sigma(X) = 4$, then $|\operatorname{Fn} X| \in \{12, 16, 20, 24, 28, 32\}$;
- if $\sigma(X) = 5$, then $|\operatorname{Fn} X| \leqslant 24$ and $|\operatorname{Fn} X| = 0 \bmod 2$;
- if $\sigma(X) = 6$, then $|\operatorname{Fn} X| \leqslant 18$ or $|\operatorname{Fn} X| = 20$;
- if $\sigma(X) = 7, 8$, then $|\operatorname{Fn} X| \leqslant 16, 12$, respectively.

This observation substantiates and refines Conjecture 1.4.

7.5. **Proof of Theorem 1.1.** In view of Proposition 7.5, we can assume that each pencil $\mathcal{P} \subset \operatorname{Fn} X$ has at most four fibers; in particular, $|\mathcal{P}| \leqslant 12$.

First, assume that $\operatorname{Fn} X$ is triangle free. If $\operatorname{Fn} X$ is also quadrangle free, then, by Proposition 7.6, either $\operatorname{Fn} X \leqslant 32$ or $\operatorname{Fn} X$ is locally elliptic; in the latter case, $\operatorname{Fn} X \leqslant 31$ by Lemma 5.4. If $\operatorname{Fn} X$ has a quadrangle, then each valency $\operatorname{val} l_i \leqslant 4$ in (5.9) and, even if $\ln(\pi_Q) = 24$, we have $|\operatorname{Fn} X| \leqslant 32$ again.

Now, assume that $\operatorname{Fn} X$ has a plane. Configurations containing a pencil of type (p, q) with either $p + q \geqslant 5$ or $p + q = 4$ and $p \geqslant 3$ are considered, respectively, in Propositions 7.5 and 7.6. Otherwise, $\operatorname{val} l = |\mathcal{P}(l)| \leqslant 9$ for any line $l \in \operatorname{Fn} X$ and, hence, $|\operatorname{Fn} X| \leqslant 28$ by (6.14). (In fact, $|\operatorname{Fn} X| \leqslant 27$, as there are no generalized quadrangles $\mathrm{GQ}(3, 2)$, see §2.5.)

There remains to show that each configuration $\operatorname{Fn} X$ of size 40 is one of those listed in the statement. In view of Proposition 7.5, we can assume that each pencil $\mathcal{P} \subset \operatorname{Fn} X$ has at most four fibers; in particular, $|\mathcal{P}| \leqslant 12$. Then, by (6.14) again, $\operatorname{Fn} X$ constitutes one of the two generalized quadrangles $\mathrm{GQ}(3,3)$, see §2.5, which we consider separately. Let T be the orthogonal complement of $\mathbf{Q}_{40}^*$ in $NS(X)$: it is a negative definite lattice of rank 6.

Let $\operatorname{Fn} X \cong \mathbb{Q}(4,3)$. Since $\operatorname{discr} \mathbf{Q}'_{40} = \langle \frac{4}{3} \rangle \oplus \mathcal{V}_2^3$, we have $\operatorname{discr} T = \langle \frac{2}{3} \rangle \oplus \mathcal{U}_2^r$, $r \leqslant 3$, see Theorem 3.1 and Proposition 2.4. In fact, $r \leqslant 2$ by Theorem 2.2, and, by an analogue of Lemma 2.5, T is an extension of $\mathbf{D}_4 \oplus \mathbf{A}_2(2)$, which is unique in its genus. This lattice has (-2)-vectors, contradicting Theorem 3.3(1). Alternatively, one can start with the lattice T' given by Lemma 2.5. Then, $\operatorname{discr}_2 T'$ is necessarily odd; hence, $T' = \mathbf{A}_2(2) \oplus \mathbf{A}_1^4$ and $T \supset T'$ is the extension by the characteristic vector $c \in \operatorname{discr}_2 T'$. This observation proves both the uniqueness in the genus and the existence of (-2)-vectors, as such vectors are already present in T'.

Let $\operatorname{Fn} X \cong W(3)$. Since $\operatorname{discr} \mathbf{Q}''_{40} = \langle \frac{2}{3} \rangle^5$, we have $\operatorname{discr} T = \langle \frac{4}{3} \rangle^5 \oplus \mathcal{U}_2^r \oplus \mathcal{V}_2$ for some $r \leqslant 2$. The lattice $\mathbf{Q}''_{40}$ has no admissible finite index extensions and, by an analogue of Lemma 2.5, T is an extension of $\bar{\mathbf{E}}_6(2)$, which is unique in its genus. Since the natural homomorphism $O(\bar{\mathbf{E}}_6) \to \operatorname{Aut} \operatorname{discr}_3 \bar{\mathbf{E}}_6$ is an isomorphism, an anti-isometry $\operatorname{discr}_3 \mathbf{Q}''_{40} \to \operatorname{discr}_3 \bar{\mathbf{E}}_6(2)$ is essentially unique, and we do obtain a quartic X as in Theorem 1.1(5). A simple computation shows that the lattice $NS(X)$ has no admissible finite index extensions. □

7.6. **Other generalized quadrangles.** For completeness, we discuss briefly the realizability (in the sense of §2.5) of the other generalized quadrangles $\mathrm{GQ}(3,t)$ by configurations of lines in nonsingular quartics.

7.6.1. *The quadrangle* $\mathrm{GQ}(3,9)$. The only realization of $\mathrm{GQ}(3,9) \cong Q(5,3)$ is that by the Fermat quartic in characteristic 3, see Theorem 1.2(1) and Remark 6.15, since for all other quartics X one has $|\operatorname{Fn} X| \leqslant 64$. In the Fermat quartic X, the four lines constituting a plane α intersect at a single point P_α, see Corollary 8.12 below. Hence, taking $\operatorname{Fn} X$ for the set of lines and all points P_α for the set of points, we also obtain a generalized quadrangle, one of type $\mathrm{GQ}(9,3)$.

7.6.2. *The quadrangle* $\mathrm{GQ}(3,5)$. This generalized quadrangle is not realized by a configuration of lines. We have $\operatorname{rk} \mathbf{Q}_{64} = 19$ and $\operatorname{discr} \mathbf{Q}_{64} = \mathcal{U}_2^2 \oplus \langle \frac{7}{4} \rangle$, and this lattice has no admissible finite index extensions. Thus, according to Theorem 3.5, this configuration could only be realized by a 2-supersingular quartic, which would contradict Proposition 7.5.

7.6.3. *The quadrangles* $\mathrm{GQ}(3,3)$. As shown in §7.5, only the generalized quadrangle $W(3)$ appears as the configuration of lines in a supersingular quartic X over a field of characteristic 2. If X is supersingular in characteristic 3, both $Q(4,3)$ and $W(3)$ can be realized:

(1) if $\operatorname{Fn} X \cong Q(4,3)$, then $\sigma X = 3$ and $(\mathbf{Q}'_{40})^\perp = \bar{\mathbf{E}}_6(2)$;
(2) if $\operatorname{Fn} X \cong W(3)$, then $3 \leqslant \sigma X \leqslant 5$ and $(\mathbf{Q}''_{40})^\perp = \bar{\mathbf{E}}_6$.

In case (1), the representation is unique, as any proper admissible extension of the lattice $NS(X) = \mathbf{Q}'_{40} \oplus \bar{\mathbf{E}}_6(2)$ contains extra lines. In case (2), the configuration is maximal with respect to inclusion: the minimal lattice $NS(X) = \mathbf{Q}''_{40} \oplus \bar{\mathbf{E}}_6$ has admissible extensions of index 3 and 9, all with the same configuration of lines.

Both $Q(4,3)$ and $W(3)$ are realizable by nonsingular quartics over $\mathbb{C}$, with the transcendental lattice isomorphic to $\mathbf{A}_2(2) \oplus \mathbf{U}^2(2)$ and $\mathbf{A}_2 \oplus \mathbf{U}^2(3)$, respectively. The former contains $\mathbf{U}(2)$, and the latter contains $\mathbf{A}_1$; hence, both quadrangles can be represented by *real* lines in a real quartic, see [6, Corollary 3.14].

7.6.4. *The quadrangle* $\mathrm{GQ}(3,1)$. We have $\operatorname{rk} \mathbf{Q}_{16} = 10$ and $\operatorname{discr} \mathbf{Q}_{16} = \mathcal{U}_4^2$. Hence, the quadrangle $\mathrm{GQ}(3,1) \cong Q(3,3)$ can be realized by a nonsingular quartic over $\mathbb{C}$, with the transcendental lattice $\mathbf{E}_8 \oplus \mathbf{U}^2(4) \supset \mathbf{A}_1$; by [6], both the quartic and the lines can be chosen real. We omit the discussion of the realizability of $\mathrm{GQ}(3,1)$ by supersingular quartics in characteristics 2 and 3.

8. Geometric arguments

In this section, we employ direct geometric arguments (at the level of defining equations) to establish the uniqueness of Schur's quartic and prove Theorem 1.3.

8.1. **Pairs of obverse pencils.** Let $\mathcal{P}_i := \mathcal{P}(l_i)$, $i = 1, 2$, $l_i \in \operatorname{Fn} X$, be a pair of obverse pencils. The pencil $\mathcal{P}_i$ defines a (quasi-)elliptic pencil $\pi_i \colon X \to \mathbb{P}^1$; hence, we have a map $\pi[l_1, l_2] := \pi_1 \times \pi_2 \colon X \to \mathbb{P}^1 \times \mathbb{P}^1$. The base $\mathbb{P}^1$ of the projection π_i can be identified with l_j, $j \neq i$. Hence, the pull-back of a point $(x, y) \in \mathbb{P}^1 \times \mathbb{P}^1$ is the intersection of X with the line connecting the points $y \in l_1$ and $x \in l_2$, excluding x, y themselves. It follows that $\pi := \pi[l_1, l_2]$ is of degree 2. The deck translation of the double covering $X \to \mathbb{P}^1 \times \mathbb{P}^1$ is known as the *Segre involution*; typically, it is not projective.

First, assume that $\operatorname{char} \Bbbk \neq 2$. Then π is a double covering ramified at a curve $D \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(4,4)$. We keep the notation (x, y) for the coordinates in the target quadric $\mathbb{P}^1 \times \mathbb{P}^1$.

Observation 8.1. The following statements are straightforward (where a curve is called *even* if its intersection index with D at each intersection point is even):

1. the line l_1 projects to an even irreducible curve of bi-degree $(3,1)$;
2. the line l_2 projects to an even irreducible curve of bi-degree $(1,3)$;
3. a line $a \in \mathcal{P}_1 \cap \mathcal{P}_2$ contracts to a singular point of D;
4. any singular point of D is a simple node of the form $\pi(a)$, $a \in \mathcal{P}_1 \cap \mathcal{P}_2$;
5. the curves $\pi(l_1)$ and $\pi(l_2)$ contain all points $\pi(a)$, $a \in \mathcal{P}_1 \cap \mathcal{P}_2$;
6. a line $a \in \mathcal{P}_1 \smallsetminus \mathcal{P}_2$ projects to an even generatrix $y = \mathrm{const}$;
7. a line $a \in \mathcal{P}_2 \smallsetminus \mathcal{P}_1$ projects to an even generatrix $x = \mathrm{const}$;
8. any other line b projects to an even irreducible curve of bidegree $(1,1)$; this curve contains a point $\pi(a)$, $a \in \mathcal{P}_1 \cap \mathcal{P}_2$, if and only if b intersects a.

As an immediate consequence, the projection π establishes a canonical one-to-one correspondence between the 3-fibers of $\mathcal{P}_1$ (respectively, 3-fibers of $\mathcal{P}_2$) and the even generatrices of the form $y = \mathrm{const}$ (respectively, $x = \mathrm{const}$) passing through a singular point of the ramification locus D.

Proposition 8.2. *Assume that* $\operatorname{char} \Bbbk \neq 2$ *and that* $\operatorname{Fn} X$ *contains a configuration* $a_1, \ldots, a_{10}, b_1, b_2$ *as in Lemma 4.3. Then, the ramification locus* D *of* $\pi := \pi[b_1, b_2]$ *splits into the union* $\pi(b_1) \cup \pi(b_2)$.

Proof. According to Observation 8.1(3), (5), the ramification locus D must have ten simple nodes, which must coincide with the ten points of $\pi(b_1) \cap \pi(b_2)$. Now, the statement is an immediate consequence of Bézout's theorem and the fact that both $\pi(b_1)$ and $\pi(b_2)$ are irreducible. □

Corollary 8.3. *Under the assumptions of Proposition 8.2, the lines b_1, b_2 are of the first kind in the sense of Segre* [21].

Proof. Recall that a line l is said to be of the second kind if a generic fiber C of the pencil $\pi[l]$ is a nonsingular cubic and the points of intersection of l and C are inflection points of C. Assume that the line b_1 is of the second kind and let $b_1 \cap C = \{P_1, P_2, P_3\}$. Then the three differences $P_i - P_j$, $1 \leqslant i < j \leqslant 3$, are 3-torsion points of the Jacobian $J(C)$. On the other hand, since b_1 is contained in the ramification locus of the fiberwise degree 2 map $\pi[b_1, b_2]$, these differences are 2-torsion points of $J(C)$. This is a contradiction.

Alternatively, one can show that, typically, P_1, P_2, P_3 are not inflection points by computing the Hessian of a generic fiber in (8.4) below. □

Under the assumptions of Proposition 8.2, the ramification locus $\pi(b_1) \cup \pi(b_2)$ determines both the abstract $K3$-surface X and polarization h, *e.g.*, as the sum of the reduced preimage of $\pi(b_1)$ (the line b_1) and the pull-back of any fiber $y = \text{const}$ (see Lemma 4.1). Hence, it determines the quartic $X \subset \mathbb{P}^3$, and one can easily see that the latter is given by the equation

$$z_1^3 z_3 f_1\Bigl(\frac{z_0}{z_1}, \frac{z_2}{z_3}\Bigr) = z_1 z_3^3 f_2\Bigl(\frac{z_0}{z_1}, \frac{z_2}{z_3}\Bigr), \tag{8.4}$$

where $f_i(x, y) = 0$ is a defining equation of the component $\pi(b_i)$, $i = 1, 2$. Clearly, this equation can be rewritten in the form

$$\bar f_1(z_0, z_1; z_2, z_3) = \bar f_2(z_0, z_1; z_2, z_3),$$

where $\bar f_i(z_0, z_1; z_2, z_3)$ is the homogenization of $f_i(x, y)$, $i = 1, 2$. Conversely, given two polynomials f_1, f_2 of bidegree $(3, 1)$ and $(1, 3)$, respectively, then, assuming that $\operatorname{char} \Bbbk \neq 2$, the quartic given by (8.4) is nonsingular if and only if the two curves $B_i := \{f_i(x, y) = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^1$ are nonsingular (equivalently, irreducible) and intersect transversally at ten points.

Proposition 8.5. *Under the assumptions of Proposition 8.2, the lines $l \in \operatorname{Fn} X$ disjoint from b_1, b_2 are in a two-to-one correspondence with quadruples of points $A_i \in \pi(b_1) \cap \pi(b_2)$, $i = 1, \ldots, 4$, whose coordinates (x_i, y_i) have equal unharmonic ratios*: $(x_1, x_2; x_3, x_4) = (y_1, y_2; y_3, y_4)$.

Proof. By Lemma 4.3 (see also Remark 4.5), any line $l \in \operatorname{Fn} X$ disjoint from b_1, b_2 intersects exactly four of the ten lines $a_1, \ldots, a_{10}$; hence, its image is an irreducible bidegree $(1, 1)$ curve L passing through the four images $A_i := \pi(a_i) \in \pi(b_1) \cap \pi(b_2)$. Conversely, any such curve is even and each of the two components of its pull-back intersects a generic plane at a single point. □

Now, assume that $\operatorname{char} \Bbbk = 2$. Then, in appropriate coordinates in $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(4, 4)$, the surface X is given by

$$z^2 + f_2(x, y) z + f_4(x, y) = 0,$$

where f_2 and f_4 are of bidegree $(2, 2)$ and $(4, 4)$, respectively. One has $f_2 \neq 0$, *i.e.*, the covering cannot be purely inseparable. Indeed, otherwise, the two pencils $\pi[l_1]$, $\pi[l_2]$ would be quasi-elliptic, which would contradict Proposition 4.8. Thus, the formal ramification locus of π is the non-reduced curve $D = 2\bar D$, where $\bar D$ is given by $f_2 = 0$ and has bidegree $(2, 2)$. Hence, any point of D is singular and any curve in $\mathbb{P}^1 \times \mathbb{P}^1$ is even. With this understood, all statements except (4) in Observation 8.1 hold literally, although most are no longer invertible.

Proposition 8.6. *If* $\operatorname{char}\Bbbk = 2$*, then* $\operatorname{Fn} X$ *cannot contain a configuration as in Lemma 4.3,* i.e.*, two disjoint lines* l_1*,* l_2 *can intersect at most eight other lines.*

Proof. The projection $\pi := \pi[l_1, l_2]$ contracts any line $a \in \mathcal{P}(l_1) \cap \mathcal{P}(l_2)$ to a point $A \in \mathbb{P}^1 \times \mathbb{P}^1$ common to the bidegree $(2, 2)$ curve $\bar{D}$ and irreducible bidegree $(3, 1)$ curve $\pi(l_1)$. By Bézout's theorem, there are at most eight such points. □

8.2. **Schur's quartic.** According to [6], if $\operatorname{char}\Bbbk = 0$, there is a unique quartic X_{64} containing 64 lines; it is called *Schur's quartic.* The lines contained in $\mathbb{X}_{64}$ span the Néron–Severi lattice $NS(X)$; we denote this lattice by $\mathbf{X}_{64}$ and call it the *Schur configuration*. The following property of $\mathbf{X}_{64}$ is easily deduced from the explicit incidence matrix of $\operatorname{Fn}\mathbf{X}_{64}$ or, alternatively, from the classical description of the 64 lines in Schur's quartic, see, *e.g.*, [2]

Lemma 8.7. *The configuration* $\operatorname{Fn}\mathbf{X}_{64}$ *contains a collection* $a_1, \ldots, a_{10}, b_1, b_2, c$ *with the following properties*:

1. *the lines* $a_1, \ldots, a_{10}, b_1, b_2$ *are as in Lemma 4.3*;
2. *the four lines* $a_1, \ldots, a_4$ *are in the* 3*-fibers of both* $\mathcal{P}(b_1)$ *and* $\mathcal{P}(b_2)$;
3. *the line* c *intersects* $a_1, \ldots, a_4$*.*

Lemma 8.8. *A nonsingular quartic* $X \in \mathbb{P}^3$ *whose configuration of lines* $\operatorname{Fn} X$ *has the properties stated in Lemma 8.7 is unique up to projective transformation. Such a quartic exists if and only if* $\operatorname{char}\Bbbk \neq 2$*.*

Proof. The possibility $\operatorname{char}\Bbbk = 2$ is ruled out by Proposition 8.6. Henceforth, we assume that $\operatorname{char}\Bbbk \neq 2$, and the existence of X is given by an explicit example, *viz.* Schur's quartic. Below, without further references, we freely use Observation 8.1.

Let $\pi := \pi[b_1, b_2]$. According to Proposition 8.2, the ramification locus D of π splits into two irreducible curves $B_i := \pi(b_i)$, $i = 1, 2$, which intersect at the ten points $A_i := \pi(a_i)$, $i = 1, \ldots, 10$. Furthermore, the irreducible bidegree $(1, 1)$ curve $C := \pi(c)$ passes through the four points $A_1, \ldots, A_4$; it follows that the coordinates of these points can be chosen to be (r, r), where $r = 0$, 1, ∞, or $\lambda \in \Bbbk \smallsetminus \{0, 1\}$.

An additional property of the irreducible bidegree $(3, 1)$ curve B_1 is that it is tangent to the four lines $y = r$ (as a special case, inflection tangent at $x = r$). To find this curve, we change the x-coordinate so that the first three *tangency* points are at $x = 0, 1, \infty$, respectively. Then, the equation has the form

$$x^3 + (u - 1)x^2 + y(v_1 x + v_0) = 0,$$

and the tangency at $(1, 1)$ gives us $v_0 = u + 1$, $v_1 = -2u - 1$.

First, assume that the degree 3 projection $B_1 \to \mathbb{P}^1$ is not purely inseparable. Then, the fourth tangency level is

$$y = \lambda(u) = -\frac{(u-1)^3(u+1)}{(2u+1)^3};$$

since $\lambda \neq 0, 1, \infty$, we have $u \neq 0, \pm 1, -2, -\frac{1}{2}$. In these coordinates, the additional points of intersection of B_1 with the four tangents $y = r$ are

$$x_0 = -u + 1, \quad x_1 = -u - 1, \quad x_\infty = \frac{u+1}{2u+1}, \quad x_\lambda = \frac{u-1}{2u+1}.$$

Now, switching back to the original coordinates, *i.e.*, sending x_0, x_1, x_∞ to $0, 1, \infty$, respectively, we arrive at the following x-coordinate of A_4:

$$\mu(u) := -\frac{(u+1)^3(u-1)}{2u+1}.$$

Equating $\mu(u) = \lambda(u)$ and disregarding the values of u ruled out above, we obtain $u^2 + u + 1 = 0$, *i.e.*, $u = \epsilon_{1,2}$ is a primitive 3-rd root of unity. In particular, since $u \neq 1$, we have $\operatorname{char} \Bbbk \neq 3$.

If $B_1 \to \mathbb{P}^1$ is purely inseparable, then $\operatorname{char} \Bbbk = 3$ and $u = 1$, *i.e.*, B_1 is given by $x^3 = y$. In this case, one easily finds that $\lambda = \mu = -1$.

The curve B_2 has similar properties, and it passes through the same quadruple of points $A_1, \ldots, A_4$; hence, its equation is that of B_1, *with the same value* of $u = 1$ (if $\operatorname{char} \Bbbk = 3$), ϵ_1, or ϵ_2, with x and y interchanged. It is easily seen that the two values $u = \epsilon_{1,2}$ can be interchanged by an appropriate change of coordinates. Thus, the ramification locus $D \subset \mathbb{P}^1 \times \mathbb{P}^1$ is unique up to isomorphism, and the quartic $X \in \mathbb{P}^3$ is given by (8.4) with f_1, f_2 as described above. □

Corollary 8.9. *A nonsingular quartic $X \subset \mathbb{P}^3$ with $\operatorname{Fn} X \cong \operatorname{Fn} \mathbf{X}_{64}$ exists if and only if $\operatorname{char} \Bbbk \neq 2$ or 3. If exists, X is isomorphic to Schur's quartic.*

Proof. The uniqueness and the restriction $\operatorname{char} \Bbbk \neq 2$ are given by Lemma 8.8. If $\operatorname{char} \Bbbk \neq 2$ or 3, the classical Schur quartic is nonsingular and contains exactly 64 lines, which follows from the classical description of these lines. If $\operatorname{char} \Bbbk = 3$, we obtain $\epsilon = 1$ in the proof of Lemma 8.8 and both pencils become quasi-elliptic; hence, X becomes supersingular and, by Theorem 1.2, X is the Fermat quartic with 112 lines. The same conclusion can as well be derived from the explicit equation. □

It is fairly easy to describe all quartics satisfying conditions (1), (2) of Lemma 8.7, *i.e.*, without assuming the existence of line c. To this end, we should not assume that A_4 has equal coordinates in the proof of Lemma 8.8. Thus, we merely start with a pair of curves B_1, B_2 with distinct values u, v of the parameter and equate $\lambda(u) = \mu(v)$ and $\lambda(v) = \mu(u)$. This gives us a 1-parameter family

$$2uv + u + v + 2 = 0$$

and a number of discrete pairs, satisfying

$$(2v + 1)^2 u^3 + 3v(2v + 1)u^2 - 3v(2v + 1)u - v(v + 2)^2 = 0$$

and the same equation with u, v interchanged. The two latter result in three sets of Galois conjugate solutions:

- $2u^4 + 4u^3 + 2u^2 + 1 = 0$ and $v = -u - 1$,
- $u^4 + 2u^2 + 4u + 2 = 0$ and $v = -u^3 + u^2 - 3u - 2$, and
- $u^4 + 4u^3 + 8u^2 + 4u + 1 = 0$ and $v = -u^3 - 4u^2 - 8u - 4$,

probably not all distinct. Strictly speaking, with u fixed, over some primes there may be other solutions for v. However, we do not investigate this issue any further, nor do we discuss the conditions under which the ramification loci obtained do give rise to nonsingular quartics.

8.3. **Proof of Theorem 1.3.** If X is not supersingular, there exists a quartic X_0 defined over a field $\Bbbk_0$, $\operatorname{char} \Bbbk_0 = 0$, with the property that $NS(X_0) \cong NS(X)$, see Theorem 3.2. Then, according to [6], either $|\operatorname{Fn} X_0| \leqslant 60$ or $NS(X_0) = \mathbf{X}_{64}$, and the same dichotomy applies to the original surface X. If $\operatorname{char} \Bbbk = 2$ or 3, the last possibility is ruled out by Corollary 8.9. □

Remark 8.10. According to [6] (and Theorem 3.2), the number $|\operatorname{Fn} X|$ of lines in a quartic X that is not supersingular takes values $\leqslant 52$, 54, 56, 60, or 64. Found in [6] is also a complete list of all configurations $\operatorname{Fn} X$ of size at least 54.

Assume that $\operatorname{char} \Bbbk = 2$ or 3. Then, the maximal value $|\operatorname{Fn} X| = 64$ is ruled out by Theorem 1.3. The next value $|\operatorname{Fn} X| = 60$ is realized by two configurations, $\mathbf{X}'_{60}$ and $\mathbf{X}''_{60}$ in the notation of [6]. An explicit defining equation of the latter quartic is obtained in [17], and it has a nonsingular reduction (still with 60 lines) over $\mathbb{F}_4$. In characteristic 3, the *known* quartics become singular. Conjecturally (M. Schütt, private communication; the conjecture is based on the arithmetical properties of the discriminant $\det NS(X) = -60$ or -55), even if a quartic X with $\operatorname{Fn} X \cong \mathbf{X}'_{60}$ or $\mathbf{X}''_{60}$ admits a nonsingular reduction modulo 3, the latter must be supersingular; by Theorem 1.2, it would be isomorphic to the Fermat quartic.

One of the three configurations with 56 lines is $\mathbf{X}_{56}$; as was recently observed by T. Shioda, over $\mathbb{C}$ this surface is a non-standard projective model of the Fermat quartic. The defining equation found by I. Shimada [23] has a nonsingular reduction in characteristic 3 and the quartic obtained has 56 lines. Thus, the only case still open is that of $\operatorname{char} \Bbbk = 3$ and $|\operatorname{Fn} X| = 60$.

8.4. **Quartics with a pencil of type $(10, 0)$.** We conclude this section with the defining equations of a few supersingular quartics in characteristic 3.

Any quartic $X \subset \mathbb{P}^3$ containing a pencil $\mathcal{P}(b_2)$ of type $(10, 0)$ is supersingular and one has $\operatorname{char} \Bbbk = 3$, see Proposition 4.8; arithmetically, such quartics are described in Proposition 6.1. If $\mathcal{P}(b_2)$ has a section $b_1 \in \operatorname{Fn} X$, this section intersects ten lines $a_1, \ldots, a_{10} \in \mathcal{P}(b_2)$. Hence, X satisfies the hypotheses of Proposition 8.2 and its equation is given by (8.4) as

$$z_1^3 z_3 f\Bigl(\frac{z_0}{z_1}, \frac{z_2}{z_3}\Bigr) = z_0 z_3^3 - z_1 z_2^3, \tag{8.11}$$

where $f(x, y)$ is an irreducible polynomial of bidegree $(3, 1)$ such that all ten roots of $f(y^3, y)$ are simple. (Equations of this form have been studied in [22].) As an immediate consequence, we have the following statement. (Note that, according to D. Veniani, private communication, the conclusion of this statement holds without the assumption that the pencil has a section.)

Corollary 8.12. *If a pencil $\mathcal{P}(b_2)$ of type $(10, 0)$ has a section, then, for any 3-fiber s_1, s_2, s_3 of the pencil, the four lines b_1, s_1, s_2, s_3 intersect at a single point.*

The projections of the ten common lines of the pencils $\mathcal{P}(b_1)$, $\mathcal{P}(b_2)$ are the points $A_i := \pi(a_i)$ with coordinates (y_i^3, y_i), $i = 1, \ldots, 10$, where y_i are the ten roots of the polynomial $f(y^3, y)$. If $\operatorname{char} \Bbbk = 3$, one has $(y_1^3, y_2^3; y_3^3, y_4^3) = (y_1, y_2; y_3, y_4)^3$; hence, Proposition 8.5 establishes a two-to-one correspondence between the lines $l \in \operatorname{Fn} X$ disjoint from b_1, b_2 and the quadruples $y_1, \ldots, y_4$ of roots of $f(y^3, y)$ satisfying the equation $(y_1, y_2; y_3, y_4) = -1$. Observe that (still assuming $\operatorname{char} \Bbbk = 3$)

- one has $(y_1, y_2; y_3, \infty) = -1$ if and only if $y_1 + y_2 + y_3 = 0$, and
- if y'_i, y''_i are the roots of $y^2 + p_i y + q_i$, $i = 1, 2$, then $(y'_1, y''_1; y'_2, y''_2) = -1$ if and only if $p_1 p_2 + q_1 + q_2 = 0$.

Below, we consider a few special cases.

Proposition 8.13. *Any quartic X containing a pair of obverse pencils of type $(10, 0)$ is projectively equivalent to the Fermat quartic.*

Proof. Since $\mathcal{P}(b_1)$ is also of type $(10, 0)$, one has $f(x, y) = x^3 - y$ in (8.11). □

Proposition 8.14. *A quartic X is as in Theorem 1.2(3) if and only if X contains a pair of obverse pencils $\mathcal{P}(b_1)$, $\mathcal{P}(b_2)$ of types $(4, 6)$ and $(10, 0)$ respectively. Up to* *projective transformation, such quartics constitute the* 1*-parameter family given by* (8.11) *with*

$$f(x,y) = x^3 - (\epsilon+1)(\epsilon y + y + \epsilon)x^2 + \epsilon^2(\epsilon+1)(y+\epsilon+1)x - \epsilon^4 y,$$

where $\epsilon \in \Bbbk \smallsetminus \mathbb{F}_3$*.*

Proof. The "only if" part is given by the explicit description of the configuration of lines in X. For the converse, we use the computation in the proof of Lemma 8.8. In the notation introduced there, let B_2 be given by $y^3 = x$; then, in appropriate coordinates, B_1 is in the 1-parameter family $B_1(u)$, $u \neq 1$, considered in the proof. To simplify the notation, we let $u := \epsilon - 1$; then $\epsilon \neq -1$ (as otherwise B_1 is purely inseparable and X is the Fermat quartic, see Proposition 8.13) and $\epsilon \neq 0, 1$ (as otherwise B_1 is reducible). By the construction, the intersection $B_1 \cap B_2$ contains the points (r, r), $r = 0, 1, \infty$, and the fourth point $(\mu, \lambda) = (\epsilon^3, \epsilon)$ found in the proof also lies in B_2. (In other words, assuming that $\mathcal{P}(b_1)$ has three 3-fibers, we obtain a fourth one, which agrees with Proposition 6.13.)

In addition to 0, 1, ϵ, ∞, the polynomial $f(y^3, y)$ has three pairs of roots (y_i', y_i'') satisfying the quadratic equations $q_i(y_i') = g_i(y_i'') = 0$, where $i = 0, 1, \epsilon$ and

$$g_0(y) := y^2 - \epsilon, \qquad g_1(y) := y^2 + y + \epsilon, \qquad g_\epsilon(y) := y^2 + \epsilon y + \epsilon.$$

Using the two observations prior to Proposition 8.13, one can see that

$$(i, \infty; y_i', y_i'') = (i, j; y_k', y_k'') = (y_i', y_i''; y_k', y_k'') = -1$$

for any permutation $\{i, j, k\}$ of $\{0, 1, \epsilon\}$. It follows that X contains at least 58 lines and, thus, is as in Theorem 1.2(3). □

Proposition 8.15. *Up to projective equivalence, the quartics as in Theorem 1.2(2) constitute the* 1*-parameter family given by* (8.11) *with*

$$f(x,y) = \epsilon y - (\epsilon+1)x + x^3,$$

where $\epsilon \in \Bbbk^\times$*.*

Proof. Let X be a quartic as in the statement; then $\operatorname{Fn} X$ is the union of the two type $(10, 0)$ pencils contained in X. Choose for $\mathcal{P}(b_2)$ one of these pencils; then, the other pencil is $\mathcal{P}(a_\infty)$ for a certain line $a_\infty \in \mathcal{P}(b_2)$, and each section of $\mathcal{P}(b_2)$ intersects a_∞. Pick a section b_1 and consider the corresponding projection $\pi := \pi[b_1, b_2]\colon X \to \mathbb{P}^1 \times \mathbb{P}^1$. The coordinates (x, y) in $\mathbb{P}^1 \times \mathbb{P}^1$ can be chosen so that $B_2 := \pi(b_2)$ is given by $x = y^3$, the line a_∞ projects to $A_\infty(\infty, \infty)$, and some other line $a_0 \in \mathcal{P}(b_1) \cap \mathcal{P}(b_2)$ projects to $(0, 0)$.

According to Corollary 8.12, the curve $B_1 := \pi(b_1)$ is inflection tangent at A_∞ to the fiber $y = \infty$; hence, its defining polynomial is of the form

$$f(x,y) = \epsilon y + ux + vx^2 + x^3, \quad \epsilon \neq 0.$$

A simple count shows that there is a line $a \in \mathcal{P}(b_1) \cap \mathcal{P}(b_2)$ other than a_∞ that intersects at least eight lines disjoint from b_1, b_2. Assuming that $a = a_0$, from the observation prior to Proposition 8.13 one concludes that the polynomial $f(y^3, y)$ must be odd; hence, $v = 0$. Up to projective transformation, we can also assume that ± 1 are among the roots; then $u = -(\epsilon + 1)$ and $f(x, y)$ is as in the statement.

In addition to 0, ∞, and ± 1, the roots of $f(y^3, y)$ are those of $y^6 + y^4 + y^2 - \epsilon$; denoting by t one of the extra roots, one can easily see that all six roots are $\pm t$ and $\pm(t \pm 1)$; they are all pairwise distinct and different from 0 and ± 1. (Since $\epsilon \neq 0$, one has $t \notin \mathbb{F}_3$). The set $\{\pm 1, \pm t, \pm(t \pm 1)\}$ contains eight triples summing up to 0. Hence, X contains at least 58 lines and, thus, is as in the statement. □

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