CONICS ON BARTH–BAUER OCTICS

ALEX DEGTYAREV

Abstract. We analyze the configurations of conics and lines on a special class of Kummer octic surfaces. In particular, we bound the number of conics by 176 and show that there is a unique surface with 176 conics, all irreducible: it admits a faithful action of one of the Mukai groups. Therefore, we also discuss conics and lines on Mukai surfaces: we discover a double plane (ramified at a smooth sextic curve) that contains 8910 smooth conics.

1. Introduction

Recall that the *Kummer surface* $\operatorname{Km}(A)$ of an abelian surface A is the quotient $A/\pm 1$ blown up at the sixteen nodes—the images of the sixteen fixed points of the involution. As is well known, $\operatorname{Km}(A)$ is a $K3$-surface equipped with a distinguished collection of sixteen pairwise disjoint smooth rational curves, *viz.* the exceptional divisors contracted by the projection $\operatorname{Km}(X) \to A/\pm 1$. Conversely (Nikulin [18]), any $K3$-surface with sixteen pairwise disjoint (-2)-curves is Kummer.

Extending the construction of Barth–Bauer [1] and Bauer [2], we define a *Barth–Bauer surface* of degree $h^2 = 2n \in 2\mathbb{Z}^+$ as a smoothly polarized Kummer surface $X \hookrightarrow \mathbb{P}^{n+1}$ such that the sixteen Kummer divisors map to sixteen irreducible conics in $\mathbb{P}^{n+1}$. Conjecturally (see [7, 6]), the maximal number of irreducible conics on a smooth quartic surface is $N_4(2) = 800$, and this maximum is attained at a certain Barth–Bauer quartic. (Recall that, unlike the well-understood maximal number $N_{2n}(1)$ of lines, the maximal number $N_{2n}(2)$ of irreducible conics on a smooth $2n$-polarized $K3$-surface $X \hookrightarrow \mathbb{P}^{n+1}$ is currently known only for sextics: $N_6(2) = 285$, see [8].) Therefore, in this paper we make an attempt to estimate the maximum $N_8(2)$ for octic surfaces by obtaining a complete classification of the Barth–Bauer octics up to *equiconical deformation*, *i.e.*, deformation in $\mathbb{P}^5$ preserving the bi-colored *full Fano graph*

$$\operatorname{Fn} X := \operatorname{Fn}_1 X \cup \operatorname{Fn}_2^* X$$

of lines and irreducible conics on X. Here and below, we use the notation

- $\operatorname{Fn}_1 X$ for the graph of lines on X,
- $\operatorname{Fn}_2 X$ for the graph of all reduced conics on X, and
- $\operatorname{Fn}_2^* X \subset \operatorname{Fn}_2 X$ for the induced subgraph of irreducible conics;

in each graph, two vertices u, v are connected by an edge of multiplicity $u \cdot v$. In addition to the Fano graphs Γ and connected components of the respective *absolute strata* $\mathcal{X}(\Gamma)$ in the space $\mathcal{B}$ of all Barth–Bauer octics, we also list the *relative strata* $\tilde{\mathcal{X}}(\Gamma, \Omega) \to \mathcal{X}(\Gamma)$ consisting of pairs (X, Ω), where X is a Barth–Bauer octic and Ω is a distinguished collection of Kummer conics on X.

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The principal results of the paper, *viz.* the complete list of deformation classes, are collected in Tables 5–8 (see Theorems 3.1, 4.1, 4.2), itemized according to the codimension of the strata in the 3-parameter family $\mathcal{B}$. (Following [8, 6], we count both conics and lines, hence both irreducible and reducible conics.) Here, in the introduction, we outline a few qualitative consequences of this classification.

Theorem 1.1 (see §4.2)**.** *The maximal number of conics on a Barth–Bauer octic is* 176*. Up to projective transformation, there is a unique Barth–Bauer octic* X_{176} *with* 176 *conics, which are all irreducible*; *it is given by*

$$z_0^2 + z_3^2 - \phi z_4^2 + \phi z_5^2 = z_1^2 - \phi z_3^2 + z_4^2 - \phi z_5^2 = z_2^2 + \phi z_3^2 - \phi z_4^2 + z_5^2 = 0,$$

where $\phi := (1 + \sqrt{5})/2$ *is the golden ratio* (*see* [3])*.*

Recall that a typical smooth octic $K3$-surface in $\mathbb{P}^5$ is a *triquadric*, *i.e.*, a regular complete intersection of three quadrics. However, the moduli space contains a divisor of *special octics*, requiring at least one cubic defining equation. Equivalently (Saint-Donat [22]), special are the octics admitting an elliptic pencil of projective degree 3. We assert that Barth–Bauer octics are never special.

Theorem 1.2 (see §2.6)**.** *Any Barth–Bauer octic is a triquadric.*

Next, we support the speculation of [8] that, although it is easier (at least, using the approach suggested in [8]) to count all, not only irreducible conics, all conics on a polarized $K3$-surface are irreducible whenever their number is large enough.

Theorem 1.3 (see Tables 5–8)**.** *Let* $X \subset \mathbb{P}^5$ *be a Barth–Bauer octic. Then*:

- *the maximal number of lines on* X *is* 28 (*a single octic, see* † *in Table 7*);
- *the maximal number of reducible conics is* 48 (*same octic as above*);
- *if* $|\mathrm{Fn}_2\, X| > 128$*, then* X *is a singular* $K3$*-surface,* i.e., $\operatorname{rk} NS(X) = 20$;
- *if* $|\mathrm{Fn}_2\, X| > 128$*, then* X *has no lines* (*hence, no reducible conics*);
- *if* $|\mathrm{Fn}_2^*\, X| > 104$*, then* X *has no lines* (*hence, no reducible conics*)*.* $\lhd$

Theorems 1.1 and 1.3 should extend to all smooth octic $K3$-surfaces, but the precise bounds may differ. For example, the sharp upper bounds on the numbers of lines and reducible conics are essentially found in [5].

Theorem 1.4 (see [5] and §4.3)**.** *The maximal number of lines on a smooth octic* $K3$*-surface in* $\mathbb{P}^5$ *is* 36*, whereas the maximal number of reducible conics is* 112*.*

Theorem 1.3 and the findings of [6, 8] suggest the following conjecture.

Conjecture 1.5. There is a number $N_{2n}^*(2) < N_{2n}(2)$ with the following property: if a smooth $2n$-polarized $K3$-surface $X \subset \mathbb{P}^{n+1}$ has more than $N_{2n}^*(2)$ conics, then X has no lines and, in particular, all conics on X are irreducible.

In conclusion, we address the question about the number of real conics on a real surface (for which, as explained in [6], Barth–Bauer octics are not likely to provide good examples). The current upper bound is as follows.

Theorem 1.6 (see §4.4)**.** *The maximal number of real conics on a real Barth–Bauer octic is* 128*. There is a unique* 1*-parameter family of real Barth–Bauer octics with* 128 *real conics, see* $*$ *in Table 7.*

TABLE 1. Conics on Mukai surfaces (see §1.1)

G	(h^2, d)	lines	conics	T	Remarks
$L_2(7)$	$(2,2)$		8526	$[14,0,28]$	
	$(4,2)$		728	$[14,0,14]$	
$\mathbb{A}_6$	$(2,2)$		$8910^?$	$[12,0,30]$	$= N_2(2)$??
	$(6,2)$		285^*	$[6,0,20]$	X_{285} in [8]
$\mathbb{S}_5$	$(6,2)$		237	$[10,0,20]$	Table 7 in [8]
M_{20}	$(4,2)$		$800^?$	$[4,0,40]$	Thm. 1.3 in [6]
	$(8,2)$		$176^?$	$[8,4,12]$	Theorem 1.1
F_{384}	$(4,1)$	48	$336+320$	$[8,0,8]$	
	$(8,1)$	32	$96+48$	$[4,0,8]$	
$\mathbb{A}_{4,4}$	$(8,2)$		144	$[12,0,12]$	
T_{192}	$(4,1)$	64^*	$576+144$	$[8,4,8]$	Remark 4.4
	$(8,2)$		160	$[4,0,24]$	Example 4.3
H_{192}	$(4,1)$	48	$336+168$	$[8,0,12]$	
	$(8,1)$	32	$96+12$	$[4,0,12]$	
N_{72}	$(6,2)$		225	$[6,0,36]$	
M_9	$(2,1)$	144^*	$5112+2988$	$[12,6,12]$	$= N_2^*(2)$??
T_{48}	$(2,1)$	108	$2862+3180$	$[16,8,16]$	

1.1. **Digression: Mukai surfaces.** The second largest number of conics is 160 and, like X_{176} in Theorem 1.1, the corresponding octic X_{160} is also characterized by the presence of a faithful projective symplectic action of a Mukai group [17], *viz.* T_{192}, see Example 4.3. It is remarkable that *Mukai surfaces* (*i.e.*, $K3$-surfaces admitting a faithful symplectic action of one of the eleven maximal groups in [17]) maximize (sometimes conjecturally) the line or conic counts in many degrees. For this reason, we use (2.10) and the known generic Néron–Severi lattices (see, *e.g.*, [13]) to compute the Fano graphs of all Mukai surfaces of degree $h^2 \leqslant 8$. Results are shown in Table 1, where we list

- the Mukai group G (in the notation of [17]), the degree h^2 of the model, and its *depth* $d := \text{g.c.d.}\{x \cdot h \mid x \in NS(X)\}$,
- the numbers of lines and conics on X; the latter is shown as a single count if all conics are irreducible, or as (reducible) + (irreducible) otherwise,
- the transcendental lattice $T(X)$.

We omit hyperelliptic models (except degree $h^2 = 2$) and those of depth $d > 2$ (as they obviously have no lines or conics). The line/conic counts known or conjectured to be maximal are marked with * or $^?$, respectively.

Some of these configurations have already appeared elsewhere (see the remark column), whereas others seem to be new. Probably, the most important discovery is the following observation (see $\mathbb{A}_6$ and M_9 in Table 1; *cf.* Conjecture 1.5).

Observation 1.7. One has $N_2(2) \geqslant 8910$ and $N_2^*(2) \geqslant 8100$ (if defined).

1.2. **Contents of the paper.** In §2, we recall a few basic facts about (polarized) Kummer surfaces (§2.1, §2.2), analyze a very general Barth–Bauer octic (§2.3), and lay the basis for the study of the equiconical strata of positive codimension (§2.4, §2.5). At the end, in §2.6, we prove Theorem 1.2.

In §3 we perform a deep case-by-case analysis resulting in the five codimension 1 strata listed in Table 5, see Theorem 3.1. Finally, in §4, we list all strata of higher codimension (see Theorems 4.1, 4.2 and Tables 6–8) and give formal proofs of the principal results of the paper stated in the introduction.

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2. Barth–Bauer octics

In this section, we recall a few basic facts about (polarized) Kummer surfaces (see §2.1 and §2.2), analyze a very general Barth–Bauer octic (see §2.3), and lay the basis for the study of the equiconical strata of positive codimension (see §2.4 and §2.5). In §2.6, we use the machinery of §2.5 to prove Theorem 1.2.

2.1. **Preliminaries.** Let Ω be a 16-element set; denote $\mathcal{C}_0 := \{\varnothing\}$, $\mathcal{C}_{16} := \{\Omega\}$. A *Kummer structure* on Ω is a collection $\mathcal{O}_8$ of 30 eight-element subsets $\mathfrak{o} \subset \Omega$ such that $\mathcal{O}_* := \mathcal{O}_0 \cup \mathcal{O}_8 \cup \mathcal{O}_{16}$ is closed under the symmetric difference $\vartriangle$. (Here and below, for a subset $\mathcal{S}_*$ of a power set, we use the convention $\mathcal{S}_n := \{\mathfrak{o} \in \mathcal{S}_* \mid |\mathfrak{o}| = n\}$. According to Nikulin [18], any Kummer structure is standard: there is a bijection between Ω and a codeword of length 16 of the (extended binary) Golay code $\mathcal{G}_*$ (see, *e.g.*, [4]) such that $\mathcal{O}_8 = \{\mathfrak{o} \in \mathcal{G}_8 \mid \mathfrak{o} \subset \Omega\}$. Then one also has

$$\mathcal{C}_* := \bigl\{\mathfrak{s} \subset \Omega \bigm| |\mathfrak{s} \cap \mathfrak{o}| = 0 \bmod 2 \text{ for all } \mathfrak{o} \in \mathcal{O}_*\bigr\} = \bigl\{\mathfrak{s} \cap \Omega \bigm| \mathfrak{s} \in \mathcal{G}_*\bigr\},$$

and the setwise stabilizer of $\mathcal{O}_*$ in $\mathbb{S}_{16}$ is the restriction to Ω of its stabilizer in the Mathieu group M_{24}. This group acts transitively on $\mathcal{C}_4$ and, hence, on the set of 8-*Kummer structures* (*cf.* [6]) defined *via*

$$\mathcal{K}_* := \mathcal{K}_*(\mathfrak{k}) := \bigl\{\mathfrak{k} \vartriangle \mathfrak{o} \bigm| \mathfrak{o} \in \mathcal{O}_*\bigr\} \quad \text{for some fixed } \mathfrak{k} \in \mathcal{C}_4.$$

Note that $\mathcal{K}_*$ is generated by any of the four elements $\mathfrak{k} \in \mathcal{K}_4$ and $\mathcal{O}_*$ is recovered back from $\mathcal{K}_*$ *via* $\mathcal{O}_* = \{\mathfrak{r} \vartriangle \mathfrak{s} \mid \mathfrak{r}, \mathfrak{s} \in \mathcal{K}_*\}$. The setwise stabilizer $\mathfrak{G}$ of $\mathcal{K}_*$ is a group of order 9216.

Throughout the paper, we use the following shortcuts (where $\mathfrak{r}, \mathfrak{s} \subset \Omega$):

$$\hbar := \tfrac{1}{2}h \in \mathbb{Q}h, \qquad \mathfrak{s} := \sum_{e \in \mathfrak{s}} e \in \mathbb{Z}\Omega, \qquad \|\mathfrak{s}/\mathfrak{r}\| := \tfrac{1}{2}(\mathfrak{s} \cap \mathfrak{r}) - \tfrac{1}{2}(\mathfrak{s} \smallsetminus \mathfrak{r}) \in \mathbb{Q}\Omega.$$

The other terminology and notation related to lattices is quite standard, *cf.* [6].

From now on, we fix an 8-Kummer structure $\mathcal{K}_*$ and consider the lattices

(2.1) $\quad \mathbf{L} := 2\mathbf{E}_8 \oplus 3\mathbf{U} \cong H_2(X;\mathbb{Z})$ for a $K3$-surface X;

(2.2) $\quad \mathbf{S} := \mathbf{S}(\mathcal{O}_*) \supset \mathbb{Z}\Omega$ is the extension *via* all $\|\mathfrak{o}/\varnothing\|$, $\mathfrak{o} \in \mathcal{O}_*$;

(2.3) $\quad \mathbf{T} := \mathbf{S}^\perp_{\mathbf{L}} \cong 3\mathbf{U}(2)$ for a fixed primitive isometry $\mathbf{S} \hookrightarrow \mathbf{L}$;

(2.4) $\quad \mathbf{S}_h := \mathbf{S}_h(\mathcal{K}_*) \supset \mathbf{S} + \mathbb{Z}h$ is the extension *via* all $\hbar + \|\mathfrak{k}/\varnothing\|$, $\mathfrak{k} \in \mathcal{K}_*$.

A primitive isometry $\mathbf{S} \hookrightarrow \mathbf{L}$ in (2.3) is unique up to isomorphism (see [18]), and in (2.4) we let $h \cdot e = 2$ for $e \in \Omega$. In particular, (2.3) implies that

(2.5) $$u^2 = 0 \bmod 4, \quad u \cdot v = 0 \bmod 2 \quad \text{for any } u, v \in \mathbf{T}.$$

We also introduce the equivalence relations

$$\mathfrak{r} \sim \mathfrak{s} \text{ iff } \mathfrak{r} \vartriangle \mathfrak{s} \in \mathcal{O}_*, \qquad \mathfrak{r} \approx \mathfrak{s} \text{ iff } \mathfrak{r} \vartriangle \mathfrak{s} \in \mathcal{O}_* \cup \mathcal{K}_*$$

TABLE 2. $\mathfrak{G}$-orbits on $\mathcal{C}_n$

n	even	odd	$\mathcal{O}_*$	$\mathcal{K}_*$
0			1×1	
4	18×4	16×4		1×4
6	12×16	16×16		
8	$18 \times (8 + 16)$	16×24	$1 \times (6 + 24)$	1×24
10	12×16	16×16		
12	18×4	16×4		1×4
16			1×1	

on $\mathcal{S}_*$ and respective equivalence classes $[\cdot]$, $[\![\cdot]\!]$.

The *parity* of a set $\mathfrak{s} \in \mathcal{C}_*$ is $|\mathfrak{s} \cap \mathfrak{k}| \bmod 2$ for some (equivalently, any) $\mathfrak{k} \in \mathcal{K}_*$. Since any $\mathfrak{s} \in \mathcal{C}_* \cup \mathcal{K}_*$ is even, the parity is preserved by $\sim$ and $\approx$. The $\mathfrak{G}$-action on $\mathcal{C}_*$ respects $^-$, $\triangle$, parity, and both $\sim$ and $\approx$; its orbits are shown in Table 2, where most nonempty cells represent a single orbit each, shown as $\#(\sim\text{-classes}) \times |\text{class}_n|$. The two exceptional cases $(\text{even})_8$ and $\mathcal{O}_8$ consist of two $\mathfrak{G}$-orbits each: an extra invariant of a set $\mathfrak{s}$ is the existence of $\mathfrak{k} \in \mathcal{K}_4$ such that $\mathfrak{k} \subset \mathfrak{s}$. However, the induced actions on $(\text{even})_8/{\sim}$ and $\mathcal{O}_8/{\sim}$ are still transitive.

The next lemma is a straightforward application of [18, 20]. We present a partial statement which is used in this paper; more details are found in [6].

Lemma 2.6. *For a Kummer structure $\mathcal{O}_*$ and primitive isometry $\mathbf{S} := \mathbf{S}(\mathcal{O}_*) \hookrightarrow \mathbf{L}$, consider an overlattice $\mathbf{S} \subset N \subset \mathbf{L}$ primitive in $\mathbf{L}$ and let $\mathbf{S}^\perp := N \cap \mathbf{T}$. Then, for each vector $u \in \mathbf{S}^\perp$, there is a class $\mathcal{U} \in \Omega/{\sim}$ such that, for each $\mathfrak{u} \in \mathcal{U}$,*

$$2|\mathfrak{u}| = u^2 \bmod 8 \quad \textit{and} \quad \tfrac{1}{2}(u + \mathfrak{u}) \in N. \qquad \triangleleft$$

2.2. **Barth–Bauer surfaces.** According to Nikulin [18], a Kummer surface (X, Ω) defines a canonical Kummer structure on the set Ω of its Kummer divisors, and the Néron–Severi lattice $NS(X)$ is a primitive extension of $\mathbf{S}$ in (2.2). If X is polarized, $NS(X) \ni h$, so that each Kummer divisor $e \in \Omega$ is a conic, $e \cdot h = 2$, then

$$\mathbf{L} \supset NS(X) \supset \mathbf{S} + \mathbb{Z}h = \mathbf{S} \oplus \mathbb{Z}\tilde{h}, \qquad \tilde{h} := \Omega + h, \quad \tilde{h}^2 = 32 + h^2; \tag{2.7}$$

in particular, $h^2 = 0 \bmod 4$ by (2.5).

From now on, we assume that h is very ample and $h^2 = 8$, even though some formulas below are written for arbitrary h^2. By Saint-Donat [22], neither h nor $\tilde{h}$ is divisible by 2 in $NS(X)$; hence, the class $\mathcal{U} \in \Omega/{\sim}$ given by Lemma 2.6 for $u = \tilde{h}$ is a certain 8-Kummer structure $\mathcal{K}_*$, so that $NS(X)$ is a primitive extension of the lattice $\mathbf{S}_h$ in (2.4). This extension must be *geometric* in the following sense.

Definition 2.8 (*cf.* Saint-Donat [22])**.** A hyperbolic overlattice $N \supset \mathbb{Z}\Omega + \mathbb{Z}h$ is called *admissible* if

(1) h is not divisible by 2 in N, and

there is no vector $r \in N$ such that either

(2) $r^2 = -2$ and $r \cdot h = 0$ (*exceptional divisor*), or
(3) $r^2 = 0$ and $r \cdot h = \pm 2$ (2-*isotropic vector*), or
(4) $r^2 = -2$, $r \cdot h = 1$, and $r \cdot e < 0$ for some $e \in \Omega$ (*missing conic*).

An admissible lattice N is called *geometric* if the isometry $\mathbf{S} \hookrightarrow \mathbf{L}$, see (2.3), extends to a primitive isometry $N \hookrightarrow \mathbf{L}$.

TABLE 3. Symplectic groups G_ω

#	$\lvert G_\omega\rvert$	index	G_ω
21	16	14	C_4^2
39	32	27	2^4C_2
49	48	50	2^4C_3
77	192	1493	T_{192}
81	960	11357	M_{20}

Remark 2.9. According to Nikulin [18], in any geometric overlattice $N \supset \mathbb{Z}\Omega+\mathbb{Z}h$ one has $N \cap (\mathbb{Q}\Omega+\mathbb{Q}h) = \mathbf{S}_h(\mathcal{K}_*)$ for some 8-Kummer structure $\mathcal{K}_*$ on Ω. For this reason we usually fix $\mathcal{K}_*$ and work with overlattices of $\mathbf{S}_h$.

Conversely, a standard chain of arguments based on the global Torelli theorem [21], surjectivity of the period map [15], and the results of Nikulin [19] and Saint-Donat [22] shows that each geometric overlattice $N \supset \mathbf{S}_h \supset \Omega$ serves as $NS(X)$ for some Barth–Bauer octic (X,Ω). Indeed, an abstract $K3$-surface X is given by the surjectivity of the period map; then, conditions (3) and (1) assert that the linear system h defines a map $\varphi_h\colon X \to \mathbb{P}^5$ which is birational onto its image, condition (2) makes the image $\varphi_h(X)$ smooth, and condition (4) is equivalent to the requirement that each class $e \in \Omega$ represent an *irreducible* (-2)-curve on X.

The moduli space of octics X obtained in this way is discussed in §2.4 below. The Fano graphs of X (see §1) can be computed in terms of the polarized lattice $N := NS(X) \ni h$ using the description of the nef cone in Huybrechts [14, § 8.1] and Vinberg's algorithm [24] (*cf.* also [6, 9]): identifying (-2) curves on X with their classes in N, we have

$$\begin{aligned}\mathrm{Fn}_n(N,h) &:= \bigl\{u \in N \bigm| u^2=-2 \text{ and } u\cdot h = n\bigr\}, \quad n=1,2,\\ \mathrm{Fn}_2^*(N,h) &:= \bigl\{u \in \mathrm{Fn}_2(N,h) \bigm| u\cdot v \geqslant 0 \text{ for all } v \in \mathrm{Fn}_1(N,h)\bigr\}.\end{aligned} \tag{2.10}$$

The inverse of (2.10) assigns to a bi-colored graph Γ the 8-polarized lattice

$$\mathcal{F}(\Gamma) := (\mathbb{Z}\Gamma+\mathbb{Z}h)/\ker, \quad h^2=8, \quad h\cdot v = \mathrm{color}(v) \text{ for } v \in \Gamma, \tag{2.11}$$

where $\mathbb{Z}\Gamma$ is freely generated by the vertices $v \in \Gamma$ and $u \cdot v = n$ whenever $u, v \in \Gamma$ are connected by an n-fold edge. *A priori*, $\mathcal{F}(\Gamma)$ is neither geometric nor admissible; in fact, id does not even need to be hyperbolic.

2.3. **Generic Barth–Bauer octics.** A very general Barth–Bauer octic $X \subset \mathbb{P}^5$ has the minimal Néron–Severi lattice $NS(X) = \mathbf{S}_h$, and a computation using (2.10) shows that X has exactly 32 conics, all irreducible:

- the 16 original Kummer conics $e \in \Omega$, and
- 16 pairwise disjoint irreducible *Barth–Bauer*, or *B*2*-conics*

$$\hbar + \|\mathfrak{k}/\mathfrak{s}\|, \qquad \mathfrak{s} \subset \mathfrak{k} \in \mathcal{K}_4,\ |\mathfrak{s}|=1; \tag{2.12}$$

  these conics have pattern 12_3 in the notation of (2.18) below.

Remark 2.13. Note that $\mathbf{S}_h = NS(X)$ is not generated over $\mathbb{Z}$ by h and conics: one has $[\mathbf{S}_h : \mathcal{F}(\mathrm{Fn}\,\mathbf{S}_h)] = 4$, see the first row in Table 5. (There are but two other strata with this property, see Tables 5 and 6.) It is for this reason that the group $O_h(\mathbf{S}_h) = \mathfrak{G} \times \mathbb{Z}/2$ is much smaller than the full group $\mathrm{Aut}(\mathrm{Fn}\,\mathbf{S}_h)$.

Denote by $\mathfrak{G}_\omega \cong (\mathbb{Z}/4)^4$ (see #21 in Table 3) the subgroup of $\mathfrak{G}$ acting identically on $\Omega/\sim$. Clearly, $\mathfrak{G}_\omega \subset O_h(\mathbf{S}_h)$ is the subgroup acting identically on $\operatorname{discr}\mathbf{S}_h$; this action extends to any overlattice $N \supset \mathbf{S}_h$ and, by the global Torelli theorem, gives rise to a projective symplectic action on any Barth–Bauer octics. All extensions of $\mathfrak{G}_\omega$ acting symplectically on (generic in their respective strata) Barth–Bauer octics are listed in Table 3, where # and "index" refer, respectively, to the list in Xiao [25] and `GAP` [11] small group library and the last column is the notation in [25].

2.4. **Connected components.** Given a lattice L, we denote by $O^+(L)$ the group of auto-isometries of L preserving a *positive sign structure*, *i.e.*, coherent orientation of all maximal positive definite subspaces of $L \otimes \mathbb{R}$.

Let $N \supset \mathbf{S}_h$ be a geometric overlattice, see Definition 2.8, and $G \subset O_h(N)$ a fixed subgroup: in what follows, we will have either $G = O_h(N)$ or $G = \operatorname{stab}\Omega$. Two isometries $\varphi_i\colon N \hookrightarrow \mathbf{L}$, $i = 1, 2$, are said to be *G-equivalent* if there exists a pair of isometries $g \in G$, $f \in O^+(\mathbf{L})$ such that $f \circ \varphi_1 = \varphi_2 \circ g$.

Fix a bi-colored graph Γ and consider geometric finite index extensions

$$N \supset \mathcal{F}(\Gamma) \ni h \quad \text{such that} \quad \operatorname{Fn}(N, h) = \Gamma. \tag{2.14}$$

Using Dolgachev's [10] coarse moduli space of lattice polarized $K3$-surfaces and factoring out the projective group, one easily concludes (see [8]) that the connected components of the equiconical stratum $\mathcal{X}(\Gamma)$ are of the form $\mathcal{X}(N \hookrightarrow \mathbf{L})$, where

- $N \supset \mathcal{F}(\Gamma) \ni h$ is a geometric finite index extension as in (2.14), regarded up to lattice isomorphism preserving h, and
- $N \hookrightarrow \mathbf{L}$ is an $O_h(N)$-equivalence class of primitive isometries.

A similar statement holds for the relative stratum $\tilde{\mathcal{X}}(\Gamma, \Omega)$, except that

- N is regarded up to isomorphism preserving h and Ω (as a set), and
- $N \hookrightarrow \mathbf{L}$ is a $(\operatorname{stab}\Omega)$-equivalence class of primitive isometries.

In both cases, a component $\mathcal{X}(\varphi\colon N \hookrightarrow \mathbf{L})$ is real if and only if φ is equivalent to $g \circ \varphi$ for some (equivalently, any) $g \in O(\mathbf{L}) \smallsetminus O^+(\mathbf{L})$.

Thus, the connected components of the strata associated to a graph Γ are in a bijection with the appropriate equivalence classes of the diagrams

$$\mathcal{F}(\Gamma) \hookrightarrow N \hookrightarrow \mathbf{L}, \tag{2.15}$$

where N is admissible, the former arrow is a finite index extension as in (2.14), and the latter arrow is a primitive isometry. At each step, there is but a finite number of choices, given by Nikulin [20]. For the auxiliary computation, we use `Digraphs` package in `GAP` [11] on the N-side and Gauss [12] and Miranda–Morrison [16] on the $N^\perp$-side; see [6, § 3.1] for further details.

2.5. **The supports of a vector.** In view of Remark 2.9, the graphs Γ to be tried for (2.15) are of the form $\Gamma := \operatorname{Fn}\mathbf{S}_h[u_i]$, where $\mathbf{S}_h[u_i] \supset \mathbf{S}_h$ is a primitive corank r extension generated by r extra lines or conics $u_1, \ldots, u_r$. The next lemma controlls such extensions by bounding the intersection indices of lines and conics.

Lemma 2.16. *Let $X \subset \mathbb{P}^5$ be a smooth $K3$-octic, $l_1, l_2 \in NS(X)$ a pair of distinct lines on X, and $c_1, c_2 \in NS(X)$ a pair of distinct conics. Then one has*

$$l_1 \cdot l_2 \leqslant 1, \quad l_1 \cdot c_1 \leqslant 2 \text{ (or 1, if } X \text{ is a triquadric)}, \quad c_1 \cdot c_2 \leqslant 2.$$

TABLE 4. Sylvester test for conics (left) and lines (right)

	0		2		4		6		8		10		12		14		16
0	○	○	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
2	×	·	·	·	·	·	·	·	·	·	·	·	·	·	·		
4	●	·	·	·	·	·	·	·	·	·	·	·	·				
6	●	·	·	·	·	·	·	·	·	·	·						
8	●	·	·	·	·	·	·	·	·								
10	●	·	·	·	·	·	·										
12	●	○	○	·	○												
14	×	×	×														
16	○																

	0		2		4		6		8		10		12		14		16
0	×	×	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
2	×	·	·	·	·	·	·	·	·	·	·	·	·	·	·		
4	●	·	·	·	·	·	·	·	·	·	·	·	·				
6	●	·	·	·	·	·	·	·	·	·	·						
8	●	·	·	·	·	·	·	·	·								
10	●	·	·	·	·	·	·										
12	●	·	·	·	·												
14	×	×	×														
16	×																

Proof. By the Hodge index theorem, the lattice $NS(X)$ is hyperbolic. Hence, for any pair of vectors $u, v \in NS(X)$, one has

$$\det(\mathbb{Z}h + \mathbb{Z}u + \mathbb{Z}v) \geqslant 0, \tag{2.17}$$

with the equality attained if and only if h, u, v are linearly dependent.

Applying (2.17) to one of the three pairs in the statement, we obtain

$$l_1 \cdot l_2 \leqslant 2, \qquad l_1 \cdot c_1 \leqslant 2, \qquad c_1 \cdot c_2 \leqslant 3,$$

and there remains to rule out the possibilities $l_1 \cdot l_2 = 2$ and $c_1 \cdot c_2 = 3$.

In the former case, $l_1 \cdot l_2 = 2$, the lattice contains the 2-isotropic vector $l_1 + l_2$, see Definition 2.8(3), and the map $X \to \mathbb{P}^5$ defined by h is two-to-one, see [22].

In the latter case, $c_1 \cdot c_2 = 3$, the determinant (2.17) vanishes and we obtain a relation $2h = c_1 + c_2$. Hence, h is divisible by 2 in $NS(X)$ and the map $X \to \mathbb{P}^5$ is also two-to-one, factoring through the Veronese embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$, see [22].

For the bound $l_1 \cdot c_1 \leqslant 1$, observe that, if $l_1 \cdot c_1 = 2$, then the vector $e := l_1 + c_1$ is 3-*isotropic*: $e^2 = 0$, $e \cdot h = 3$. According to [22] (see also [9]), the presence of such a vector in $NS(X)$ is equivalent to the fact that X is special. □

In view of Lemma 2.16, if e is an irreducible conic on X, then $u \cdot e \in \{0, 1, 2\}$ for any line or conic $u \neq e$. It follows that a 1-vector extension

$$\mathbf{S}_h[u] := (\mathbf{S}_h + \mathbb{Z}u)/\ker$$

(not necessarily proper) is uniquely determined by the degree $u \cdot h$ and two *supports*

$$\operatorname{supp}_i u := \bigl\{e \in \Omega \bigm| u \cdot e = i\bigr\} \subset \Omega, \quad i = 1, 2,$$

which are two disjoint subsets of Ω. Letting $p := |\operatorname{supp}_1 u|$ and $q := |\operatorname{supp}_2 u|$, we will say that

$$u \text{ has } \textit{pattern } p_q^\star \text{ (if it is a line) or } p_q \text{ (if it is a conic).} \tag{2.18}$$

Assuming that $\mathbf{S}_h[u]$ is an integral lattice, we also have

$$\operatorname{supp}_1 u \in \mathcal{C}_* \text{ is an even (resp. odd) set if } u \cdot h \text{ is even (resp. odd).} \tag{2.19}$$

Finally, denoting by $u_{\mathbf{S}}$ the orthogonal projection of u to $\mathbf{S}_h \otimes \mathbb{Q}$, we find that

$$u_{\mathbf{S}}^2 = -\frac{p}{2} - 2q + \frac{(p + 2q + \varepsilon)^2}{h^2 + 32}, \tag{2.20}$$

where p, q are as above and $\varepsilon := u \cdot h$. The lattice $\mathbf{S}_h[u]$ is hyperbolic and of corank 1 over $\mathbf{S}_h$ if and only if $u_{\mathbf{S}}^2 > u^2 = -2$. This inequality results in Table 4 (the pairs marked with a $\cdot$ are ruled out), where, in view of (2.19), only even values of p are

TABLE 5. Strata of codimension $\leqslant 1$ (see Theorem 3.1)

Name	Patterns	δ_2^2	δ_5	Lines	Conics	$\lvert G\rvert$	i_Ω	G_ω	$\lvert\det\rvert$	(r,c)
open					32	$18432 \cdot 864$	2	21	640^4	$(1,0)$
$1^\star$	$4_0^\star$	$5/8$	0	4	32	1152	2	21	400	$(1,0)$
$2^\star$	$6_0^\star, 12_0^\star, 4_0$	$5/8$	± 1	20	$16+20$	576	1	21	144	$(1,0)$
3	$4_0, 12_0$	$2/4$	± 3		40	$1024 \cdot 16$	2	21	576^2	$(1,0)$
4	$6_0, 10_0$	$2/2$	± 4		64	3072	4	21	384	$(1,0)$
5	8_0	$2/4$	0		80	2048	2	21^2	320	$(1,0)$

considered. For the reader's convenience, the pairs ruled out by (2.19) and Table 2 are marked with a $\times$, and those prohibited in §3.2 below are marked with a $\circ$.

2.6. **Proof of Theorem 1.2.** As already mentioned, [22] (see also [9]) states that a smooth $K3$-octic is special if and only if the lattice $NS(X)$ contains a 3-*isotropic vector*, *i.e.*, a vector u such that $u^2 = 0$ and $u \cdot h = 3$. Applying (2.17) to $v = e \in \Omega$, we get $u \cdot e \in \{0, 1, 2\}$. Hence, similar to §2.5, an extension $\mathbf{S}_h[u]$ by a 3-isotropic vector u is determined by the pair of supports $\operatorname{supp}_i u \subset \Omega$, $i = 1, 2$. Arguing as in §2.5, we arrive at $u_{\mathbf{S}}^2 \geqslant u^2 = 0$, where $u_{\mathbf{S}}^2$ is given by (2.20) with $\varepsilon = 3$. This inequality results in $\lvert\operatorname{supp}_1 u\rvert \in \{0, 14, 16\}$. On the other hand, $\operatorname{supp}_1 u \in \mathcal{C}_*$ is an odd set, see (2.19), contradicting to Table 2. □

3. STRATA OF CODIMENSION 1

The goal of this section is the description of the codimension 1 strata in the space $\mathcal{B}$ of Barth–Bauer octics. The following theorem is proved in §3.3 below.

Theorem 3.1. *The space $\mathcal{B}$ has five irreducible equiconical strata of codimension* 1, viz. *those listed in Table 5. Each stratum consists of a single real component.*

For completeness, in the first row of Table 5 we also show the open stratum of codimension 0, *i.e.*, the one consisting of generic Barth–Bauer octics.

3.1. **Notation in Tables 5–8.** The rows of each table represent the isomorphism classes of pairs (Γ, Ω), where Γ is a Fano graph and $\Omega \subset \Gamma$ is a distinguished set of 16 irreducible Kummer conics. The rows corresponding to isomorphic abstract bi-colored graphs Γ are prefixed with equal superscripts. Listed in Table 5 are

- the name of the stratum (for further references),
- the patterns of the extra lines and conics, see (2.18), and
- a description of the images $\delta_p(u) \in \operatorname{discr}_p \mathbf{S}_h$, $p = 2, 5$ (see §3.4 below), of a distinguished generator u.

Instead, the first column of the other tables merely lists

- the types of the clusters (see §3.4 below), as references to Table 5.

The rest of the data is common to Tables 5–8; they apply to a very general member $X \in \mathcal{X}$ of the respective stratum:

- the numbers of lines and conics on X, in the same form as in Table 1,
- the order of the group $G := \operatorname{Aut}\operatorname{Fn}(X, h)$; if $N := NS(X)$ is *not* generated by lines and conics, it is shown in the form $\lvert O_h(N)\rvert \cdot [G : O_h(N)]$,
- the index $i_\Omega := [G : G_\Omega]$ of the setwise stabilizer $G_\Omega := \operatorname{stab}\Omega \subset G$,

- the group (as a reference to Table 3) G_ω of symplectic automorphisms of X and the index $[\operatorname{Aut}_h X : G_\omega]$, if greater than 1, as a superscript,
- the determinant $|\det NS(X)| = |\det T(X)|$ and the index $[NS(X) : \mathcal{F}(\Gamma)]$, if greater than 1, as a superscript (see also Remark 3.2),
- the numbers (r, c) of, respectively, real components and pairs of complex conjugate components of the stratum, see Remark 3.2.

Remark 3.2. In Tables 7 and 8 listing the singular octics, instead of $\det T(X)$ we show the isomorphism classes of the transcendental lattice $T(X)$, each class in a separate row. The counts (r, c) are itemized accordingly.

Given a pair (Γ, Ω) and a class $T \in \operatorname{genus} T(X)$, the counts $(\tilde{r}, \tilde{c})$ for the relative stratum $\tilde{\mathcal{X}}_T(\Gamma, \Omega)$ may differ from the respective counts (r, c) for $\mathcal{X}_T(\Gamma)$. If this is the case, the counts are shown in the form $(r, c) \to (\tilde{r}, \tilde{c})$.

3.2. **Restrictions on extra lines and conics.** We start with a few further (*i.e.*, beyond those found in §2.5) restrictions on the supports of an extra line or conic u. Note that the statement and proof of Lemma 3.3, as well as those of Lemma 3.4 concerning the case $\operatorname{supp}_1 u \in \mathcal{O}_*$, are valid for any degree $h^2 \in 4\mathbb{Z}^+$.

Lemma 3.3 (see [6]). *Let $u \notin \mathbf{S}_h$ be an extra conic (line), and let*

$$\mathfrak{u} := \operatorname{supp}_1 u,\ p := |\mathfrak{u}|, \quad \textit{and} \quad \mathfrak{u}' := \operatorname{supp}_2 u,\ q := |\mathfrak{u}'|.$$

Then, for any pair $\mathfrak{v}, \mathfrak{v}' \subset \Omega$ such that

$$\mathfrak{v} \in [\mathfrak{u}]_p, \quad \mathfrak{v}' \subset \Omega \smallsetminus \mathfrak{v}, \quad |\mathfrak{v}'| = q,$$

there is a conic (resp. line) $v \in \mathbf{S}_h[u]$ such that $\operatorname{supp}_1 v = \mathfrak{v}$ and $\operatorname{supp}_2 v = \mathfrak{v}'$.

Proof. For completeness, we cite the proof found in [6]. A set $\mathfrak{v}$ as in the statement has the form $\mathfrak{v} = \mathfrak{u} \vartriangle \mathfrak{o}$ for some $\mathfrak{o} \in \mathcal{C}_*$ such that $2|\mathfrak{o} \cap \mathfrak{u}| = |\mathfrak{o}|$. Let $\mathfrak{s}_+ := \mathfrak{o} \cap \mathfrak{u}'$ and pick $\mathfrak{s}_- \subset \mathfrak{o} \cap \mathfrak{u}$ so that $|\mathfrak{s}_-| = |\mathfrak{s}_+|$. Then, the vector

$$w := u + \|\mathfrak{o}/\mathfrak{u}\| + \mathfrak{s}_+ - \mathfrak{s}_- \in \mathbf{S}_h[u]$$

has $\operatorname{supp}_1 w = \mathfrak{u} \vartriangle \mathfrak{o}$ and $\mathfrak{w}' := \operatorname{supp}_2 w = \mathfrak{u}' \vartriangle (\mathfrak{s}_+ \cup \mathfrak{s}_-)$, so that $|\mathfrak{w}'| = |\mathfrak{u}'|$. There remains to let

$$v := w + (\mathfrak{v}' \smallsetminus \mathfrak{w}') - (\mathfrak{w}' \smallsetminus \mathfrak{v}'). \qquad \square$$

Lemma 3.4 (*cf.* [6]). *If $u \notin \mathbf{S}_h$ is an extra conic and $\mathfrak{u} := \operatorname{supp}_1 u \in \mathcal{O}_* \cup \mathcal{K}_*$, then any geometric extension of the lattice $\mathbf{S}_h[u]$ is generated by lines over $\mathbf{S}_h$.*

Proof. Assuming the contrary, let $\mathfrak{u}' := \operatorname{supp}_2 u$ and consider the vector

$$\hat{u} := \begin{cases} u - \|\mathfrak{u}/\varnothing\| + \mathfrak{u}', & \text{if } \mathfrak{u} \in \mathcal{O}_*, \\ \hbar - u - \|\bar{\mathfrak{u}}/\mathfrak{u}'\|, & \text{if } \mathfrak{u} \in \mathcal{K}_*. \end{cases}$$

We have $\hat{u} \in \mathbf{T}$, see (2.3), and, respectively,

$$\begin{aligned} \hat{u}^2 &= \tfrac{1}{2}|\mathfrak{u}| + 2|\mathfrak{u}'| - 2, & \hat{u} \cdot h &= 2 + |\mathfrak{u}| + 2|\mathfrak{u}'| & &\text{if } \mathfrak{u} \in \mathcal{O}_*, \\ \hat{u}^2 &= -\tfrac{1}{2}|\mathfrak{u}| + \tfrac{1}{4}h^2 + 4, & \hat{u} \cdot h &= 14 - |\mathfrak{u}| - 2|\mathfrak{u}'| + \tfrac{1}{2}h^2 & &\text{if } \mathfrak{u} \in \mathcal{K}_*. \end{aligned}$$

In view of (2.5), the presence of this vector $\hat{u} \in \mathbf{T}$ rules out the patterns p_0, $p = 0$, 8, 16. The few remaining cases (see Table 4) are considered below.

The patterns 12_q, $q = 2, 4$: we have $\hat{u}^2 = 0$ and $\hat{u} \cdot h = \pm 2$, *i.e.*, $\hat{u}$ is a 2-isotropic vector, see Definition 2.8(3).

The patterns 0_1 *and* 12_q, $q = 0, 1$: we have $\hat{u}^2 = 0$ and $\hat{u} \cdot h = 6$ or 4. Therefore, by Lemma 2.6, any geometric extension of $\mathbf{S}_h[u]$ must contain a vector of the form $v := -\frac{1}{2}\hat{u} - \|\mathfrak{s}/\varnothing\|$ for some $\mathfrak{s} \in \mathcal{C}_0 \cup \mathcal{C}_4$. If $\mathfrak{s} \in \mathcal{C}_0$, *i.e.*, $\mathfrak{s} = \varnothing$, then $\hat{u}$ is divisible by 2; due to (2.5), this is only possible if $\hat{u} \cdot h = 4$, making $\frac{1}{2}\hat{u}$ a 2-isotropic vector, see Definition 2.8(3). Otherwise, if $\mathfrak{s} \in \mathcal{O}_4$, we obtain

$$v^2 = -2, \quad v \cdot h = 1 \text{ or } 2, \quad v \cdot e = -1 \text{ for each } e \in \mathfrak{s},$$

resulting in a missing conic, see Definition 2.8(4), or exceptional divisor $v - e$, see Definition 2.8(2), respectively.

The pattern 4_0: we have $\hat{u}^2 = 4$ and $\hat{u} \cdot h = 14$; by Lemma 2.6, any geometric extension of $\mathbf{S}_h[u]$ must contain a line of the form $\frac{1}{2}\hat{u} + \|\mathfrak{s}/\varnothing\|$, $\mathfrak{s} \in \mathcal{C}_6$. Observe that, in fact, this is the only case where the lattice $\mathbf{S}_h[u]$ as in the statement does admit a geometric extension, *cf.* Lemma 3.8 below. □

Lemma 3.5. *Let* $u \notin \mathbf{S}_h$ *be an extra conic and assume that* $\mathfrak{u}' := \operatorname{supp}_2 u \neq \varnothing$. *Then the lattice* $\mathbf{S}_h[u]$ *has no geometric extensions.*

Proof. According to Tables 2, 4 and Lemma 3.4, we can assume that

$$\mathfrak{u} := \operatorname{supp}_1 u \in \mathcal{C}_{12} \smallsetminus \mathcal{K}_*;$$

hence, there is a set $\mathfrak{k} \in \mathcal{K}_4$ such that $|\mathfrak{k} \cap \mathfrak{u}| = 2$. Using Lemma 3.3, we can change the set $\mathfrak{u}'$ so that $|\mathfrak{k} \cap \mathfrak{u}'| \geqslant \min\{2, |\mathfrak{u}'|\}$. Pick a singleton $\mathfrak{s} \subset \mathfrak{k}$ as follows:

- $\mathfrak{s} \subset \mathfrak{k} \smallsetminus (\mathfrak{u} \cup \mathfrak{u}')$ if $|\mathfrak{u}'| = 1$, or
- $\mathfrak{s} \subset \mathfrak{k} \cap \mathfrak{u}$ if $|\mathfrak{u}'| \geqslant 2$.

Then, for the $B2$-conic $v := \hbar + \|\mathfrak{k}/\mathfrak{s}\|$, we have $v \cdot u = -1$ and, hence, $u - v$ is an exceptional divisor, see Definition 2.8(2). □

Lemma 3.6. *Let* $u \notin \mathbf{S}_h$ *be an extra conic,* $\mathfrak{u} := \operatorname{supp}_1 u$, *and* $p := |\mathfrak{u}|$. *Then, for any set* $\mathfrak{w} \in [\![\mathfrak{u}]\!]_{16-p} \smallsetminus [\mathfrak{u}]$, *there is a conic* $w \in \mathbf{S}_h[u]$ *such that* $\operatorname{supp}_1 w = \mathfrak{w}$.

Proof. Any set $\mathfrak{w}$ as in the statement is of the form $\overline{\mathfrak{v} \vartriangle \mathfrak{s}}$, where $\mathfrak{v} := \operatorname{supp}_1 v \in [\mathfrak{u}]_p$ for an appropriate vector v given by Lemma 3.3 and $\mathfrak{s} \in \mathcal{K}_4$, $|\mathfrak{s} \cap \mathfrak{v}| = 2$. Besides, by Lemma 3.5 we can assume that $\operatorname{supp}_2 u = \operatorname{supp}_2 v = \varnothing$. Then, it is immediate that the conic $w := \hbar - \|\mathfrak{s}/\mathfrak{v}\| - v$ is as required. □

Lemma 3.7. *Let* $u \notin \mathbf{S}_h$ *be an extra line and* $\mathfrak{u} := \operatorname{supp}_1 u \in \mathcal{C}_8$. *Then, the lattice* $\mathbf{S}_h[u]$ *is not admissible.*

Proof. There exists a subset $\mathfrak{r} \in \mathcal{K}_{12}$ such that $|\mathfrak{r} \cap \mathfrak{u}| = 7$; then, it is immediate that $-\hbar + \|\mathfrak{r}/\mathfrak{u}\| + 2u$ is an exceptional divisor, see Definition 2.8(2). □

Lemma 3.8. *Let* $u \notin \mathbf{S}_h$ *be an extra line and* $\mathfrak{u} := \operatorname{supp}_1 u \in \mathcal{C}_6$. *Then*:

(1) *all sixteen* $B2$*-conics, see* (2.12)*, are reducible in* $\mathbf{S}_h[u]$;
(2) *for each* $\mathfrak{v} \in \mathcal{K}_4$*, there is an irreducible conic* $v \in \mathbf{S}_h[u]$ *with* $\operatorname{supp}_1 v = \mathfrak{v}$;
(3) *for each* $\mathfrak{v} \in [\![\mathfrak{u}]\!]_{12}$*, there is a line* $v \in \mathbf{S}_h[u]$ *with* $\operatorname{supp}_1 v = \mathfrak{v}$.

Proof. For statement (1), observe that, for each pair $\mathfrak{s} \subset \mathfrak{k} \in \mathcal{K}_4$ as in (2.12), there is $\mathfrak{w} \in [\mathfrak{u}]_6$ such that $\mathfrak{w} \cap \mathfrak{k} = \mathfrak{k} \smallsetminus \mathfrak{s}$; then, $w \cdot k = -1$, where $w \in \mathbf{S}_h[u]$ is the line with $\operatorname{supp}_1 w = \mathfrak{w}$ given by Lemma 3.3 and $k = \hbar + \|\mathfrak{k}/\mathfrak{s}\|$ is the $B2$-conic (2.12).

For each pair $\mathfrak{k}$, $\mathfrak{w}$ as above, the line $v := k - w$ has support $\mathfrak{v} := \overline{\mathfrak{w} \vartriangle \mathfrak{k}} \in [\![\mathfrak{u}]\!]_{12}$, and all lines as in statement (3) can be obtained in this way.

Finally, the four extra conics as in statement (2) are

$$\hbar - \|\mathfrak{r}/\mathfrak{w}\| - (\mathfrak{v} \smallsetminus \mathfrak{r}) - 2w,$$

where $\mathfrak{w}$ and w are as above and $\mathfrak{r} \in \mathcal{K}_{12}$, $|\mathfrak{r} \cap \mathfrak{w}| = 3$; *cf.* the last case 4_0 in the proof of Lemma 3.4. □

3.3. **Proof of Theorem 3.1.** According to Tables 2, 4 and Lemmas 3.4, 3.5, 3.7, there are but five (pairs of) patterns that need to be considered:

$$4_0, 12_0;\ 6_0, 10_0;\ 8_0 \quad \text{or} \quad 4_0^\star;\ 6_0^\star, 10_0^\star.$$

Here, two patterns constitute a pair, *e.g.*, $4_0, 12_0$, if they result in identical 1-vector extensions: in the example, the extension $\mathbf{S}_h[u]$ by a vector with pattern 4_0 contains one with pattern 12_0 (see Lemma 3.6 or, for lines, Lemma 3.8) and *vice versa*.

Furthermore, Lemma 3.4 asserts that $\mathfrak{u} := \operatorname{supp}_1 u \notin (\mathcal{C}_* \cup \mathcal{K}_*)$: the case $\mathfrak{u} \in \mathcal{K}_4$ can be ignored as the lattice $\mathbf{S}_h[u]$ itself is not geometric whereas any geometric extension thereof is generated by lines, *viz.* the pair of patterns $6_0^\star, 10_0^\star$. Obviously, the $\mathfrak{G}$-isomorphism class of $\mathbf{S}_h[u]$ depends only on the $\mathfrak{G}$-orbit of $\mathfrak{u}$; by Lemma 3.3, this can further be replaced with the $\mathfrak{G}$-orbit of $[\mathfrak{u}]$. Hence, referring to Table 2 and parity condition (2.19), we conclude that each of the five (pairs of) patterns above results in a single $\mathfrak{G}$-isomorphism class of extensions. A computation shows that

- each of the five lattices $N := \mathbf{S}_h[u]$ obtained in this way is geometric,
- there are no proper geometric finite index extensions $N' \supset N$, and
- each lattice $N \supset \mathbf{S}_h \supset \Omega$ admits a unique $O_h(N, \Omega)$-isomorphism class of primitive isometries $N \hookrightarrow \mathbf{L}$ (see §2.4).

Thus, there are five strata, each consisting of a single real component (see §2.4), and using (2.10) one can show that, in addition to Ω and $B2$-conics (2.12), the lines and conics in N are precisely those given by Lemmas 3.3, 3.6, and 3.8. The precise counts are given in Table 5. □

3.4. **Clusters.** The discriminant $\operatorname{discr} \mathbf{S}_h$ has 2- and 5-torsion:

$$\operatorname{discr}_2 \mathbf{S}_h \cong \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \oplus \left[\tfrac{5}{8}\right], \qquad \operatorname{discr}_5 \mathbf{S}_h \cong \left[\tfrac{8}{5}\right].$$

The groups $2 \operatorname{discr}_2 \mathbf{S}_h \cong \mathbb{Z}/4$ and $\operatorname{discr}_5 \mathbf{S}_h \cong \mathbb{Z}/5$ have distinguished generators $\eta_2 := \frac{1}{4}\tilde{h}$ and $\eta_5 := \frac{1}{5}\tilde{h}$, respectively, see (2.7).

Consider a geometric extension $N \supset \mathbf{S}_h$. Following [6], define a *cluster* in N as a collection of all lines and conics $u \in N$ sent to the same point of the projective space $\mathbb{P}((N/\mathbf{S}_h) \otimes \mathbb{Q})$. Consider also the canonical homomorphism

$$\delta = \delta_2 \oplus \delta_5 \colon N \to \mathbf{S}_h^\vee \to \operatorname{discr} \mathbf{S}_h = \mathbf{S}_h^\vee / \mathbf{S}_h.$$

Directly by the definition, the image $\delta(C)$ of each cluster $C \subset N$ generates a cyclic subgroup in $\operatorname{discr} \mathbf{S}_h$. More precisely, since each cluster is contained in a 1-vector extension, Theorem 3.1 and Lemmas 3.3, 3.6, 3.8 used in its proof imply that the image of each cluster consists of

- a single element α, as in stratum $1^\star$ in Table 5, or
- a pair of elements $\pm\alpha$, as in strata 3, 4, 5, or
- a pair $\pm\alpha$ and common element $2\alpha = \eta_2 \oplus 2\eta_5$, as in stratum $2^\star$.

TABLE 6. Strata of codimension 2 (see Theorem 4.1)

Clusters	Lines	Conics	$\lvert G\rvert$	i_Ω	G_ω	$\lvert\det\rvert$	(r,c)
$1^\star, 1^\star$	8	32	384	2	21	240	$(1,0)$
$1^\star, 1^\star, 5$	8	$8+72$	256	2	21^2	160	$(1,0)$
$1^\star, 2^\star, 4$	24	$32+36$	192	2	21	80	$(1,0)$
[1] $1^\star, 3$	4	40	64	1	21	320	$(1,0)$
[1] $1^\star, 3$	4	40	64	1	21	320	$(1,0)$
$1^\star, 4$	4	64	384	4	21	240	$(1,0)$
$2^\star, 3$	20	$16+28$	64	1	21	128	$(1,0)$
$3, 3$		48	256	2	21	416	$(1,0)$
$3, 3$		48	$512\cdot 16$	2	21	512^2	$(1,0)$
$3, 3$		48	512	2	21	512	$(1,0)$
$3, 3, 4$		80	512	4	21	288	$(1,0)$
$3, 4$		72	512	4	21	320	$(1,0)$
$3, 5$		88	256	2	21^2	288	$(1,0)$
$4, 4$		96	2304	6	49	224	$(1,0)$
$4, 5$		112	1024	4	21^2	192	$(1,0)$
$*$ $5, 5$		128	1024	2	39^2	160	$(1,0)$

The generating images $\delta(u) = \delta_2(u) \oplus \delta_5(u)$ are shown in Table 5, in the form of the square $\delta_2^2 = r/s \bmod 2\mathbb{Z}$ (where s is the order of δ_2) and coefficient of δ_5 in the basis η_5. Computing the orbits of the $\mathfrak{G}$-action on $\operatorname{discr} \mathbf{S}_h$, we conclude that, with the extra restriction that

$$\delta_2(u) \cdot \eta_2 = \tfrac{1}{4}(\epsilon + p) \bmod \mathbb{Z} \text{ for } u \text{ with pattern } p_0\ (\varepsilon = 2) \text{ or } p_0^\star\ (\varepsilon = 1),$$

$$\delta_2(u) \neq \pm\eta_2 \text{ unless } u \text{ is a non-generating conic of pattern } 4_0 \text{ in stratum } 2^\star,$$

these data determine the $\mathfrak{G}$-orbit of $\delta(u)$. On the other hand, by comparison to Table 2, the vector $\delta(u)$ determines $[\operatorname{supp}_1 u]$ and, hence, the extension $\mathbf{S}_h[u]$.

4. Strata of higher codimension

In this section, we complete the proofs of the principal results of the paper by analyzing the double and triple (self-)intersections of the five strata found in §3.

Theorem 4.1. *The space $\mathcal{B}$ has 15 irreducible equiconical strata of codimension 2, see Table 6. Each stratum consists of a single real component; one of the absolute strata splits into two relative ones (prefixed with [1] in Table 6).*

In a stratum of codimension 3, each octic X is a so-called *singular $K3$-surface* ($\operatorname{rk} NS(X) = 20$ is maximal); hence, X is *rigid*, *i.e.*, X is projectively equivalent to any equiconical deformation thereof. In other words, modulo the group $PGL(\mathbb{C}, 6)$, the union of the codimension 3 strata is a finite collection of points, and it is these points that are listed in Tables 7 and 8.

Theorem 4.2. *All equiconically rigid Barth–Burau octics are listed in Tables 7, 8; altogether, there are*

- 36 *isomorphism classes of abstract Fano graphs Γ,*
- 41 *isomorphism classes of pairs (Γ, Ω),*
- 33 *real and* 14 *pairs of complex conjugate octics X, and*
- 38 *real and* 38 *pairs of complex conjugate pairs (X, Ω).*

Table 7. Rigid octics with > 80 conics (see Theorem 4.2)

Clusters	Lines	Conics	$\lvert G\rvert$	i_Ω	G_ω	T	(r,c)
$5, 5, 5$		176	15360	10	81^2	$[8, 4, 12]$	$(1, 0)$
$4, 5, 5$		160	3072	12	77^2	$[4, 0, 24]$	$(1, 0)$
$3, 5, 5$		136	512	2	39^2	$[4, 0, 36]$	$(1, 0)$
$1^\star, 1^\star, 1^\star, 1^\star, 5, 5$	16	$32 + 96$	256	2	39^2	$[4, 2, 16]$	$(1, 0)$
$3, 3, 3, 4, 4$		120	384	6	49	$[8, 4, 20]$	$(1, 0) \to (0, 1)$
$3, 4, 5$		120	256	4	21^2	$[8, 0, 20]$	$(1, 0) \to (2, 0)$
$1^\star, 1^\star, 4, 5$	8	$8 + 104$	256	4	21^2	$[4, 0, 24]$	$(1, 0) \to (0, 1)$
† $1^\star, 1^\star, 2^\star, 4, 4$	28	$48 + 52$	288	3	49	$[4, 2, 12]$	$(1, 0)$
$1^\star, 4, 4$	4	96	576	6	49	$[4, 2, 36]$	$(1, 0) \to (2, 0)$
$3, 3, 5$		96	256	2	21^2	$[8, 4, 28]$	$(1, 0)$
$3, 3, 5$		96	256	2	21^2	$[8, 0, 32]$	$(1, 0)$
$3, 3, 5$		96	256	2	21^2	$[8, 0, 32]$	$(1, 0)$
$3, 3, 3, 3, 4$		96	256	4	39	$[8, 0, 24]$	$(1, 0) \to (0, 1)$
$3, 3, 3, 4$		88	128	4	21	$[8, 4, 32]$	$(1, 0) \to (0, 2)$
$1^\star, 1^\star, 3, 5$	8	$8 + 80$	32	2	21^2	$[4, 2, 32]$	$(0, 1)$
						$[8, 2, 16]$	$(0, 2)$
$1^\star, 2^\star, 3, 3, 4$	24	$32 + 52$	64	2	21	$[8, 2, 8]$	$(1, 0) \to (2, 0)$

4.1. **Proof of Theorems 4.1 and 4.2.** We use the approach of [6, §3].

For Theorem 4.1, we consider all corank 2 extensions $\mathbf{S}_h[u, v]$ by a pair of vectors, each as in Table 5; an extra piece of data is the product $u \cdot v$, which must satisfy Lemma 2.16. (We adopt Convention 3.9 in [6] and assume that the generating set has the maximal number of lines; then, we can also assume that all generating conics are irreducible and, hence, $u \cdot v \geqslant 0$.) The vast majority of possibilities are ruled out by the Hodge index theorem, as in §2.5, leaving but 30 $\mathfrak{G}$-orbits of triples $([\mathfrak{u}], [\mathfrak{v}], u \cdot v)$. Each triple is analyzed in the spirit of §3, and only 20 of them admit a geometric finite index extension (which is always trivial). There remains to observe that some of the lattices obtained are isomorphic: in fact, each geometric lattice $\mathbf{S}_h[u, v]$ is generated over $\mathbf{S}_h$ by appropriate representatives of any pair of clusters contained in $\mathbf{S}_h[u, v]$.

Theorem 4.2 is proved similarly, by extending one of the 16 geometric lattices $\mathbf{S}_h[u, v]$ given by Theorem 4.1 by a third extra line or conic w. □

4.2. **Proof of Theorem 1.1.** The bound $\lvert \mathrm{Fn}_2 X\rvert \leqslant 176$ and the uniqueness of the Barth–Bauer octic X_{176} at which this bound is attained are given by Theorems 3.1, 4.1, 4.2. Furthermore, X_{176} admits a faithful projective symplectic action of the Mukai group M_{20} (see [17]; #81 in Table 3). On the other hand, according to [6, Corollary 7.3] (see also [3], where a slightly stronger assumption is used), this property characterizes a unique octic $K3$-surface $X \subset \mathbb{P}^5$. The defining equations cited in Theorem 1.1 are found in [3]. □

Example 4.3. It is remarkable that the only Barth–Bauer octic X_{160} realizing the next largest number 160 of conics (the second raw in Table 7) is also characterized by the presence of a faithful projective symplectic action of a Mukai group, this time T_{192} (#77 in Table 3). The uniqueness of a T_{192}-octic in $\mathbb{P}^5$ is easily proved similar to [6, § 7.1].

Table 8. Other rigid octics (see Theorem 4.2)

Clusters	Lines	Conics	$\lvert G\rvert$	i_Ω	G_ω	T	(r,c)
$1^\star,3,3,4$	4	80	64	4	21	$[8,2,20]$	$(0,1)\to(0,4)$
$3,3,4$		80	256	4	39	$[12,4,20]$	$(0,1)\to(0,2)$
$1^\star,1^\star,1^\star,1^\star,5$	16	$16+64$	512	2	39[2]	$[8,4,12]$	$(1,0)$
$1^\star,2^\star,3,4$	24	$32+44$	64	2	21	$[4,0,16]$	$(1,0)\to(0,1)$
[1] $1^\star,3,4$	4	72	64	2	21	$[4,0,44]$	$(1,0)\to(0,1)$
						$[12,4,16]$	$(0,1)\to(0,2)$
[1] $1^\star,3,4$	4	72	64	2	21	$[4,0,44]$	$(1,0)\to(0,1)$
						$[12,4,16]$	$(0,1)\to(0,2)$
$1^\star,1^\star,4$	8	64	256	4	39	$[12,0,12]$	$(0,1)\to(0,2)$
$3,3,3,3$		64	256	2	39	$[8,0,32]$	$(1,0)\to(2,0)$
$3,3,3$		56	384	2	49	$[4,0,68]$	$(1,0)\to(2,0)$
						$[8,4,36]$	$(1,0)\to(0,1)$
$3,3,3$		56	64	2	21	$[8,4,48]$	$(1,0)\to(0,1)$
						$[16,4,24]$	$(0,1)\to(0,2)$
$2^\star,3,3$	20	$16+36$	64	1	21	$[8,4,16]$	$(1,0)$
$2^\star,3,3$	20	$16+36$	64	1	21	$[8,4,16]$	$(1,0)$
$2^\star,3,3$	20	$16+36$	32	1	21	$[4,2,24]$	$(1,0)$
						$[8,2,12]$	$(0,1)$
$1^\star,1^\star,3,3$	8	48	64	2	21	$[8,2,20]$	$(0,1)\to(0,2)$
$1^\star,3,3$	4	48	64	2	21	$[16,0,16]$	$(0,1)\to(0,2)$
$1^\star,3,3$	4	48	64	2	21	$[16,0,16]$	$(0,1)\to(0,2)$
[2] $1^\star,3,3$	4	48	64	1	21	$[8,4,32]$	$(2,0)$
[2] $1^\star,3,3$	4	48	64	1	21	$[8,4,32]$	$(2,0)$
[3] $1^\star,3,3$	4	48	64	1	21	$[8,4,32]$	$(2,0)$
[3] $1^\star,3,3$	4	48	64	1	21	$[8,4,32]$	$(2,0)$
[4] $1^\star,3,3$	4	48	32	1	21	$[4,2,56]$	$(2,0)$
						$[16,6,16]$	$(0,1)$
[4] $1^\star,3,3$	4	48	32	1	21	$[4,2,56]$	$(2,0)$
						$[16,6,16]$	$(0,1)$
[5] $1^\star,1^\star,3$	8	40	64	1	21	$[4,0,44]$	$(1,0)$
						$[12,4,16]$	$(0,1)$
[5] $1^\star,1^\star,3$	8	40	64	1	21	$[4,0,44]$	$(1,0)$
						$[12,4,16]$	$(0,1)$
$1^\star,1^\star,1^\star$	12	32	576	2	49	$[4,2,36]$	$(1,0)\to(2,0)$

First, the Néron–Severi lattice S of a very general (non-algebraic) $K3$-surface with a faithful symplectic T_{192}-action (*cf.* [13]) can be found as $h^\perp \subset NS(X_{160})$. One has

$$\operatorname{discr}_2 S = \left[\tfrac{5}{4}\right] \oplus \left[\tfrac{5}{4}\right] \oplus \left[\tfrac{5}{4}\right], \qquad \operatorname{discr}_3 S = \left[\tfrac{4}{3}\right],$$

and the image of the natural homomorphism $\operatorname{Aut}(\operatorname{Fn} X_{160}) \hookrightarrow O(S) \to \operatorname{Aut}(\operatorname{discr} S)$ is an index 12 subgroup preserving one of the 12 vectors α_i of square $\frac{3}{2} \bmod 2\mathbb{Z}$.

On the other hand, each of the twelve vectors α_i as above gives rise to an index 4 extension of $S \oplus \mathbb{Z}h$, which is the Néron–Severi lattice of a Barth–Bauer octic with 160 conics. By Theorem 4.2, we conclude that all these extensions are isomorphic; hence, all 12 vectors constitute a single $O(S)$-orbit and the natural homomorphism $O(S) \to \operatorname{Aut}(\operatorname{discr} S)$ is surjective.

From the last statement, using the techniques of [20] and the uniqueness of

$$S^\perp \cong \begin{bmatrix} 4 & 0 & 0 \\ 0 & 8 & 4 \\ 0 & 4 & 8 \end{bmatrix}$$

in its genus, we conclude that there is a single $O(S)$-equivalence class of primitive isometries $S \hookrightarrow \mathbf{L}$; furthermore, any element of $O(S^\perp)$ extends to an autoisometry of $\mathbf{L}$. Since the group $O^+(S^\perp)$ acts transitively on the six square 8 vectors in $S^\perp$, the uniqueness of a T_{192}-octic surface follows, *cf.* §2.4.

Remark 4.4. The same argument shows that there is a unique T_{192}-quartic in $\mathbb{P}^3$. It is the famous Schur [23] quartic X_{64} maximizing the number of lines: it has 64 lines and 576 reducible + 144 irreducible = 720 conics.

4.3. **Proof of Theorem 1.4.** The bound on the number of lines is explicitly stated in [5]. To estimate the number of reducible conics (*i.e.*, pairs of intersecting lines), recall the bound

$$\operatorname{val} v \leqslant \begin{cases} 7, & \text{if } X \text{ is a triquadric,} \\ 8, & \text{if } X \text{ is a special octic} \end{cases}$$

on the valency of a line in the graph $\operatorname{Fn}_1 X$, see [5, Proposition 2.12]. It follows that the number of reducible conics does not exceed

$$\begin{cases} 30 \cdot 7/2 = 105, & \text{if } X \text{ is a triquadric and } |\operatorname{Fn}_1 X| \leqslant 30, \\ 26 \cdot 8/2 = 104, & \text{if } X \text{ is special and } |\operatorname{Fn}_1 X| \leqslant 26. \end{cases}$$

On the other hand, the Fano graphs of the triquadrics with more than 30 lines and special octics with more than 26 lines are listed in [5] (see Theorems 1.2 and 1.4 respectively), and the number of reducible conics in these graphs is easily computed: the maximum is 112, attained at a unique triquadric (Θ'_{36} in [5]). □

4.4. **Proof of Theorem 1.6.** As explained in [6], an equiconical stratum of Barth–Bauer octics contains a real octic with all lines and conics real if and only if the respective generic transcendental lattice has a direct summand isomorphic to $\mathbf{U}(2)$. In particular, this stratum must have codimension at most 2. On the other hand, according to Theorems 3.1 and 4.1, the maximal number of conics on a Barth–Bauer octic of Picard rank $\rho \leqslant 19$ is 128 (see the line marked with a $*$ in Table 6), the typical transcendental lattice being $T \cong \mathbf{U}(2) \oplus [40]$, as required.

To show that this is the maximum, we have to consider singular octics given by Theorem 4.2 and Tables 7, 8 and, for each such octic X, compute the actions c_* induced on $NS(X)$ by all possible real structures $c\colon X \to X$. (See [6] for details.) This computation gives us at most 56 real conics, all maximal configurations corresponding to certain real structures on the octic X_{176} introduced in Theorem 1.1 (the first row in Table 7). □

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Department of Mathematics, Bilkent University, 06800 Ankara, TURKEY
Email address: `degt@fen.bilkent.edu.tr`