SLOPES OF LINKS AND SIGNATURE FORMULAS

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Abstract. We present a new invariant, called *slope*, of a colored link in an integral homology sphere and use this invariant to complete the signature formula for the splice of two links. We develop a number of ways of computing the slope and a few vanishing results. Besides, we discuss the concordance invariance of the slope and establish its close relation to the Conway polynomials, on the one hand, and to the Kojima–Yamasaki η-function (in the univariate case) and Cochran invariants, on the other hand.

1. Introduction

The principal goal of this short note is to present a brief account of our recent results and announce a few further advances concerning the new invariant of colored links, called *slope*. Originally introduced as a means of handling an extra correction term in our signature formula (*cf.* Theorem 2.3 *vs.* Theorem 2.4), the slope proved to be interesting on its own right. It is closely related to, but not strictly dependent on, the Conway potentials of the links involved (*cf.* Theorem 3.7 *vs.* Example 3.9) and, as such, it can be regarded as a generalization, to both multicomponent links and those with nonvanishing linking numbers, of the Kojima η-function (see Corollary 3.8). In the case of two-component links, we observe a similar relation to Milnor's $\bar{\mu}$-invariants (see Proposition 5.4 *vs.* Example 5.5); we expect to generalize these results to more components. The slope is easily computable in many ways, see §3.2, Theorem 3.7, or §4 (new results). The most important new aspect is the concordance invariance of the slope, see §5.

The principal object of our study is a μ-*colored link*, *i.e.*, an oriented link L in an integral homology sphere $\mathbb{S}$ equipped with a surjective map $\pi_0(L) \twoheadrightarrow \{1, \dots, \mu\}$, called *coloring*. The union of the components of L given the same color $i = 1, \dots, \mu$ is denoted by L_i. We denote by $X := \mathbb{S} \smallsetminus T_L$ the complement of a small open tubular neighborhood of L. The group $H_1(X)$ is free abelian, generated by the classes m_C of the meridians of the components $C \subset L$. The coloring induces an epimorphism

$$\varphi : \pi_1(X) \twoheadrightarrow H := \bigoplus_{i=1}^{\mu} \mathbb{Z} t_i \tag{1.1}$$

sending m_C to t_i whenever $C \subset L_i$. A multiplicative character $\omega \colon \pi_1(X) \to \mathbb{C}^\times$ is determined by its values on the meridians, and the torus of characters preserving the coloring (*i.e.*, those that factor through φ) is naturally identified with $(\mathbb{C}^\times)^\mu$. Often, we split the components of L into two groups, $L = L' \cup L''$, on which the coloring takes, respectively, μ' and μ'' values, $\mu' + \mu'' = \mu$; then, we regard a character as a "vector" $\omega = (\omega', \omega'') \in (\mathbb{C}^\times)^{\mu'} \times (\mathbb{C}^\times)^{\mu''}$, *cf.* §2.3.

Given a topological space X and a multiplicative character $\omega \colon \pi_1(X) \to \mathbb{C}^\times$, we denote by $H_*(X; \mathbb{C}(\omega))$ the homology of X with coefficient in the local system $\mathbb{C}(\omega)$ twisted by ω.

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2. The signature formula

2.1. **Signature of colored links.** Let L be a μ-colored link in an integral homology sphere $\mathbb{S}$. A *spanning pair* for $(\mathbb{S}, L)$ is a pair (N, F), where N is a compact smooth oriented 4-manifold such that $\partial N = \mathbb{S}$ and $F = F_1 \cup \ldots \cup F_\mu \subset N$ is a properly immersed surface (see [8]) such that $\partial F_i = F_i \cap \partial N = L_i$ for all $i = 1, \ldots, \mu$. Fix an open tubular neighborhood T_F of F and let $W_F := N \smallsetminus T_F$. We assume that the group $H_1(N \smallsetminus F)$ is freely generated by the meridians of the components of F. Then, any character ω on $\mathbb{S} \smallsetminus L$ extends to a unique character on W_F, also denoted by ω. If $\omega \in (S^1)^\mu$ is unitary, the space $H_2(W_F; \mathbb{C}(\omega))$ has a Hermitian intersection form, and we denote by $\operatorname{sign}^\omega(W_F)$ its signature.

Definition 2.1. The *signature* and *nullity* of a μ-colored link $L \subset \mathbb{S}$ are the maps

$$\begin{aligned} \sigma_L \colon (S^1 \smallsetminus 1)^\mu &\longrightarrow \mathbb{Z}, & n_L \colon (S^1 \smallsetminus 1)^\mu &\longrightarrow \mathbb{Z}, \\ \omega &\longmapsto \operatorname{sign}^\omega(W_F) - \operatorname{sign}(W_F), & \omega &\longmapsto \dim H_1(\mathbb{S} \smallsetminus L; \mathbb{C}(\omega)), \end{aligned}$$

respectively. By convention, we extend these functions to all unitary characters $\omega \in (S^1)^\mu$ by patching the components L_i on which $\omega_i = 1$. According to [8] or [16], σ_L is independent of the choice of a spanning pair.

2.2. **Splice of colored links.** A $(1, \mu)$-*colored link* is a $(1 + \mu)$-colored link of the form

$$K \cup L = K \cup L_1 \cup \ldots \cup L_\mu,$$

where the *knot* K is the only component given the distinguished color 0. We define the *linking vector* $\overline{\ell k}(K, L) = (\lambda_1, \ldots, \lambda_\mu) \in \mathbb{Z}^\mu$, where $\lambda_i := \ell k(K, L_i)$ and ℓk is the linking number in the integral homology sphere $\mathbb{S}$. In the following definition, for a $(1, \mu^*)$-colored link $K^* \cup L^* \subset \mathbb{S}^*$, $* = {}'$ or ${}''$, we denote by $T_{K^*} \subset \mathbb{S}^*$ a small tubular neighborhood of K^* disjoint from L^*. Let $m^*, \ell^* \subset \partial T_{K^*}$ be, respectively, its meridian and *Seifert longitude*, *viz.* the longitude unlinked with K^* in the homology sphere $\mathbb{S}^*$. We orient the meridian m^* so that $m^* \circ \ell^* = 1$ *with respect to the orientation of* ∂T_{K^*} *induced from* T_{K^*}.

Definition 2.2. Given two $(1, \mu^*)$-colored links $K^* \cup L^* \subset \mathbb{S}^*$, $* = {}'$ or ${}''$, their *splice* is the $(\mu' + \mu'')$-colored link $L' \cup L''$ in the integral homology sphere

$$\mathbb{S} := (\mathbb{S}' \smallsetminus \operatorname{int} T_{K'}) \cup_\varphi (\mathbb{S}'' \smallsetminus \operatorname{int} T_{K''}),$$

where the gluing homeomorphism $\varphi \colon \partial T_{K'} \to \partial T_{K''}$ takes m' and ℓ' to ℓ'' and m'', respectively.

2.3. **(Non)-additivity of the signature.** The *index* of $x \in \mathbb{R}$ is $\operatorname{ind}(x) := \lfloor x \rfloor - \lfloor -x \rfloor \in \mathbb{Z}$. The Log-*function* $\operatorname{Log} \colon S^1 \to [0, 1)$ sends $\exp(2\pi i t)$ to $t \in [0, 1)$. For an integral vector $\lambda \in \mathbb{Z}^\mu$, $\mu \geqslant 0$, we define the *defect function*

$$\begin{aligned} \delta_\lambda \colon (S^1)^\mu &\longrightarrow \mathbb{Z} \\ \omega &\longmapsto \operatorname{ind}\bigl(\textstyle\sum_{i=1}^\mu \lambda_i \operatorname{Log} \omega_i\bigr) - \textstyle\sum_{i=1}^\mu \lambda_i \operatorname{ind}(\operatorname{Log} \omega_i). \end{aligned}$$

Theorem 2.3 (see [7])**.** *For $* = {}'$ or ${}''$, consider a $(1, \mu^*)$-colored link $K^* \cup L^* \subset \mathbb{S}^*$, and let $L \subset \mathbb{S}$ be the splice of the two links. Let $\lambda^* := \overline{\ell k}(K^*, L^*)$ and, for characters $\omega^* \in (S^1)^{\mu^*}$, denote $v^* := (\omega^*)^{\lambda^*} \in S^1$. Then,* assuming that $(v', v'') \neq (1, 1)$*, one has*

$$\begin{aligned} \sigma_L(\omega', \omega'') &= \sigma_{K' \cup L'}(v'', \omega') + \sigma_{K'' \cup L''}(v', \omega'') + \delta_{\lambda'}(\omega')\delta_{\lambda''}(\omega''), \\ n_L(\omega', \omega'') &= n_{K' \cup L'}(v'', \omega') + n_{K'' \cup L''}(v', \omega''). \end{aligned}$$

In the special case $v' = v'' = 1$, the formulas of Theorem 2.3 are no longer valid. If $v^* = 1$, the character ω^* is *admissible* (see Definition 3.1 below) and, hence, there is a well-defined *slope* $\kappa^* := (K^*/L^*)(\omega^*) \in \mathbb{R} \cup \infty$ (see Proposition 3.3; recall that all characters are *unitary*). These

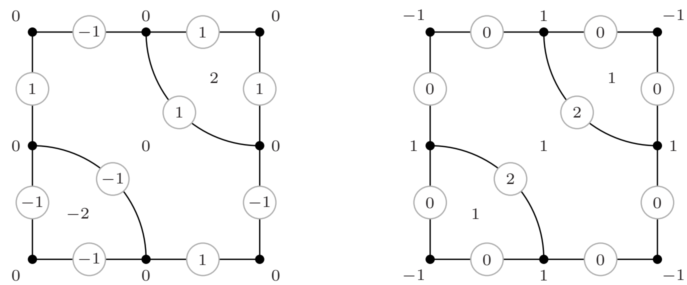

FIGURE 1. The correction terms $\Delta\sigma$ (left) and Δn (right) in Theorem 5.1.

two slopes give rise to extra correction terms shown in Figure 1. Here, the domain of $\Delta\sigma(\kappa', \kappa'')$ and $\Delta n(\kappa', \kappa'')$ is the square $[-\infty, \infty]^2 \ni (\kappa', \kappa'')$, and the curve in the figures is the hyperbola $\kappa'\kappa'' = 1$. For $\Delta\sigma$, there also is an "explicit" formula

$$\Delta\sigma(\kappa', \kappa'') = \operatorname{sg}\kappa' - \operatorname{sg}\left(\frac{1}{\kappa'} - \kappa''\right),$$

where

$$\operatorname{sg} x = \begin{cases} 0, & \text{if } x = 0 \text{ or } \infty, \\ 1, & \text{if } x > 0, \\ -1, & \text{if } x < 0, \end{cases}$$

and we disambiguate $\infty - \infty = 0$. It is worth mentioning that, in the next statement, the terms $\Delta\sigma$ and Δn are the *only* contribution of the knots K^*; the rest depends on the links L^* only.

Theorem 2.4 (see [8])**.** *For $* = {}'$ or $''$, consider a $(1, \mu^*)$-colored link $K^* \cup L^* \subset \mathbb{S}^*$, and let $L \subset \mathbb{S}$ be the splice of the two links. Let $\omega^* \in (S^1)^{\mu^*}$ be two admissible characters (so that $\upsilon' = \upsilon'' = 1$), and denote $\kappa^* = (K^*/L^*)(\omega^*)$. Then, with $\Delta\sigma, \Delta n \in \{0, \pm1, \pm2\}$ as in Figure 1, one has*

$$\sigma_L(\omega', \omega'') = \sigma_{L'}(\omega') + \sigma_{L''}(\omega'') + \delta_{\lambda'}(\omega')\delta_{\lambda''}(\omega'') + \Delta\sigma(\kappa', \kappa''),$$
$$n_L(\omega', \omega'') = n_{L'}(\omega') + n_{L''}(\omega'') + \Delta n(\kappa', \kappa'').$$

3. THE SLOPE

In this section we define the slope of a $(1, \mu)$-colored link: it is a function defined on a subset of the character torus and taking values in the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \infty$. The domain of the slope contains all *unitary* admissible (see Definition 3.1) characters, on which the slope takes *real* values (see Proposition 3.3); this fact is used in Theorem 2.4.

3.1. **Definition of the slope.** Fix a $(1, \mu)$-colored link $K \cup L \subset \mathbb{S}$ and let $X := \mathbb{S} \smallsetminus T_L$ and $\bar{X} = \mathbb{S} \smallsetminus T_{K \cup L}$. We denote by m and ℓ, respectively, the meridian of K and its Seifert longitude (*cf.* the convention in §2.2).

Definition 3.1. A character $\omega : \pi_1(X) \longrightarrow \mathbb{C}^\times$ is called *admissible* if $\omega([K]) = 1$.

The variety of admissible characters is denoted

$$\mathcal{A}(K/L) = \left\{\omega \in (\mathbb{C}^\times)^\mu \;\middle|\; \omega^\lambda = 1\right\},$$

where $\lambda := \overline{\ell k}(K, L)$ is the linking vector. If $\lambda = 0$, we have $\mathcal{A}(K/L) = (\mathbb{C}^\times)^\mu$; otherwise, letting $N :=$ g.c.d.(λ) and $\nu := \lambda/N$, the irreducible over $\mathbb{Q}$ components of $\mathcal{A}(K/L)$ are the zero sets

$\mathcal{A}_d(K/L)$ of the cyclotomic polynomials $\Phi_d(\omega^\nu)$, $d \mid N$. We will mainly deal with *nonvanishing* admissible characters

$$\omega \in \mathcal{A}^\circ(K/L) := \mathcal{A}(K/L) \cap (\mathbb{C}^\times \smallsetminus 1)^\mu.$$

Any character $\omega \in \mathcal{A}^\circ(K/L)$ induces a character $\omega\colon \pi_1(\bar{X}) \to \mathbb{C}^\times$ sending m and ℓ to 1.

Consider the intersection $\partial_K \bar{X} = \partial T_K$ of $\partial \bar{X}$ with the closure of T_K and the inclusion

$$i : \partial_K \bar{X} \hookrightarrow \bar{X}.$$

Denote $Z(\omega) := \ker i_*$, where

$$i_* : H_1(\partial_K \bar{X}; \mathbb{C}(\omega)) \longrightarrow H_1(\bar{X}; \mathbb{C}(\omega)).$$

If $\omega \in \mathcal{A}^\circ(K/L)$ is admissible, m and ℓ form a basis of $H_1(\partial_K \bar{X}; \mathbb{C}(\omega)) = \mathbb{C}^2$.

Definition 3.2. Let $\omega \in \mathcal{A}^\circ(K/L)$ and assume that $\dim Z(\omega) = 1$, so that $Z(\omega)$ is generated by a single vector $am + b\ell$ for some $[a : b] \in \mathbb{P}^1$. The *slope* of $K \cup L$ at $\omega \in \mathcal{A}^\circ(K/L)$ is the quotient

$$(K/L)(\omega) := -\frac{a}{b} \in \mathbb{P}^1 = \mathbb{C} \cup \infty.$$

As in Definition 2.1, we extend the slope to the whole set $\mathcal{A}(K/L)$ by patching the components of L on which ω vanishes.

Thus, strictly speaking, the slope is defined on a *subset* of $\mathcal{A}(K/L)$. This subset is dense (see Theorem 3.6 below); moreover, according to the next statement, the slope is always well defined and *real* on the *unitary* admissible characters, so that Theorem 2.4 makes sense.

Proposition 3.3 (see [8])**.** *Let $\omega \in \mathcal{U}(K/L) := \mathcal{A}(K/L) \cap (S^1 \smallsetminus 1)^\mu$ be a* unitary *character. Then $\dim Z(\omega) = 1$ and, hence, $(K/L)(\omega)$ is well defined. Moreover, $(K/L)(\omega) \in \mathbb{R} \cup \infty$.*

Proposition 3.4 (essentially, [8])**.** *For a $(1, \mu)$-colored link $K \cup L \subset \mathbb{S}$ one has*:

1. $(-K/L)(\omega) = (K/{-L})(\omega) = (K/L)(\omega) = (K/L)(\omega^{-1})$;
2. $(K/L')(\omega_1, \omega_2, \ldots) = (K/L)(\omega_1^{-1}, \omega_2, \ldots)$, *where* $L' := -L_1 \cup L_2 \cup \ldots \cup L_\mu$;
3. $(\bar{K}/\bar{L}) = -(K/L)$, *where $\bar{K} \cup \bar{L}$ is the mirror of $K \cup L$; thus, if $K \cup L$ is amphichiral, then K/L takes values in* $\{0, \infty\}$;
4. *if K lies in a ball disjoint from L, then* $K/L \equiv 0$.
5. *if K bounds a Seifert surface disjoint from a C-complex for L, then* $K/L \equiv 0$;

3.2. **Fox calculus.** The slope can be computed by means of Fox calculus from a presentation of the fundamental group $\pi_1(\bar{X})$ of the link complement, together with the classes $m, \ell \in \pi_1(\bar{X})$. In the case of links in S^3, both pieces of data can be derived from the link diagram. Indeed, for the group one can choose the Wirtinger presentation, where meridians are the generators. For ℓ, we trace a curve C parallel to K and such that $\ell k(K, C) = 0$; then, starting from the segment corresponding to the chosen meridian of K and moving along C in the positive direction, we write down the corresponding generator (or its inverse) each time when undercrossing positively (respectively, negatively) the diagram of $K \cup L$. Thus, let

$$m, \ell \in \pi_1(\bar{X}) = \langle x_1, \ldots, x_p \mid r_1, \ldots, r_q \rangle$$

and, using the epimorphisms $F := \langle x_1, \ldots, x_p \rangle \twoheadrightarrow \pi_1(\bar{X}) \twoheadrightarrow \pi_1(X) \twoheadrightarrow H$, *cf.* (1.1), specialize the Fox derivatives to $\partial/\partial x_i \colon F \to \Lambda := \mathbb{Z}H$. Consider the complex of Λ-modules

$$S_*\colon \quad S_2 \xrightarrow{\partial_1} S_1 \xrightarrow{\partial_0} S_0 \longrightarrow 0,$$

where

$$S_2 = \bigoplus_{i=1}^{q} \Lambda r_i, \quad S_1 = \bigoplus_{i=1}^{p} \Lambda dx_i, \quad S_0 = \Lambda$$

and dx_i stands for a formal generator corresponding to x_i. The "differential" of a word $w \in F$ is

$$dw := \sum_{i=1}^{p} \frac{\partial w}{\partial x_i} dx_i \in S_1;$$

then, letting

$$\partial_1 \colon r_i \mapsto dr_i, \qquad \partial_0 \colon dx_i \mapsto (\text{the image of } x_i \text{ in } H \subset \Lambda) - 1,$$

we obtain a complex computing the homology $H_{\leqslant 1}$ of the H-covering of $\bar{X}$.

Now, pick an admissible nonvanishing character $\omega \in \mathcal{A}^\circ(K/L)$ and consider the specialization $S_*(\omega) := S_* \otimes_\Lambda \mathbb{C}(\omega)$. Then, it is straightforward that

$$Z(\omega) = \operatorname{Ker}\bigl[H_1(\partial_K \bar{X}; \mathbb{C}(\omega)) = \mathbb{C}m \oplus \mathbb{C}\ell \xrightarrow{\text{in}_*} S_1(\omega)/\operatorname{Im}\partial_1(\omega)\bigr],$$

where the inclusion homomorphism in_* is the specialization of $m \mapsto dm$, $\ell \mapsto d\ell$. (Note that, by the assumption that $\omega \in \mathcal{A}^\circ(K/L)$, this homomorphism lands into $\operatorname{Ker}\partial_0(\omega)$.) Computing the above kernel in the basis m, ℓ, we can also compute the slope whenever it is defined.

Example 3.5 (the Whitehead link, see [8])**.** Consider the $(1,1)$-colored Whitehead link $K \cup L$, see L5a1 in [1]. Since $\ell k(K, L) = 0$, we have $\mathcal{A}^\circ(K/L) = S^1 \smallsetminus 1$. The "standard" presentation of $\pi_1(\bar{X})$ (derived from the Wirtinger presentation) is

$$\pi_1(\bar{X}) = \bigl\langle m, m_1, \ell \bigm| [m, \ell] = 1, \ell = m_1 m^{-1} m_1^{-1} m m_1^{-1} m^{-1} m_1 m \bigr\rangle,$$

where m and m_1 are meridians of K and L, respectively, and ℓ is a Seifert longitude of K. We can further specialize Λ to the group ring $\mathbb{Z}H_1(\bar{X}) = \mathbb{Z}[t^{\pm 1}, t_1^{\pm 1}]$, sending $m \mapsto t$, $m_1 \mapsto t_1$, $\ell \mapsto 1$. Then, denoting by x, y the two relations in the presentation above, we have

$$dx = (t-1)d\ell, \quad dy = d\ell - t^{-1}(1-t_1)(1-t_1^{-1})dm - (1-t^{-1})(1-t_1^{-1})dm_1.$$

The specialization at a character $\omega \in \mathcal{A}^\circ(K/L)$ means sending $t_1 \mapsto \omega$ and $t \mapsto 1$, so that the image $\operatorname{Im}\partial_1(\omega)$ is generated by $dy \mapsto d\ell - (1-\omega)(1-\omega^{-1})dm$. Thus,

$$(K/L)(\omega) = (1-\omega)(1-\omega^{-1}).$$

In particular, notice that the slope is not invariant under link homotopies: the Whitehead link is link homotopic to the unlink of two components, which has trivial slope.

3.3. **The rationality.** The *characteristic varieties* $\mathcal{V}_r(X)$ of X (related to φ) are defined *via*

$$\mathcal{V}_r(X) := \bigl\{\omega \in (\mathbb{C}^\times)^\mu \bigm| \dim H_1(X; \mathbb{C}(\omega)) \geqslant r\bigr\}, \quad r \geqslant 0.$$

They are algebraic varieties in $(\mathbb{C}^\times)^\mu$, which are nested ($\mathcal{V}_r \supset \mathcal{V}_{r+1}$) and depend only on the fundamental group π of X (and φ). The irreducible components of $\mathcal{V}_r(X)$ of codimension $\leqslant 1$ are closely related to the $\mathbb{Z}H$-module $H_1(X; \mathbb{Z}H)$. They constitute the zero locus of the $(r-1)$-*st order*

$$\Delta_{X,r-1} := \text{g.c.d.}\, E_{r-1}(H_1(X; \mathbb{Z}H)) = \text{g.c.d.}\, E_r(S_1/\operatorname{Im}\partial_1) \in \mathbb{Z}H,$$

where $E_s(M) \subset \mathbb{Z}H$ is the s-th elementary ideal. In particular, the 0-th order $\Delta_X := \Delta_{X,0}$ is called the *multivariate Alexander polynomial* of X and φ. As usual, if X and φ are as in (1.1), we abbreviate $\Delta_{L,r} := \Delta_{X,r}$ and $\Delta_L := \Delta_X$.

Theorem 3.6 (see [8])**.** *Pick a component $\mathcal{A} \subset \mathcal{A}(K/L)$ and let r be the minimal integer such that $\Delta_{L,r}|_{\mathcal{A}} \neq 0$,* i.e.*, $\mathcal{A} \smallsetminus \mathcal{V}_{r+1}(L)$ is dense in $\mathcal{A}$. Denote by R the coordinate ring of $\mathcal{A}$ and fix a normalization of $\Delta_{L,r}$. Then, either*

(1) *there exists a unique polynomial* $\Delta_{\mathcal{A}} \in R$ *such that*

$$(K/L)(\omega) = \frac{\Delta_{\mathcal{A}}(\omega)}{\Delta_{L,r}(\omega)}$$

holds for each character $\omega \in \mathcal{A}^\circ \smallsetminus \mathcal{V}_{r+1}(L)$, *or*

(2) *the slope* $(K/L)(\omega) = \infty$ *is well defined and infinite at each character* ω *in a certain dense Zariski open subset of* $\mathcal{A}$.

Case (2) *cannot occur if* $r = 0$, i.e., *if* $\Delta_L|_{\mathcal{A}} \neq 0$, cf. *Theorem 3.7 below.*

According to this theorem, the slope gives rise to a rational function, possibly identical ∞, on the variety of admissible characters, *i.e.*, an element of $\mathbb{Q}(\mathcal{A}(K/L)) \cup \infty$. Note that this function does not depend on the coloring too much: one can start with the maximal coloring (assigning its own color to each component), upon which it suffices to identify the variables (components of ω) corresponding to the components of L given the same color.

In the next theorem, we evaluate the Conway potential at the radical $\sqrt{\omega} := (\sqrt{\omega_1}, \ldots, \sqrt{\omega_\mu})$, which is not quite well defined. We use the convention that one of the values of each radical is chosen and used consistently *throughout the whole formula*. The nature of the formula guarantees that the result is independent of the initial choice.

Theorem 3.7 (see [8])**.** *For a* $(1, \mu)$*-colored link* $K \cup L \subset \mathbb{S}$, *denote* $\nabla' := \dfrac{\partial}{\partial t}\nabla_{K\cup L}$, *where* t *is the first variable, corresponding to* K. *Then, for a character* $\omega \in \mathcal{A}(K/L)$, *one has*

$$(K/L)(\omega) = -\frac{\nabla'(1, \sqrt{\omega})}{2\nabla_L(\sqrt{\omega})} \in \mathbb{C} \cup \infty,$$

provided that the expression in the right hand side makes sense, i.e., $\nabla'(1, \sqrt{\omega})$ *and* $\nabla_L(\sqrt{\omega})$ *do not vanish simultaneously. In particular, the slope is well defined in this case.*

The η-function was defined by Kojima and Yamasaki [11] for two-component links in S^3 with linking number zero, generalizing Goldsmith's invariants [9]. The next corollary follows from Jin [10] computing the η-function in terms of the Alexander polynomials of $K \cup L$ and L.

Corollary 3.8 (see [8])**.** *For any two-component link* $K \cup L \subset S^3$, $\ell k(K, L) = 0$, *and character* $\omega \in \mathcal{A}(K/L)$ *such that* $\Delta_L(\omega) \neq 0$, *one has*

$$(K/L)(\omega) = \eta_{K\cup L}(\omega).$$

As an example, for the $(1, 1)$-colored Whitehead link we have $\nabla_{K\cup L}(t, t_1) = (t - t^{-1})(t_1 - t_1^{-1})$ and $\nabla_L(t_1) = 1/(t_1 - t_1^{-1})$. Hence, for any $\omega \in \mathbb{C}^\times$,

$$(K/L)(\omega) = -\big(\sqrt{\omega_1} - \sqrt{\omega_1}^{-1}\big)^2 = (1 - \omega_1)(1 - \omega_1^{-1}),$$

which agrees with Example 3.5. This example illustrates also the independence of the ratio in Theorem 3.7 of the choice of $\sqrt{\omega}$. Theorem 3.7 is inconclusive if $\nabla_L(\sqrt{\omega}) = \nabla'(1, \sqrt{\omega}) = 0$, *cf.* the next example. Furthermore, even in the univariate case and at an isolated common root, l'Hôpital's rule does *not* apply, *cf.* Example 4.6 below.

Example 3.9 (equal higher orders, see [8])**.** Let $K \cup L_1 \cup L_2$ and $K' \cup L_1' \cup L_2'$ be the links L11n353 and L11n384 (see [1]), respectively. Both have 11 crossings and 3 components, and all their orders are equal:

$$\Delta_{K\cup L} = \Delta_{K'\cup L'} = (t_2 - 1)(t - 1)^3(t_1 - 1), \quad \Delta_L = \Delta_{L'} = 0, \quad \Delta_{L,1} = \Delta_{L',1} = 1.$$

Note that Theorem 3.7 is inconclusive. Since $\overline{\ell k}(K, L) = \overline{\ell k}(K', L') = (0, 0)$, one has $\mathcal{A}^\circ(K/L) = \mathcal{A}^\circ(K'/L') = (S^1 \smallsetminus 1)^2$, and, by Fox calculus (from the link diagrams), for any character $\omega :=$

$(\omega_1, \omega_2) \in (\mathbb{C}^\times)^2$, we obtain

$$(K/L)(\omega) = -\frac{\omega_1\omega_2^2 + \omega_1^2 - 4\omega_1\omega_2 + \omega_2^2 + \omega_1}{\omega_1\omega_2}, \quad (K'/L')(\omega) = -\frac{(\omega_1 - 1)(\omega_1\omega_2^2 - 1)}{\omega_1\omega_2}.$$

Thus, the slope can distinguish links with equal Alexander polynomials and higher orders.

4. Slopes via C-complexes

A C-complex (see [3]) for a μ-colored link $L = L_1 \cup \ldots \cup L_\mu \subset S^3$ is a collection $F = \bigcup_i F_i$ of Seifert surfaces F_i, $\partial F_i = L_i$, disjoint except for a finite number of *clasps*, *i.e.*, disjoint segments connecting pairs of points $a_i \in L_i$, $a_j \in L_j$, $i \neq j$ (and disjoint from L otherwise) along which F_i and F_j intersect transversally. Each homology class in $H_1(F)$ can be represented by a collection of *proper loops*, *i.e.*, lops $\alpha\colon S^1 \to F$ such that the pull-back of each clasp is a single segment (possibly empty). We routinely identify classes, loops, and their images.

Given a vector $\varepsilon \in \{\pm 1\}^\mu$, the *push-off* α^ε of a proper loop α is the loop in $S^3 \smallsetminus F$ obtained by a slight shift of α off each surface F_i in the direction of ε_i (if α runs along a clasp $F_i \cap F_j$, the shift respects both directions ε_i and ε_j). Due to [3] (notice that our conventions differ), this operation gives rise to a well-defined homomorphism $\Theta^\varepsilon\colon H_1(F) \to H_1(S^3 \smallsetminus F)$ and we can define the *Seifert forms*

$$\theta^\varepsilon\colon H_1(F) \otimes H_1(F) \to \mathbb{Z}, \qquad \alpha \otimes \beta \mapsto \ell k(\alpha, \beta^\varepsilon).$$

Now, given a character $\omega \in (\mathbb{C}^\times \smallsetminus 1)^\mu$, we define

$$\Pi(\omega) := \prod_{i=1}^{\mu}(1 - \omega_i) \in \mathbb{C}, \qquad A(\omega) := \sum_{\varepsilon \in \{\pm 1\}^\mu} \prod_{i=1}^{\mu} \varepsilon_i \omega_i^{(1-\varepsilon_i)/2} \Theta^\varepsilon\colon H_1(F; \mathbb{C}) \to H^1(F; \mathbb{C}).$$

Let also

$$E(\omega) := \Pi(\omega)^{-1} A(\omega)\colon H_1(F; \mathbb{C}) \to H^1(F; \mathbb{C});$$

if $\omega \in (S^1 \smallsetminus 1)^\mu$ is unitary, this differs from $H(\omega)$ considered in [3] by the positive real factor $\Pi(\bar{\omega})^{-1}\Pi(\omega)^{-1}$. This operator computes ∇_L (see [2]), σ_L, n_L and the Blanchfield pairing (see [6]).

Theorem 4.1 (see [3]). *If $\omega \in (S^1 \smallsetminus 1)^\mu$ is a nonvanishing unitary character, then the operator $E(\omega)$ is Hermitian and one has $\sigma_L(\omega) = \operatorname{sign} E(\omega)$ and $n_L(\omega) = \dim \operatorname{Ker} E(\omega) + b_0(F) - 1$.*

Now, let $K \cup L$ be a $(1, \mu)$-colored link, $\overline{\ell k}(K, L) = 0$. Then, we have a well-defined class

$$\kappa \in H^1(F; \mathbb{C}), \qquad \kappa\colon \alpha \mapsto \ell k(\alpha, K).$$

Theorem 4.2 (to appear). *In the notation introduced above, for $\omega \in (\mathbb{C}^\times \smallsetminus 1)^\mu$, one has*

$$(K/L)(\omega) = \begin{cases} -\langle \alpha, \kappa \rangle, & \textit{if } \kappa \in \operatorname{Im} E(\omega) \cap \operatorname{Ker} E(\omega)^\perp, \\ \infty, & \textit{if } \kappa \notin \operatorname{Im} E(\omega) \cup \operatorname{Ker} E(\omega)^\perp, \\ \text{undefined}, & \textit{otherwise}, \end{cases}$$

where, in the first case, $\alpha \in H_1(F)$ is any class such that $E(\omega)(\alpha) = \kappa$.

Corollary 4.3. *Let $K \cup L \subset S^3$ be a $(1,1)$-colored link, $\ell k(K, L) = 0$. Pick a Seifert surface F for L disjoint from K and consider the Seifert form $\theta := \theta^+$, the associated map $\Theta\colon H_1(F) \to H^1(F)$, and the linking coefficient $\kappa \in H^1(F; \mathbb{C})$ with K. For $\omega \in \mathbb{C}^\times \smallsetminus 1$, let $A(\omega) := \Theta - \omega\Theta^*$. Then*

$$(K/L)(\omega) = \begin{cases} -(1-\omega)\langle \alpha, \kappa \rangle, & \textit{if } \kappa \in \operatorname{Im} A(\omega) \cap \operatorname{Ker} A(\omega)^\perp, \\ \infty, & \textit{if } \kappa \notin \operatorname{Im} A(\omega) \cup \operatorname{Ker} A(\omega)^\perp, \\ \text{undefined}, & \textit{otherwise}, \end{cases}$$

where, in the first case, $\alpha \in H_1(F; \mathbb{C})$ is any class such that $A(\omega)(\alpha) = \kappa$.

The homomorphism $\Theta\colon H_1(F) \to H^1(F)$ associated to the Seifert form has the property that

$$\text{all nonzero invariant factors of } \Theta - \Theta^* \text{ are } \pm 1. \tag{4.4}$$

(Indeed, $\Theta - \Theta^*$ is the intersection index, which is unimodular modulo boundary.) Conversely, any matrix satisfying (4.4) can be realized as the Seifert form of a certain link L [12]. Since also any homomorphism $H_1(F) \to \mathbb{Z}$ can obviously be realized by a knot algebraically unlinked with L, we have the following corollary, describing all univariate slopes in purely algebraic terms.

Corollary 4.5. *Let G be a finitely generated free abelian group, $\Theta\colon G \to G^\vee$ a homomorphism satisfying (4.4), and $\kappa \in G^\vee$. Denote $A(\omega) := \Theta - \omega\Theta^*\colon G\otimes\mathbb{C} \to G\otimes\mathbb{C}$. Then, there exists a $(1,1)$-colored link whose slope at $\omega \in \mathbb{C}^\times \smallsetminus 1$ is given by*

$$(K/L)(\omega) = \begin{cases} -(1-\omega)\langle\alpha,\kappa\rangle, & \textit{if } \kappa \in \operatorname{Im} A(\omega) \cap \operatorname{Ker} A(\omega)^\perp, \\ \infty, & \textit{if } \kappa \notin \operatorname{Im} A(\omega) \cup \operatorname{Ker} A(\omega)^\perp, \\ \text{undefined}, & \textit{otherwise}, \end{cases}$$

where, in the first case, $\alpha \in G\otimes\mathbb{C}$ is any class such that $A(\omega)(\alpha) = \kappa$.

Choosing a standard geometric basis for the unimodular (modulo kernel) skew-symmetric bilinear form $\theta\colon G\otimes G \to \mathbb{Z}$ associated to Θ in Corollary 4.5, one can restate (4.4) as follows: there is a constant $r \leqslant \frac{1}{2}\operatorname{rk} G$ such that the matrix θ satisfies the condition

$$\theta_{ij} - \theta_{ji} = \begin{cases} 1, & \text{if } j = i-1 \in \{1,3,\ldots,2r-1\}, \\ 0, & \text{for all other pairs } j \leqslant i. \end{cases}$$

In this case, one can find a link L with $(\operatorname{rk} G - 2r + 1)$ components. Using this observation and Corollary 4.5, one can easily construct examples of univariate slopes with prescribed properties.

Example 4.6 (l'Hôpital's rule, *cf.* Theorem 3.7)**.** Fix integers $b \neq 0, -1$, $c \neq 0$, x, y and let

$$\theta = \begin{bmatrix} 0 & b \\ b+1 & c \end{bmatrix}, \qquad \kappa = [x,y]^t.$$

By Corollary 4.5, there exists a two-component link $K \cup L$ whose slope is given by

$$(K/L)(\omega) = -\frac{(\omega-1)^2 x(2by + y - cx)}{(\omega b - b - 1)(\omega b + \omega - b)}$$

for all $\omega \neq \omega_\pm := (1 + 1/b)^{\pm 1}$. Letting $\kappa_+ := [2b+1, c]^t$ and $\kappa_- := [0,1]^t$, we have:

- if $\kappa = \kappa_\pm$, then $K/L \equiv 0$ except that $(K/L)(\omega_\pm)$ is undefined;
- if $\kappa = 0$, then $K/L \equiv 0$;
- for all other values of κ, we have $(K/L)(\omega_\pm) = \infty = \lim_{\omega\to\omega_\pm}(K/L)(\omega)$.

Now, take for θ the direct sum of two copies of the above matrix and let $\kappa := \kappa_+ \oplus \kappa_-$. For the new link $K \cup L$ we have $K/L \equiv 0$ away from $\omega_\pm$, whereas $(K/L)(\omega_\pm) = \infty \neq \lim_{\omega\to\omega_\pm}(K/L)(\omega)$, *i.e.*, l'Hôpital's rule does not apply in Theorem 3.7.

5. Concordance invariance

This section contains most of our new results. Proofs will appear elsewhere.

5.1. **Slopes and concordance.** Two $(1, \mu)$-colored links $K \cup L$ and $K' \cup L'$ in the same integral homology sphere $\mathbb{S}$ are said to be *concordant* if they bound a union $D \cup A = D \cup A_1 \cup \cdots \cup A_\mu$ of pairwise disjoint locally flat cylinders embedded in $\mathbb{S} \times I$, where $I = [0, 1]$. More precisely, D is a single cylinder with $\partial D = (K \times 0) \sqcup (-K \times 1)$ and each A_i, $i = 1, \ldots, \mu$, is a disjoint union of cylinders such that $\partial A_i = (L_i \times 0) \sqcup (-L_i \times 1)$.

Let

$$U_\mu := \{ p \in \mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}] \mid p(1, \ldots, 1) = \pm 1 \}.$$

A character $\omega \in (\mathbb{C}^\times)^\mu$ is called a *concordance root* if $p(\omega) = 0$ for some $p \in U_\mu$ (*cf.* the *Knotennullstelle* [14]). We denote by $\mathcal{A}_!(K/L) \subset \mathcal{A}(K/L)$ the subset of the characters which are *not* concordance roots.

For example, for $\mu = 1$, a root of unity ω is *not* a concordance root if and only if the order of ω is a prime power (including $\omega = 1$ of order 1). It follows that the *closure* $\bar{\mathcal{A}}_!(K/L)$ is equal to $\mathcal{A}(K/L)$ if $\overline{\ell k}(K, L) = 0$; otherwise, $\bar{\mathcal{A}}_!(K/L)$ is the union of the $\mathbb{Q}$-components $\mathcal{A}_d(K/L)$ corresponding to the divisors $d \mid \text{g.c.d.}\, \overline{\ell k}(K, L)$ that are prime powers (including $d = 1$), see §3.1.

Theorem 5.1. *Let $K \cup L$ and $K' \cup L'$ be two concordant $(1, \mu)$-colored links. Then, there is a natural identification $\mathcal{A}_!(K/L) = \mathcal{A}_!(K'/L')$ and, for all $\omega \in \mathcal{A}_!(K/L)$,*

$$(K/L)(\omega) = (K'/L')(\omega).$$

Corollary 5.2. *If two $(1, \mu)$-colored links $K \cup L$ and $K' \cup L'$ are concordant, then the slopes K/L and K'/L',* restricted to $\mathbb{Q}(\bar{\mathcal{A}}_!(K/L)) \cup \infty$ (cf. *Theorem 3.6*)*, coincide.*

Example 5.3. Note that, for a maximally colored μ-component link, Corollary 5.2 gives us μ *a priori* distinct concordance invariants, as one can take for K any of the μ components.

For example, for the two-component link L10n2 in [1] we have that $K/L \neq L/K$. Indeed, using Theorem 3.7 we compute $K/L \equiv 0$, whereas

$$(L/K)(\omega) = -\frac{(\omega - 1)^4}{\omega^4 - 3\omega^3 + 5\omega^2 - 3\omega + 1}.$$

It follows, in particular, that this link is not slice, *i.e.*, it is not concordant to the unlink of two components.

5.2. **Two-component links in the sphere.** Let $K \cup L \subset S^3$ be a two-component link with $\ell k(K, L) = 0$. The Sato–Levine invariant of this link is defined from the intersection of Seifert surfaces of K and L [15]. Cochran extended this construction and defined a sequence of concordance invariants β^i as the Sato–Levine invariants of successive derivations of $K \cup L$ [4]. He proved that the β^i are certain well-defined canonical lifts of Milnor's linking numbers of the form $\bar{\mu}(1^{2i}2^2) := \bar{\mu}(1 \ldots 122)$ [5, Theorem 6.10] (for the definition and notation of Milnor linking numbers see [5, Definition 5.1]). Moreover, he showed that the sequence of β^i is equivalent to the η-function up to a certain change of variable [4, Theorem 7.1]; thus, Corollaries 3.8 and 4.3 show that the slope $K/L \in \mathbb{Q}(\mathbb{C}^\times)$ can be used to compute Cochran's β^i. Note that the relation between β^i and the slope can be shown directly from the computation with Seifert surfaces, see Corollary 4.3.

As far as the $\bar{\mu}$ Milnor invariants are concerned, note also that they *per se* do *not* determine K/L (*cf.* Example 5.5 below); however, the *vanishing* of all $\bar{\mu}$-invariants does imply the vanishing of the slope away from the concordance roots (in fact, from the roots of $\Delta_L \in U_1$). Indeed, in this case the $\bar{\mu}$ are well defined in $\mathbb{Z}$ and the above "lift" to β^i is redundant. Example 5.3 shows that the converse does not hold: $K/L \equiv 0$ does not imply $\bar{\mu} \equiv 0$.

Proposition 5.4 (see [5, Theorem 5.4] and Theorem 3.7)**.** *Let $K \cup L \subset S^3$ be a two-component link such that all $\bar{\mu}(1^m 2^n)$ vanish. Then $(K/L)(\omega) = 0$ for all $\omega \in \mathbb{C}^\times \smallsetminus \Delta_L^{-1}(0) \supset \mathcal{A}_!(K/L)$.*

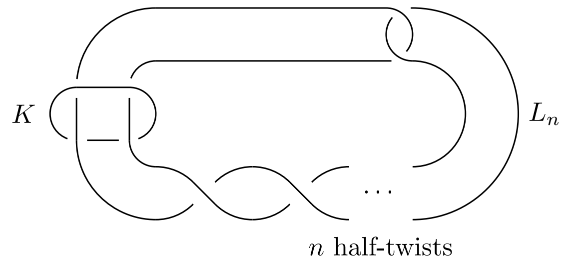

FIGURE 2. The links $K \cup L_n \subset S^3$ in Example 5.5.

Example 5.5 (equal $\bar{\mu}$-invariants, *cf.* [11])**.** Let $K \cup L_n \subset S^3$ be the family of links in Figure 2, $n \geqslant 2$. They all have the same $\bar{\mu}$-invariants, see [11, Application 3]. Using the "obvious" Seifert surface, we immediately obtain $\kappa = [1, 0]$ and

$$\theta = \begin{bmatrix} k & 0 \\ 1 & -1 \end{bmatrix} \quad \text{for } n = 2k \text{ even}, \qquad \theta = \begin{bmatrix} k+1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{for } n = 2k+1 \text{ odd}.$$

Hence, by Corollary 4.3, the slopes

$$(K/L_n)(\omega) = -\frac{(\omega - 1)^2}{\lfloor \frac{n+1}{2} \rfloor \omega^2 - \bigl(2\lfloor \frac{n}{2} \rfloor + 1\bigr)\omega + \lfloor \frac{n+1}{2} \rfloor}$$

are pairwise distinct, which agrees with the computation of η in [11], *cf.* Corollary 3.8.

In conclusion, it is worth emphasizing that, unlike the η-function, the slope can be defined and distinguish links even if the two components K and L are algebraically linked, that is, if $\ell k(K, L) \neq 0$.

Example 5.6 (linked components)**.** Let $K \cup L$ and $K' \cup L'$ be, respectively, the links L4a1 and L7n1 in [1]. Both have $\ell k(K, L) = 2$ (so that $\mathcal{A}^\circ = \mathcal{A}^\circ_! = \{-1\}$) and equal $\bar{\mu}$-invariants:

$$\bar{\mu}(12) = 2, \quad \bar{\mu}(112) = \bar{\mu}(122) = 1 \bmod 2, \quad \text{and} \quad \bar{\mu}(i_1 i_2 \dots i_n) = 0 \quad \text{for } n \geqslant 4,$$

see [13]. A simple computation using Fox calculus (see §3.2) shows that

$$(K/L)(-1) = (L/K)(-1) = -2 \quad \text{whereas} \quad (K'/L')(-1) = \frac{2}{3}, \quad (L'/K')(-1) = 6.$$

Since -1 is not a concordance root, the two links are not concordant.

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