CONICS ON SMOOTH QUARTIC SURFACES

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Abstract. We prove that the maximal number of conics, *a priori* irreducible of reducible, on a smooth spatial quartic surface is 800, realized by a unique quartic. We also classify quartics with many (at least 720) conics. The maximal number of real conics on a real quartics is between 656 and 718.

1. Introduction

Unless stated otherwise, all algebraic varieties in this paper are over $\mathbb{C}$.

1.1. **Principal results.** Following [2], denote by $N_{2n}(d)$ the maximal number of smooth rational degree d curves that a smooth $2n$-polarized $K3$-surface $X \subset \mathbb{P}^{n+1}$ may contain. The numbers $N_{2n}(1)$ of lines, most notably $N_4(1)$, have a long history going back at least to F. Schur [31] and currently are well known, see [5, 9, 14, 29, 31, 32, 33] and further references therein. However, even the very next case, the numbers $N_{2n}(2)$ of conics, is still wide open, with just a few sporadic examples and conjectures [1, 2, 7, 8, 11, 10].

The principal result of the present paper is Theorem 1.1 below. In the spirit of [5, 14] and other papers based on the global Torelli theorem and lattice theory, we do not just prove a bound but also classify all large configurations of conics. Note also that, in spite of the definition of $N_{2n}(2)$, *a priori* we do not distinguish between irreducible and reducible conics; thus, a *conic* is either a smooth (planar) curve of projective degree 2 or a pair of distinct intersecting lines. Throughout the paper, if reducible conics are present, we describe the total number of conics as

$$(\text{irreducible}) + (\text{reducible}) = (\text{total}).$$

Theorem 1.1 (see §7.1). *A smooth quartic $X \subset \mathbb{P}^3$ with at least 720 conics* (a priori *irreducible or reducible*) *is projectively equivalent to one of the following surfaces*:

1. *the M_{20}-quartic (5.8), see [23], with 800 irreducible conics,*
$$z_0^4 + z_1^4 + z_2^4 + z_3^4 + 12 z_0 z_1 z_2 z_3 = 0, \quad \text{or}$$
2. *a $4^2\mathfrak{A}_4$-quartic (5.10), with 736 irreducible conics* (cf. *Addendum 1.4*), *or*
3. *one of the two $L_2(7)$-quartics (5.11) with 728 irreducible conics, or*
4. *the T_{192}-,* aka Schur's quartic, *with 64 lines and $144 + 576 = 720$ conics,*
$$z_0(z_0^3 - z_1^3) = z_2(z_2^3 - z_3^3).$$

The quartics in items (1), (2), (3) have no lines; hence, all conics are irreducible.

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Thus, Theorem 1.1 states that $N_4(2) = 800$, as conjectured in [7, 10]. It states also that the conjectural bound $N_4^*(2)$ is well defined and equals 720. (Recall that, conjecturally, see, *e.g.*, [11], there is a bound $N_{2n}^*(2) < N_{2n}(2)$ such that any smooth $K3$-surface $X \subset \mathbb{P}^{n+1}$ of degree $2n$ with more than $N_{2n}^*(2)$ conics has no lines. For the moment, we only know that $249 \leqslant N_6^*(2) < 261$ is well defined, see [8]. *Cf.* also a similar speculation on $N_{2n}^*(1)$ in [15, 16]. The existence and even the meaning of the more general bounds $N_{2n}^*(d)$ are not quite clear yet.)

The quartic in item (1) was discovered (in conjunction with the conic counting problem) in [7], upon which X. Roulleau (private communication) observed that the surface admits a faithful symplectic action of M_{20} (hence the equation above) and as such it was studied by C. Bonnafé and A. Sarti [3]; later, B. Naskręcki [24] found explicit equations of all 800 conics. Quartic (2) has previously appeared in [10], whereas (3) and (4) appeared in [11], as examples of quartics with many conics. Of course, Schur's quartic (4) has been known ever since Schur [31], and it is famous for quite a few other extremal properties. Thus, it is the only smooth quartic

- maximizing the number $N_4^*(2) = 720$ of conics in the presence of lines,
- maximizing the number of *reducible* conics on a smooth quartic, see [10],
- admitting a faithful symplectic action of Mukai's group T_{192}, see [11],
- maximizing the number $N_4(1) = 64$ of lines, see [14],
- minimizing the determinant $\det T \geqslant 48$ of the transcendental lattice of a singular $K3$-surface admitting a smooth quartic model, see [6].

Remark 1.2. The notation/terminology according to the group $\mathrm{Sym}_h X \subset \mathrm{Aut}_h X$ of symplectic projective automorphisms of X used in Theorem 1.1 is mainly due to the lack of imagination. This is justified for M_{20} and T_{192}, as quartics admitting symplectic actions of these groups are indeed unique, see [3, 10] and [11], respectively. However, $4^2\mathfrak{A}_4$-quartics constitute a 1-parameter family (see Addendum 1.4), and only one of them has 736 conics. Similarly, there are two $PGL(4, \mathbb{C})$-classes of $L_2(7)$-quartics (see, essentially, [19]): one of them is item (3) in Theorem 1.1 whereas the other has neither lines nor conics as the polarization h splits as a direct summand of the Néron–Severi lattice $NS(X)$.

1.2. **Further observations and examples.** Our principal results are proved by a computation and, in order to produce some interesting examples, we saved and analyzed all large (at least 600 conics) configurations encountered in the course of the proof. The total conic counts observed are

$$800, 736, 728, 720, 704, 680, 664, 660, 656, 640, 636, 628, 624, 622, 620, 616, 608, 600,$$

some of which are represented by multiple configurations/quartics. We do not assert that this list is complete, but it must be close to such; in particular, it does contain all Barth–Bauer quartics with at least 600 conics, see [10].

In particular, one of the examples found beats the record for the number of *real* conics on a *real* quartic set in [10], see Addendum 1.3; the next known configuration is $188 + 448 = 636$ real conics on Y_{56} in [14], see Remark 7.4.

Addendum 1.3 (see §7.2)**.** *There exists a real smooth quartic with* 656 *real conics, see* (5.14)*; moreover, all conics are irreducible and have non-empty real part. Hence, we have the bounds*

$$656 \leqslant N_4(2; \mathbb{R}) \leqslant 718$$

on the maximal number of real conics on a real smooth quartic.

Yet another example is the largest known configuration in a 1-parameter family. This family connects quartics (1) and (2) in Theorem 1.1.

Addendum 1.4 (see §7.3)**.** *There exists an equiconical* 1*-parameter family* $\mathcal{X}$ *of (generically)* $4^2\mathfrak{A}_4$*-quartics with* 608 *irreducible conics, see* (5.16)*. The closure* $\bar{\mathcal{X}}$ *of this family contains quartics* (1) *and* (2) *in Theorem 1.1, as well as the Barth–Bauer quartic with* 640 *conics, see* [10] *and* (5.15)*.*

The previously known record for both the number of real conics and the number of conics in a family was 560 (see [11]; same quartic).

A few other examples are discussed in §7.4.

1.3. **Idea of the proof.** Similar to [8, 9, 13], we establish that, regarded as an abstract graph, the configuration of conics (irreducible or reducible) on a smooth quartic surface can be realized as a certain special set of square 4 vectors in a 4-polarized Niemeier lattice, *i.e.*, positive definite even unimodular lattice of rank 24. There are but 24 Niemeier lattices, all of which are well known, see [26, 4]; this already implies that there is indeed a uniform bound on the number of conics, a fact that is not immediately obvious *a priori*.

The principal novelty is the fact that, due to the much larger numbers involved, in order to make the computation feasible we have to shift the paradigm and deal with sublattices (or even rational subspaces) rather that just subsets. This shift lets us revise and substantially refine the combinatorial estimates on the number of conics, immediately resulting in the bound of 1736, which is much better than the previously known 5016 (see [2], with a reference to S. A. Strømme). Reducing this further down to 718 (with a few exceptions stated in Theorem 1.1), still computer aided, requires a drastic revision of all parts of the algorithm and the underlying mathematics: in addition to working with subspaces rather than subsets, we employ the full orthogonal group rather than just reflections, treat more carefully iterated index 2 subgroups in §6.1, *etc*.

1.4. **Contents of the paper.** In §2 we reduce the conic counting problem to the study of the so-called *geometric* sets of square 4 vectors in 4-polarized Niemeier lattices. The precise conditions are formalized in §3.1, and in the rest of §3 we deal with the combinatorial background necessary for §4, where we rule out 18 out of the 24 Niemeier lattices. Unlike a few earlier papers [9, 13], here we consider primitive sublattices, hence so-called *saturated* sets only; this fact lets us refine the concept of admissibility and improve the combinatorial estimates. Our principal goal is making this part of the proof as human comprehendible as possible; several examples are worked out in full detail.

In §5, we treat the remaining five lattices rationally generated by roots, and this part of the proof, based essentially on Lemma 4.3, is heavily computer aided. To save space and keep the code separate from the underlying mathematical ideas, we have moved the details to the companion text [12]. Here, we merely state the result, outline the proof, and list exceptional or otherwise interesting configurations of conics in a form from which they can easily be reconstructed.

The Leech lattice is treated in a similar manner in §6. The heart of our approach is Lemma 6.2. We give a complete proof of this statement since it is followed closely by the new version of the code, which is much more efficient than that used in [8, 13], see §6.1 and [12, §7]. We discover that all extremal configurations of conics listed in Theorem 1.1 can be embedded to the Leech lattice, as described in §6.2.

In §7 we collect the output of the computation in §4–§6 and complete the formal proof of the principal results of the paper stated in the introduction.

All technical details and a plethora of further examples of configurations of conics are found in the companion text [12]: it is available electronically, from the author's web page or as an ancillary file for this paper in `arXiv:2407.00493`.

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Most algorithms used in the paper were implemented using `GAP` [18].

2. Embedding to a Niemeier lattice

In this section we exploit the fact that a smooth spatial quartic is a $K3$-surface and reduce the conic counting problem to the study of large collections of certain square 4 vectors in a Niemeier lattice.

2.1. **Smooth quartics as $K3$-surfaces.** Recall that a smooth quartic $X \subset \mathbb{P}^3$ is a $K3$-surface; in particular, there is an isometry

$$H_2(X) \cong \mathbf{L} := -2\mathbf{E}_8 \oplus 3\mathbf{U}.$$

(Here and below, we use Poincaré duality to identify $H^2(X)$ and $H_2(X)$ and, *via* the intersection index form, regard the latter as a unimodular integral lattice.) The Néron–Severi lattice $NS(X) \subset H_2(X)$ is a primitive hyperbolic sublattice of *Picard rank* $\rho := \operatorname{rk} NS(X) \leqslant 20$; this lattice is naturally 4-polarized by the class $h \in NS(X)$, $h^2 = 4$, of hyperplane section.

The following fact is essentially contained in Saint-Donat [30] (see also [15] for an accurate restatement in terms of homology classes rather than linear systems); regarding the homology, we also refer to the surjectivity of the period map [21].

Theorem 2.1 (Saint-Donat [30]). *Let $S \ni h$ be a 4-polarized hyperbolic lattice and $S \hookrightarrow \mathbf{L}$ a primitive isometric embedding. Then there is a* smooth *quartic $X \subset \mathbb{P}^3$ and an isometry*

$$\bigl(\mathbf{L} \supset S \ni h\bigr) \cong \bigl(H_2(X) \supset NS(X) \ni h\bigr)$$

if and only if there is no class $e \in S$ such that

1. $e^2 = -2$ *and* $e \cdot h = 0$ (exceptional divisor), *or*
2. $e^2 = 0$ *and* $e \cdot h = 2$ (2-isotropic vector). ◁

Next, we recall Vinberg's algorithm [34] for computing the fundamental polyhedron of the group generated by reflections in the hyperbolic space

$$\bigl\{x \in S \otimes \mathbb{R} \bigm| x^2 > 0\bigr\}/\mathbb{R}^\times$$

containing the class of h or, for short, just the *fundamental polyhedron of $S \ni h$*. For each $n \in \mathbb{N}$, introduce inductively the sets

$$\Delta_n(S,h) := \bigl\{x \in S \bigm| x^2 = -2 \text{ and } x \cdot h = n\bigr\},$$
$$\Delta_n^\circ(S,h) := \bigl\{x \in \Delta_n \bigm| x \cdot y \geqslant 0 \text{ for all } y \in \Delta_m^\circ,\ m < n\bigr\}.$$

We use the fact that X is assumed smooth and, by Theorem 2.1, $\Delta_0^\circ = \Delta_0 = \varnothing$; this lets us avoid the subtlety with the choice of a Weyl chamber for the root system in $h^\perp \subset S$ (*cf.*, *e.g.*, [15]). Then, the fundamental polyhedron of $S \ni h$ is the set

$$\Delta(S,h) := \bigcup_{n=0}^{\infty} \Delta_n^\circ(S,h).$$

(Strictly speaking, the fundamental polyhedron is the intersection of the half-spaces $x \cdot v \geqslant 0$, $v \in \Delta$, but, since we are interested in the set of its walls only, we abuse the language and refer to Δ itself as the fundamental polyhedron.)

Now, assume that $(S \ni h) = (NS(X) \ni h)$ for a quartic $X \subset \mathbb{P}^3$. Then, according to [20, § 8.1], the map $C \mapsto [C] \in S$ establishes a bijection between the set of smooth rational curves on X and Δ. Taking into account the projective degree $[C] \cdot h$ of a curve and still assuming that X is smooth, we have canonical bijections

$$\begin{aligned} \operatorname{Fn}_1(X,h) &:= \{\text{lines on } X\} &&= \Delta_1^\circ(S,h) = \Delta_1(S,h),\\ \operatorname{Fn}_2^\circ(X,h) &:= \{\text{irreducible conics on } X\} &&= \Delta_2^\circ(S,h),\\ \operatorname{Fn}_2(X,h) &:= \{\text{all conics on } X\} &&= \Delta_2(S,h). \end{aligned} \tag{2.2}$$

All three sets, as well as the *graph* $\operatorname{Fn}_*(X,h) := \operatorname{Fn}_1 \cup \operatorname{Fn}_2^\circ$ *of lines and conics*, are regarded as (multi-)graphs, with two vertices represented by curves C_1, C_2 connected by an edge of multiplicity $C_1 \cdot C_2$ (no edge if $C_1 \cdot C_2 = 0$). It is easily seen that all multiplicities are non-negative, even if both $C_1, C_2 \in \operatorname{Fn}_2(X,h)$ are (distinct) reducible conics. The graph Fn_* is also *colored* according to the projective degree of its vertices.

The four Fano graphs above can be defined for any 4-polarized hyperbolic lattice $S \ni h$ provided that S does not contain a vector e as in Theorem 2.1(1) or (2): it is this condition that keeps the multiplicities non-negative, *cf.* Lemma 2.5 below.

2.2. **The modified Néron–Severi lattice.** Start with a 4-polarized hyperbolic lattice $S \ni h$ and consider the sublattice

$$S_{\text{even}} := \bigl\{x \in S \bigm| x \cdot h = 0 \bmod 2\bigr\}.$$

Clearly, $S_{\text{even}} = S$ or $[S : S_{\text{even}}] = 2$. This lattice is still hyperbolic and it is h-*even* in the sense that it is 4-polarized (by the same vector $h \in S_{\text{even}}$) and

$$h \in 2S_{\text{even}}^\vee, \ \textit{i.e.}, \ x \cdot h = 0 \bmod 2 \text{ for each } x \in S_{\text{even}}. \tag{2.3}$$

Now, given an h-even 4-polarized lattice $S \ni h$, we define $(S \ni h)^\sharp$ to be the same pair $S \ni h$ with the bilinear form on S modified *via*

$$x \otimes y \mapsto \tfrac{1}{2}(x \cdot h)(y \cdot h) - (x \cdot y).$$

Since this operation makes sense for h-even lattices only, we will use the shorthand $(S \ni h)^\sharp := (S_{\text{even}} \ni h)^\sharp$ in the general case (*cf.* also the alternative construction in §2.3 below, where the passage to S_{even} is automatic).

Lemma 2.4. *The map $(S \ni h) \mapsto (S \ni h)^\sharp$ is an involutive operation on the set of h-even 4-polarized even lattices. A lattice S is hyperbolic if and only if $(S \ni h)^\sharp$ is positive definite.*

Proof. For the last assertion, observe that the modification preserves $h^2 = 4$ and reverses the form on $h^\perp$. All other statements are straightforward. ∎

For a positive definite 4-polarized h-even lattice $S^\sharp \ni h$ we modify the notion of its *Fano graph*, letting

$$\underline{\mathrm{Fn}}(S^\sharp, h) := \bigl\{x \in S^\sharp \bigm| x^2 = 4 \text{ and } x \cdot h = 2\bigr\}$$

and connecting two vertices x, y by an edge of multiplicity $2 - x \cdot y$. As above, this notion is mostly useful if $S^\sharp$ is root free; then, all multiplicities are non-negative due to the following lemma.

Lemma 2.5. *If a positive definite 4-polarized lattice $S \ni h$ is root free, then, for any two vectors $l_1, l_2 \in \underline{\mathrm{Fn}}(S, h)$ one has*

$$l_1 \cdot l_2 = -2 \text{ (iff } l_1 + l_2 = h),\ 0,\ 1,\ 2, \text{ or } 4 \text{ (iff } l_1 = l_2).$$

Proof. Consider the sublattice $R := \mathbb{Z}h + \mathbb{Z}l_1 + \mathbb{Z}l_2$ and assert that $\det R \geqslant 0$. This results in $-2 \leqslant l_1 \cdot l_2 \leqslant 4$. In the two border cases, $\det R = 0$ and the vectors are linearly dependent, as stated in the lemma. If $l_1 \cdot l_2 = 3$, then $l_1 - l_2$ is a root, and if $l_1 \cdot l_2 = -1$, then $h - l_1 - l_2$ is a root. □

Proposition 2.6. *Assume that a 4-polarized hyperbolic lattice $S \ni h$ admits a primitive embedding to* $\mathbf{L}$. *Then*:

1. *$S \ni h$ is isomorphic to the Néron–Severi lattice $NS(X) \ni h$ of a smooth quartic $X \subset \mathbb{P}^3$ if and only if the modified lattice $(S, h)^\sharp$ is root free*;
2. *if $(S, h)^\sharp$ is root free, the graph $\mathrm{Fn}_2(X, h)$ of all conics of any quartic X as in item (1) is canonically isomorphic to the Fano graph $\underline{\mathrm{Fn}}(S^\sharp, h)$.*

Proof. For the first statement, observe that any vector $e \in S$ as in Theorem 2.1(1) or (2) would survive to S_{even} and give rise to a root $e \in S^\sharp$ such that $e \cdot h = 0$ or 2, respectively. Conversely, any such root in $S^\sharp$ is a prohibited vector in S. On the other hand, since $S^\sharp$ is positive definite, $|e \cdot h| \leqslant 2$ for any root $e \in S^\sharp$. Since $S^\sharp$ is also h-even, this leaves $e \cdot h \in \{0, \pm 2\}$, *i.e.*, at least one of $\pm e$ is necessarily a prohibited vector.

The second statement follows from (2.2) and the obvious observation that

$$\Delta_2(S, h) = \Delta_2(S_{\text{even}}, h) = \underline{\mathrm{Fn}}(S^\sharp, h).$$ □

2.3. **Embedding $S^\sharp$ to a Niemeier lattice.** The chain

$$(S \ni h) \mapsto (S_{\text{even}} \ni h) \mapsto (S_{\text{even}} \ni h)^\sharp$$

transforming a hyperbolic 4-polarized lattice to a positive definite h-even one can alternatively be described as follows:

1. consider the orthogonal complement $h^\perp \subset S$; change the sign of the form;
2. consider the orthogonal direct sum $S' = (-h^\perp) \oplus \mathbb{Z}h$, $h^2 = 4$;
3. define $S^\sharp$ as the extension of S' *via* all/any vector of the form

$$\tfrac{1}{2}(2l - h) \oplus \tfrac{1}{2}h \in S' \otimes \mathbb{Q},$$

where $l \in S$ is such that $l \cdot h = 2 \bmod 4$.

If $l \in S$ as in Step (3) does not exist, we merely leave $S^\sharp := S'$; this case is not interesting as one obviously has $\Delta_2(S, h) = \varnothing$. Otherwise, both $S_{\text{even}} \supset S'$ (as an abelian group) and $S^\sharp \supset S'$ are extensions of index 2, and

$$l \mapsto \tfrac{1}{2}(2l - h) \oplus \tfrac{1}{2}h, \quad l \in S, \quad l \cdot h = 2 \bmod 4,$$

is a canonical bijection between the cosets $S_{\text{even}} \smallsetminus S'$ and $S^\sharp \smallsetminus S'$.

It follows that the positive definite 4-polarized lattice $S^\sharp \ni h$ constructed in §2.2 from a hyperbolic lattice $S \ni h$ is the lattice $S(S, h)$ considered in [8]. Thus, we have the following statement, based on Nikulin's criterion [27].

Proposition 2.7 (see [8, Proposition 2.10]). *If a 4-polarized hyperbolic lattice $S \ni h$ admits a primitive embedding to $\mathbf{L}$, then the modified lattice $S^\sharp$ admits a primitive embedding to at least one of the 24 Niemeier lattices.* ◁

3. Combinatorial bounds

In this section we discuss a few simple (and very rough) combinatorial bounds that rule out (in §4 below) the vast majority of Niemeier lattices. The few remaining ones will be treated in the subsequent sections.

3.1. **Niemeier lattices** (see [4, 26]). Recall that, with one exception (the Leech lattice, see §6 below), a Niemeier lattice N is rationally generated by roots and is determined up to isomorphism by its maximal root system D. Therefore, we use the notation

$$N := N(D) = N\bigl(\textstyle\bigoplus_k D_k\bigr), \quad k \in \Omega, \tag{3.1}$$

where D_k are the indecomposable $\mathbf{A}$–$\mathbf{D}$–$\mathbf{E}$ components of D and Ω is the index set. There are well-defined orthogonal projections $N \to D_k^\vee$, $k \in \Omega$, and we also fix the notation $l = \sum_k l|_k$, $l|_k \in D_k^\vee$, $k \in \Omega$, for the decomposition of a vector $l \in N$. The vector l or its projection $l|_k$ is called *integral* if $l \in D$ or $l|_k \in D_k$, respectively; otherwise, if $l \in N \smallsetminus D$ or $l|_k \in D_k^\vee \smallsetminus D_k$, they are called *rational*. If $l^2 = 4$, then each projection $l|_k$ is either integral, and then $l|_k^2 \in \{0, 2, 4\}$, or a shortest vector in its discriminant class $(l|_k \bmod D_k) \in \operatorname{discr} D_k$, as otherwise a shorter representative would give rise to a root $l' \notin D$. Furthermore, if $l^2 = 4$ and at least one projection $l|_k \neq 0$ is integral, then so are l and all other projections.

We fix a polarization $h \in N$, $h^2 = 4$, and abbreviate

$$\mathfrak{O} := \mathfrak{O}_h := \underline{\operatorname{Fn}}(N, h) = \bigl\{ l \in N \bigm| l^2 = 4,\ l \cdot h = 2 \bigr\}.$$

The elements of $\mathfrak{O}_h$ are called *conics*. We have a fixed point free *duality* involution

$$*\colon \mathfrak{O}_h \to \mathfrak{O}_h, \quad l \mapsto l^* := h - l.$$

Geometrically, l and l^* represent a complementary pair of conics constituting a hyperplane section of the quartic. A subset $\mathfrak{C} \subset \mathfrak{O}_h$ is *self-dual* if $\mathfrak{C}^* = \mathfrak{C}$.

Definition 3.2. For subsets $\mathfrak{C} \subset \mathcal{S} \subset \underline{\operatorname{Fn}}(N, h)$, we define the *spans*

$$\operatorname{span}_{\mathbb{Z}} \mathfrak{C} := \mathbb{Z}\mathfrak{C} + \mathbb{Z}h \subset N, \quad \operatorname{span} \mathfrak{C} := N \cap (\mathbb{Q}\mathfrak{C} + \mathbb{Q}h) \subset N$$

and *saturations*

$$\operatorname{sat}_{\mathcal{S}} \mathfrak{C} := \mathcal{S} \cap \operatorname{span} \mathfrak{C} \subset \mathcal{S}, \quad \operatorname{sat} \mathfrak{C} := \mathfrak{O}_h \cap \operatorname{span} \mathfrak{C} \subset \mathfrak{O}_h.$$

A subset $\mathfrak{C}$ is called

- *saturated* (resp. *$\mathcal{S}$-saturated*) if $\operatorname{sat} \mathfrak{C} = \mathfrak{C}$ (resp. $\operatorname{sat}_{\mathcal{S}} \mathfrak{C} = \mathfrak{C}$);
- *admissible* if $\operatorname{span} \mathfrak{C}$ is h-even and root free.

Note that a saturated set is automatically self-dual; if $\mathcal{S}$ is self-dual, then so are all $\mathcal{S}$-saturated sets.

Definition 3.2 extends to any 4-polarized positive definite lattice $N \ni h$. By an implicit reference to $\operatorname{span} \mathfrak{C}$ we also extend to subsets $\mathfrak{C} \subset \mathfrak{O}_h$ such lattice-theoretic notions as rank, discriminant, *etc.*, so that, *e.g.*, $\operatorname{rk} \mathfrak{C} := \operatorname{rk} \operatorname{span} \mathfrak{C}$.

Definition 3.3. An admissible subset $\mathfrak{C} \subset \underline{\mathrm{Fn}}(N,h)$ is called *geometric* if there is an isometry

$$\varphi\colon (\operatorname{span}\mathfrak{C} \ni h)^\sharp \hookrightarrow \mathbf{L}$$

such that the primitive hull of the image $\operatorname{Im}\varphi$ is either $\operatorname{Im}\varphi$ itself or a certain h-odd index 2 overlattice $S \supset \operatorname{Im}\varphi$.

Since, for a $K3$-surface X, one has $\sigma_+H_2(X) = 19$ and $NS(X)$ is hyperbolic, an obvious necessary condition for a set $\mathfrak{C} \subset \mathfrak{O}_h$ to be geometric is that

$$\operatorname{rk}\mathfrak{C} \leqslant 20; \tag{3.4}$$

this is an essential part of all algorithms, even if we choose to skip the expensive [8, Proposition 2.10], possibly followed by the analysis of index 2 extensions.

Note that, since $\operatorname{span}\mathfrak{C}$ in Definition 3.3 is assumed root free, both $(\operatorname{span}\mathfrak{C} \ni h)^\sharp$ and any h-odd index 2 extension thereof are automatically free of vectors $e \in S$ as in Theorem 2.1(1) or (2). Furthermore, according to Lemma 2.5, an admissible subset $\mathfrak{C} \subset \underline{\mathrm{Fn}}(N,h)$ can be regarded as a graph. Thus, Proposition 2.6 and Proposition 2.7 imply the following statement.

Proposition 3.5. *An abstract graph Γ is realizable as the graph $\mathrm{Fn}_2(X,h)$ of conics of a smooth quartic $X \subset \mathbb{P}^3$ if and only if Γ is graph-isomorphic to a saturated geometric subset $\mathfrak{C} \subset \underline{\mathrm{Fn}}(N,h)$ in a 4-polarized Niemeier lattice $N \ni h$.* ◁

3.2. **Orbits and combinatorial orbits.** We make extensive use of the following groups:

- $O(N)$, the full orthogonal group of N,
- $R(N) \subset O(N)$, the subgroup generated by reflections $r_e\colon x \mapsto x - (x\cdot e)e$ against the roots $e \in D$,
- $O_h(N)$, the stabilizer of h in $O(N)$, and
- $R_h(N) := O_h(N) \cap R(N)$.

Since $R(N)$ acts simply transitively on the set of its fundamental polyhedra, we conclude that

(3.6) $R_h(N)$ is generated by reflections against the roots orthogonal to h.

Besides, for any overlattice or inner product $\mathbb{Q}$-vector space $B \supset N$,

(3.7) the action of $R_h(N) \subset R(N)$ extends to $B \supset N$ *via* reflections.

The orbits of the action on $\mathfrak{O}_h$ of the groups $O_h(N)$ and $R_h(N)$ are called *orbits* and *combinatorial orbits*, respectively; typically, each orbit $\bar{\mathfrak{o}}$ splits into a number of combinatorial orbits $\mathfrak{o}_1, \mathfrak{o}_2, \ldots$, all of the same cardinality. The duality $*$ induces an involution (no longer free) on the sets of orbits and combinatorial orbits.

We define the *count* $c(\mathfrak{o})$ and *bound* $b(\mathfrak{o})$ of a combinatorial orbit $\mathfrak{o}$ *via*

$$c(\mathfrak{o}) := |\mathfrak{o}|, \qquad b(\mathfrak{o}) := \max|\mathfrak{C}|,$$

where $\mathfrak{C}$ runs over all *admissible* subsets $\mathfrak{C} \subset \mathfrak{o}$. It is obvious that $c(\mathfrak{o})$ and $b(\mathfrak{o})$ are constant within each orbit and that $c(\mathfrak{o}^*) = c(\mathfrak{o})$, $b(\mathfrak{o}^*) = b(\mathfrak{o})$.

These naïve bounds are extended to unions of combinatorial orbits, *e.g.*, orbits $\bar{\mathfrak{o}}$ or the whole set $\mathfrak{O}_h$, *by additivity*:

$$c(\mathfrak{o}_1 \cup \ldots) = c(\mathfrak{o}_1) + \ldots, \qquad b(\mathfrak{o}_1 \cup \ldots) := b(\mathfrak{o}_1) + \ldots.$$

3.3. **The lattices $\mathbf{H}_n$, $\mathbf{A}_n$, and $\mathbf{D}_n$.** Given an index set $\mathcal{I} := \mathcal{I}_n := \{1, \ldots, n\}$, we consider the (odd unimodular) Euclidean lattice

$$\mathbf{H}_n := \bigoplus_{i\in\mathcal{I}} \mathbb{Z}e_i, \quad e_i^2 = 1.$$

When $\mathbf{H}_n$ and, hence, $\mathcal{I}$ are fixed, we denote by $\bar{o} := \mathcal{I} \smallsetminus o$ the complement of $o \subset \mathcal{I}$ and let $\mathbf{1}_o := \sum e_i \in \mathbf{H}_n$, $i \in o$. The notation $r \vartriangle s := (r \cup s) \smallsetminus (r \cap s)$ stands for the symmetric difference of sets. We have

$$\text{(3.8)} \qquad \text{(a)}\ \mathbf{1}_u \cdot \mathbf{1}_v = |u \cap v|, \qquad \text{(b)}\ \mathbf{1}_u^2 - \mathbf{1}_u \cdot \mathbf{1}_v = \frac{1}{2}|u \vartriangle v| \quad \text{if } |u| = |v|.$$

It is this and a few similar relations below in this section that explain the relevance of bounds (3.23), (3.24) in §3.5 below.

The group $O(\mathbf{H}_n)$ is generated by permutations of the basis vectors (equivalently, reflections against the roots $e_i - e_j$) and reflections $r_i \colon a \mapsto a - 2(a \cdot e_i)e_i$ against the generators e_i. All roots in $\mathbf{H}_n$ are of the form $\pm e_i \pm e_j$, $i, j \in \mathcal{I}$, $i \neq j$.

The root lattice $\mathbf{A}_n$ is defined as $\mathbf{1}_{\mathcal{I}}^{\perp} \subset \mathbf{H}_{n+1}$:

$$\mathbf{A}_n = \bigl\{\textstyle\sum_i \alpha_i e_i \in \mathbf{H}_{n+1} \bigm| \sum_i \alpha_i = 0\bigr\}.$$

The discriminant group is $\mathbb{Z}/(n+1)$, and the shortest vectors in the discriminant class $\underline{m} = \underline{0}, \ldots, \underline{n}$ (we use the notation of [4], underlined to avoid confusion) are

$$[o] := \frac{1}{n+1}\bigl(|\bar{o}|\mathbf{1}_o - |o|\mathbf{1}_{\bar{o}}\bigr), \quad [o]^2 = \frac{|o||\bar{o}|}{n+1}, \quad \text{for } o \subset \mathcal{I},\ |o| = m,$$

so that $[\bar{o}] = -[o]$ and

$$\text{(3.9)} \quad \text{(a)}\ [r] \cdot [s] = |r \cap s| - \frac{|r||s|}{n+1}, \qquad \text{(b)}\ [r]^2 - [r] \cdot [s] = \frac{1}{2}|r \vartriangle s| \quad \text{if } |r| = |s|.$$

The $R(\mathbf{A}_n)$-orbits of integral vectors of square 4 or 2 are

$$4_\bullet := \{\mathbf{1}_u - \mathbf{1}_v\}, \quad 2_\circ := \{\mathbf{1}_u - \mathbf{1}_v\} \quad \text{for } u, v \subset \mathcal{I},\ u \cap v = \varnothing,\ |u| = |v| = 2 \text{ or } 1,$$

respectively, and we have

$$\text{(3.10)} \qquad [o] \cdot (\mathbf{1}_u - \mathbf{1}_v) = |u \cap o| - |v \cap o| \quad \text{whenever } |u| = |v|.$$

Likewise, the root lattice $\mathbf{D}_n$ is the maximal even sublattice of $\mathbf{H}_n$:

$$\mathbf{D}_n = \bigl\{\textstyle\sum_i \alpha_i e_i \in \mathbf{H}_n \bigm| \sum_i \alpha_i = 0 \bmod 2\bigr\}.$$

The discriminant $\operatorname{discr} \mathbf{D}_n$ is $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ (if n is even) or $\mathbb{Z}/4$ (if n is odd); the shortest vectors are (in the notation $\underline{1}, \underline{2}, \underline{3}$ of [4] for the discriminant classes, which we underline to avoid confusion)

$$\underline{2} \ni e_i \text{ for } i \in \mathcal{I} \quad \text{and} \quad \underline{1}, \underline{3} \ni [o] := \frac{1}{2}(\mathbf{1}_o - \mathbf{1}_{\bar{o}}), \quad [o]^2 = \frac{n}{4}, \quad \text{for } o \subset \mathcal{I}$$

(the class $[o] \bmod \mathbf{D}_n = \underline{1}$ or $\underline{3}$ depends on the parity of $|o|$), and we have an almost literal analogue of (3.9b):

$$\text{(3.11)} \qquad [r]^2 - [r] \cdot [s] = \frac{1}{2}|r \vartriangle s|.$$

If $d \geqslant 5$, the $R(\mathbf{D}_n)$-orbits of integral vectors of square $m = 4$ or 2 are

$$4_\bullet^2 := \bigl\{\pm 2e_i\bigr\}, \quad 4_\bullet := \bigl\{\textstyle\sum_{i\in u} \pm e_i\bigr\}, \quad 2_\circ := \bigl\{\textstyle\sum_{i\in u} \pm e_i\bigr\} \quad \text{for } u \subset \mathcal{I},\ |u| = m.$$

If $d = 4$, then $4_\bullet$ splits into two orbits according to the parity of the number of the $+$-signs. On the other hand, $4_\bullet$ and $4_\bullet^2$ constitute a single $O(\mathbf{D}_4)$-orbit.

We intentionally use the same notation $[\,\cdot\,]$ for the shortest discriminant vectors in both $\mathbf{A}_n$ and $\mathbf{D}_n$: in addition to (3.9b), (3.11) *vs.* (3.8b), these vectors have the following common property:

$$\sum_{o\subset\mathcal{I}}\alpha_o[o]=\sum_{o\subset\mathcal{I}}\mathbf{1}_o\quad\text{provided that}\quad\sum_{o\subset\mathcal{I}}\alpha_o=0 \tag{3.12}$$

for a collection of coefficients $\alpha_o\in\mathbb{Q}$, $o\subset\mathcal{I}$. This identity follows from the fact that the coefficients of e_i, $i\in\bar{o}$, differ from those of e_i, $i\in o$, by -1. In fact, (3.12) explains also (3.8b), (3.9b), and (3.11): given two vectors $a,b\in\mathbf{H}_n\otimes\mathbb{Q}$ of the same length, $a^2=b^2$, one obviously has $(a-b)^2=2a^2-2(a\cdot b)$.

3.4. **Counts and bounds *via* blocks.** Assume that an $R_h(N)$-invariant, *cf.* (3.7), orthogonal decomposition of the form

$$N\subset B_1\oplus B_2$$

has been fixed, where B_r, $r=1,2$, are certain positive definite inner product spaces over $\mathbb{Q}$, called *blocks*. (Often, we take for B_r sums of some of $D_k\otimes\mathbb{Q}$ in (3.1), but this is not assumed; in fact, we do not even assume that $\operatorname{rk}N=\dim B_1+\dim B_2$, *cf.* §3.6 below.) The orthogonal projection $N\to B_r$ is denoted by $|_r$, $r=1,2$, *cf.* the similar notation in §3.1.

In view of (3.6), the $R_h(N)$-invariance of the block decomposition is equivalent to the assertion that each root $e\in h^\perp\subset N$ is either in B_1 or in B_2; hence, in the obvious notation, $R_h(N)=R_h(B_1)\times R_h(B_2)$. It follows that each combinatorial orbit $\mathfrak{o}$ has the property that

(1) $\mathfrak{o}=\mathfrak{o}|_1\times\mathfrak{o}|_2=\bigl\{l_1+l_2\bigm|l_r\in\mathfrak{o}|_r,\ r=1,2\bigr\}$ and

(2) the pointwise stabilizer of $\mathfrak{o}|_r$ in $R_h(N)$ is transitive on $\mathfrak{o}|_s$, $\{r,s\}=\{1,2\}$.

Thus, we can define the *partial count* and *bound* of $\mathfrak{o}$ *via*

$$c_r(\mathfrak{o}):=\bigl|\mathfrak{o}|_r\bigr|,\qquad b_r(\mathfrak{o}):=\max|\mathfrak{C}_r|, \tag{3.13}$$

the maximum running over all subsets $\mathfrak{C}_r\subset\mathfrak{o}|_r$ such that, for some (equivalently, any) fixed $l_s\in\mathfrak{o}|_s$, $\{r,s\}=\{1,2\}$, the set $\bigl\{l_s+l_r\bigm|l_r\in\mathfrak{C}_r\bigr\}$ is admissible. In view of (2) above, this latter condition is indeed independent of the choice of $l_s\subset B_s$. As an immediate consequence of these definitions and (1), (2) above, we have

$$c(\mathfrak{o})=c_1c_2,\qquad b(\mathfrak{o})\leqslant\min\{b_1c_2,c_1b_2\}=c(\mathfrak{o})\min_r\frac{b_r}{c_r}.$$

By induction, we easily extend these relations to any number of pairwise orthogonal $R_h(N)$-invariant blocks B_r, $r\in I$, arriving at

$$c(\mathfrak{o})=\prod_r c_r(\mathfrak{o}),\qquad b(\mathfrak{o})\leqslant c(\mathfrak{o})\min_r\frac{b_r(\mathfrak{o})}{c_r(\mathfrak{o})}. \tag{3.14}$$

Since the block decomposition is $R_h(N)$-invariant, the following quantities are constant for each $r=1,2$ and each combinatorial orbit $\mathfrak{o}$ (where $l\in\mathfrak{o}|_r$):

$$\mathfrak{o}|_r^2:=l^2,\qquad\mathfrak{o}|_r\cdot h|_r:=l\cdot h|_r,\qquad h|_r^2\ \text{(independent of }\mathfrak{o}).$$

Since reflections act identically on the discriminant, the discriminant class

$$l\bmod(N\cap B_r)\in\operatorname{discr}(N\cap B_r),\quad l\in\mathfrak{o}|_r,$$

is also constant within each combinatorial orbit $\mathfrak{o}$. If this class is zero, then either $\mathfrak{o}|_r=\{0\}$, or $\mathfrak{o}|_r$ consists of roots, or $\mathfrak{o}|_r^2=4$, and then necessarily $\mathfrak{o}|_r\cdot h|_r=2$ as the projection of $\mathfrak{o}$ to the other block is trivial.

For a subset $\mathfrak{C}_r \subset \mathfrak{o}|_r$, we can define the *relative span*

$$\operatorname{span}_0 \mathfrak{C}_r := \bigl\{ \textstyle\sum_l \alpha_l l \bigm| l \in \mathfrak{C}_r,\ \alpha_l \in \mathbb{Q},\ \sum_l \alpha_l = 0 \bigr\} \subset B_r.$$

Alternatively, $\operatorname{span}_0 \mathfrak{C}_r$ can be defined as the vector space spanned by all differences $l - l_0$, where $l, l_0 \in \mathfrak{C}_r$ and l_0 is fixed. Then, necessary (but typically not sufficient) conditions for the admissibility of $\mathfrak{C}_r$ are (see Lemma 2.5 and Definition 3.2)

(3.15) $\qquad l_1^2 - l_1 \cdot l_2 = 0$ (iff $l_1 = l_2$), 2, 3, 4, or 6 for all $l_1, l_2 \in \mathfrak{C}_r$,

(3.16) $\qquad$ the lattice $N \cap \operatorname{span}_0 \mathfrak{C}_r$ is root free.

In the special case where $h|_r^2 = 4$ (or, equivalently, $h \in B_r$), there is a somewhat stronger restriction still "local" with respect to B_r:

(3.17) $\qquad$ the lattice $N \cap (\operatorname{span}_0 \mathfrak{C}_r + \mathbb{Q}h|_r)$ is root free.

Clearly, (3.17) implies (3.16) which, in turn, implies *part* of (3.15): $l_1^2 - l_1 \cdot l_2 = 1$ if and only if $l_1 - l_2$ is a root. The other part of (3.15), $l_1^2 - l_1 \cdot l_2 \neq 5$, is the local version of the requirement that $h - l_1 - l_2$ should not be a root.

The fact that conditions (3.15) and (3.16) are not sufficient is illustrated by the next lemma, whose proof is not quite local. The lemma is particularly meaningful if $h|_1 = 0$ or, more generally, $\mathfrak{o}|_1 \cdot h|_1 = 0$; otherwise, there are better bounds that are explored below on a case-by-case basis.

Lemma 3.18. *Consider a block decomposition $N \subset B_1 \oplus B_2$, let $N_r := N \cap B_r$, and denote by $\mathfrak{rt}_r \subset N_r$ the sublattice generated by all* integral *roots in N_r, $r = 1, 2$. If, for a combinatorial orbit $\mathfrak{o}$, we have $\mathfrak{o}|_1 \subset \mathfrak{rt}_1$, then $b_1(\mathfrak{o}) \leqslant \operatorname{rk} \mathfrak{rt}_1$; moreover, the adjacency graph of any admissible set $\mathfrak{C}_1 \subset \mathfrak{o}|_1$ is a Dynkin diagram.*

Proof. It follows from the assumption that also $\mathfrak{o}|_2 \subset \mathfrak{rt}_2$. Let $\mathfrak{C}_1 \subset \mathfrak{o}|_1$ be a set as in (3.13), so that, for some fixed $v \in \mathfrak{o}|_2 \subset \mathfrak{rt}_2$, the set $\mathfrak{C} := \bigl\{ u + v \bigm| u \in \mathfrak{C}_1 \bigr\}$ is admissible. By (3.15), we have $u_1 \cdot u_2 \in \{0, -1, -2\}$ for any distinct $u_1, u_2 \in \mathfrak{C}_1$, *i.e.*, the elements of $\mathfrak{C}_1$ constitute the vertices of a certain Dynkin diagram, *a priori* elliptic or affine. However, if $u_1, \ldots, u_m$ constitute an *affine* Dynkin diagram, there is a relation $\sum_{i=1}^m \alpha_i u_i = 0$ with all $\alpha_i > 0$, so that $\alpha := \sum_{i=1}^m \alpha_i \neq 0$. It follows that the root $v = \bigl(\sum_{i=1}^m \alpha_i(u_i + v)\bigr)/\alpha$ is in $\operatorname{span} \mathfrak{C}$ and $\mathfrak{C}$ is not admissible.

Thus, the set $\mathfrak{C}_1$ is linearly independent, and the statement follows. □

The count $c_1(\mathfrak{o})$ in Lemma 3.18 is the total number of roots in $\mathfrak{rt}_1$, which, for indecomposable root lattices, is as follows (*cf.* $2_\circ$ in §3.3):

$$\mathbf{A}_n\colon\ 2\mathcal{C}_{2,n+1}, \quad \mathbf{D}_n\colon\ 4\mathcal{C}_{2,n}, \quad \mathbf{E}_6\colon\ 72, \quad \mathbf{E}_7\colon\ 126, \quad \mathbf{E}_8\colon\ 240,$$

where $\mathcal{C}_{2,n} = n(n-1)/2$ are the binomial coefficients.

3.5. **Combinatorial estimates.** In this section we improve and refine some of the combinatorial bounds introduced in [9, 13]. The refinement is based on the extra condition (3.20), which is derived from (3.16) and is due to the fact that we only consider primitive sublattices of Niemeier lattices.

Let $\Omega := \Omega_n$ be a finite set, $|\Omega| = n$. Given a collection $\mathfrak{S}$ of subsets of Ω, we can consider the $\mathbb{Q}$-vector space

$$\operatorname{span}_0 \mathfrak{S} := \bigl\{ \textstyle\sum_s \alpha_s \mathbf{1}_s \bigm| s \in \mathfrak{S},\ \alpha_s \in \mathbb{Q},\ \sum_s \alpha_s = 0 \bigr\} \subset \mathbf{H}_n \otimes \mathbb{Q}.$$

The collection $\mathfrak{S}$ is called *admissible* if

(3.19) $\qquad |r \mathbin{\triangle} s| \in \Delta := \{0, 4, 6, 8, 12\}$ for each pair $r, s \in \mathfrak{S}$, and

(3.20) the lattice $\mathbf{D}_n \cap \operatorname{span}_0 \mathfrak{S}$ is root free, where $\mathbf{D}_n \subset \mathbf{H}_n$ is as in §3.3.

As explained in §3.4, (3.20) implies *part* of (3.19), *viz.* the fact that $|r \vartriangle s| \neq 2$. Note though that our estimates almost do not use the other part, $|r \vartriangle s| \neq 8$.

Remark 3.21. In view of (3.12), the vectors $\mathbf{1}_s$ in the definition of $\operatorname{span}_0 \mathfrak{S}$ can be replaced with $[s] \in \mathbf{A}_{n-1}^\vee$ or $[s] \in \mathbf{D}_n^\vee$, see §3.3. This interpretation explains the relevance of admissible collections: they represent conditions (3.15), (3.16) that are necessary for the admissibility of the restriction of a combinatorial orbit to a block of type $\mathbf{A}_{n-1} \otimes \mathbb{Q}$ or $\mathbf{D}_n \otimes \mathbb{Q}$.

The interpretation *via* shortest discriminant vectors $[s] \in \mathbf{D}_n^\vee$ implies also that the admissibility property is invariant under

(3.22)
- (a) the natural action of the permutation group $\mathbb{S}_n$ on Ω_n,
- (b) transformations $\mathfrak{S} \mapsto \mathfrak{S} \vartriangle o := \bigl\{ s \vartriangle o \bigm| s \in \mathfrak{S} \bigr\}$ with $o \subset \Omega$ fixed,
- (c) the setwise complement $\mathfrak{S} \mapsto \mathfrak{S}^- := \{ \Omega \smallsetminus s \,|\, s \in \mathfrak{S} \} = \mathfrak{S} \vartriangle \Omega$.

Most statements are obvious. The invariance of (3.19) under (3.22b) follows from the relation $(r \vartriangle o) \vartriangle (s \vartriangle o) = r \vartriangle s$, and for (3.20) we observe that $\mathfrak{S} \mapsto \mathfrak{S} \vartriangle \{i\}$ is induced by the reflection r_i (see §3.3) which preserves $\mathbf{D}_n \subset \mathbf{H}_n$.

For $0 \leqslant m \leqslant n$, define

$$\mathcal{A}_{m,n} := \max |\mathfrak{S}_m|, \qquad \mathcal{D}_n := \max |\mathfrak{S}_*|,$$

where $\mathfrak{S}_m$ runs over all admissible collections of m*-element* subsets of Ω and $\mathfrak{S}_*$ runs over all admissible collections. Then, for $1 \leqslant m \leqslant n$, we have

$$\begin{gathered} \mathcal{A}_{0,n} = \mathcal{A}_{1,n} = 1, \quad \mathcal{A}_{3,6} = 3^*, \quad \mathcal{A}_{3,7} = 5^*, \quad \mathcal{A}_{3,8} = 6^*, \quad \mathcal{A}_{4,8} = 8^*, \\ \text{(a)}\ \mathcal{A}_{m,n} = \mathcal{A}_{n-m,m}, \qquad \text{(b)}\ \mathcal{A}_{m,n} \leqslant \mathcal{A}_{m-1,n-1} + \mathcal{A}_{m,n-1}, \\ \text{(c)}\ \mathcal{A}_{m,n} \leqslant \Bigl\lfloor \frac{n}{m} \mathcal{A}_{m-1,n-1} \Bigr\rfloor, \end{gathered} \tag{3.23}$$

$$\begin{gathered} \mathcal{D}_0 = \ldots = \mathcal{D}_3 = 1, \quad \mathcal{D}_4 = \mathcal{D}_5 = 2, \quad \mathcal{D}_6 = 3^*, \quad \mathcal{D}_7 = 5^*, \quad \mathcal{D}_8 = 8^*, \\ \text{(a)}\ \mathcal{D}_n \leqslant \sum_{m \in \Delta} \mathcal{A}_{m,n}, \qquad \text{(b)}\ \mathcal{D}_n \leqslant 2\mathcal{D}_{n-1}, \end{gathered} \tag{3.24}$$

where marked with a * are the values depending on (3.20). These assertions are proved below.

Remark 3.25. Relations (3.23), (3.24) are used to estimate $\mathcal{A}_*$, $\mathcal{D}_*$ inductively, selecting *at each step* the best bound found. For example, once $\mathcal{A}_{2,6} = \mathcal{A}_{3,6} = 3$ has been established, (3.23c) gives us $\mathcal{A}_{3,7} \leqslant \lfloor (7/3) \cdot 3 \rfloor = 7$. However, using also (3.23a), we get the sharp bound $\mathcal{A}_{3,7} = \mathcal{A}_{4,7} \leqslant \lfloor (7/4) \cdot 3 \rfloor = 5$.

To make the notation uniform, introduce also

$$\mathcal{C}_{m,n} := \binom{n}{m} \quad \text{and} \quad \mathcal{P}_0 := 1, \quad \mathcal{P}_n := 2^{n-1} \quad \text{for } n > 0.$$

(The count $\mathcal{P}_n$ is interpreted as the number of subsets $s \in \Omega$ of the same parity.) We extend $\mathcal{A}_{m,n} = \mathcal{C}_{m,n} = 0$ unless $0 \leqslant m \leqslant n$ and $\mathcal{D}_n = \mathcal{P}_n = 0$ unless $n \geqslant 0$.

Proof of (3.23) *and* (3.24). Relation (3.23a) is the invariance of the admissibility under (3.22c). The proofs of (3.23b) and (3.24b) mimic the standard proofs of the similar combinatorial identities for $\mathcal{C}_*$ and $\mathcal{P}_*$: we pick a point $* \in \Omega$, break an admissible set $\mathfrak{S}$ into the subsets

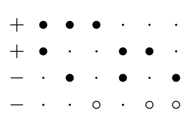
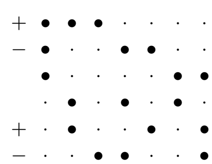
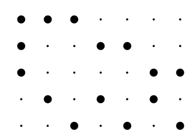
$$\mathfrak{S} - * := \bigl\{ s \smallsetminus * = s \vartriangle \{*\} \bigm| * \in s \in \mathfrak{S} \bigr\} \quad \text{and} \quad \mathfrak{S}|_* := \bigl\{ s \bigm| * \notin s \in \mathfrak{S} \bigr\},$$

and estimate the two separately, without any further analysis.

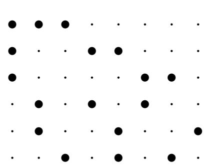
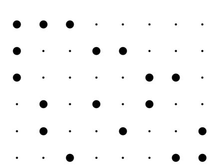
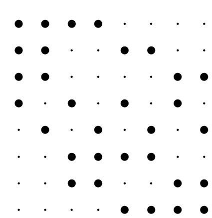
For (3.23c), using $|\mathfrak{S} - *| \leqslant \mathcal{A}_{m-1,n-1}$ again, we conclude that each point $*$ is contained in at most $\mathcal{A}_{m-1,n-1}$ sets $s \in \mathfrak{S}$. Counting the total cardinality of the sets $s \in \mathfrak{S}$, we get $m|\mathfrak{S}| \leqslant n\mathcal{A}_{m-1,n-1}$, as stated. For (3.24a) we observe that, replacing $\mathfrak{S}$ with $\mathfrak{S} \vartriangle o$ for a fixed $o \in \mathfrak{S}$, see (3.22b), we can assume that $\varnothing \in \mathfrak{S}$ and, by (3.19), $|s| \in \Delta$ for each $s \in \mathfrak{S}$, upon which we use $\mathcal{A}_{m,n}$ to estimate the number of sets of each cardinality separately.

It remains to compute the not immediately obvious values for small values of m, n. Below, all uniqueness assertions are modulo the applicable equivalence given by (3.22), *i.e.*, up to (3.22a) and, for $\mathcal{D}_*$-type bounds, also up to (3.22b), (3.22c). We use the term (m,n)-*collection* for a collection of m-element subsets of Ω_n.

It is easily seen that there are but two maximal (with respect to inclusion) $(3,6)$-collections satisfying (3.19), *viz.* a pair of disjoint triplets and (3.26a):

(3.26) (a)  (b)  (c) 

In the latter case, the combination shown in the figure is twice a root; hence, (any) one of the sets (shown as a "ghost" in the figure) must be removed and we arrive at $\mathcal{A}_{3,6} = 3$, realized by a unique collection. Then we mimic the proof of (3.23b), trying to construct a larger collection from a pair of smaller ones. The only pair of extremal $(2,6)$- and $(3,6)$-collections satisfying (3.19) is (3.26b): it has several prohibited combinations like the one shown in the figure, and *all* of them cannot be destroyed by removing just one set. Hence, we need to use the other maximal $(3,6)$-collection, arriving at (3.26c). The passage to $\mathcal{A}_{3,8}$ and $\mathcal{A}_{4,8}$ is similar, but tedious; we leave details to the reader (or, rather, computer). There are but two extremal $(3,8)$-collections, see (3.27a), (3.27b), and a unique extremal $(4,8)$-collection (3.27c).

(3.27) (a)  (b)  (c) 

The proof of the $\mathcal{D}$-type bounds is similar. Up to the new equivalence, (3.26a) and (3.26c) are still the only sets representing $\mathcal{D}_6 = 3$ and $\mathcal{D}_7 = 5$, respectively. The bound $\mathcal{D}_7 = 8$ is represented by several classes, including (3.27c). $\square$

For yet another combinatorial bound we replace the pair $N \ni h$ with

$$N := \mathbf{D}_{n+1} \subset \mathbf{H}_{n+1} \quad \text{and} \quad h := 2e_{n+1}$$

and consider the combinatorial orbit $\mathfrak{o} \subset 4_\bullet$. Since $\mathfrak{o} \cdot h = 2$, we have

$$\mathfrak{o} = \bigl\{\pm e_i \pm e_j \pm e_k + e_{n+1} \bigm| 1 \leqslant i < j < k \leqslant n\bigr\},$$

so that we can merely speak about square 3 vectors $\pm e_i \pm e_j \pm e_k \in H_n$. (If $n = 3$, the set $\mathfrak{o}$ splits into two combinatorial orbits, but we combine them together.)

Denoting by $\mathcal{T}_n := \max|\mathfrak{S}|$ the maximal cardinality of a set $\mathfrak{S} \subset \mathfrak{o}$ that is admissible (as in Definition 3.2) and self-dual, $\mathfrak{S}^* = \mathfrak{S}$, we have, for $n \geqslant 2$,

$$\mathcal{T}_0 = \mathcal{T}_1 = \mathcal{T}_2 = 0, \quad \mathcal{T}_3 = 2, \quad \mathcal{T}_4 = 4, \quad \mathcal{T}_5 = 8, \quad \mathcal{T}_6 = 12, \quad \mathcal{T}_7 = 18,$$

$$\text{(a)}\ \mathcal{T}_n \leqslant 2\Bigl\lfloor \frac{1}{3} n(n-2) \Bigr\rfloor, \qquad \text{(b)}\ \mathcal{T}_n \leqslant 2(n-2) + \mathcal{T}_{n-1}. \tag{3.28}$$

Proof of (3.28). The proof of (3.28a) follows that of (3.23c): we estimate the total cardinality of the elements of $\mathfrak{o}$ regarded as 3-element subsets of $\{\pm\} \times \mathcal{I}_n$. We can take 2 out of $\lfloor\cdot\rfloor$ since $\mathcal{T}_n$ is obviously even. The other bound (3.28b) is proved similar to (3.23b): we break an admissible set into *three* subsets, according to the coefficient of e_n, which can be 0 or ± 1.

In both cases, the rôle of $\mathcal{A}_{m-1,n-1}$ should be played by the bound $b = n - 1$ given by Lemma 3.18. However, if roots constituting a Dynkin diagram D span $\mathbf{D}_n \otimes \mathbb{Q}$, each component of D must be of type $\mathbf{D}_k$ (including $\mathbf{D}_3 := \mathbf{A}_3 \subset \mathbf{H}_3$ or $\mathbf{D}_2 := 2\mathbf{A}_1 \subset \mathbf{H}_2$); then, there is a pair u_1, u_2 of roots such that $u_1 - u_2 = 2e_i$ and $\frac{1}{2}(h + u_1 - u_2) \in \operatorname{span}\mathfrak{C}$ is a root. Thus, we can reduce b down to $n - 2$.

The values $\mathcal{T}_n$ for small values of n are found by brute force. □

3.6. **Blocks of type $\mathbf{A}_n$.** In this and next sections, we use the material of §3.5 to estimate the partial bounds $b_k(\mathfrak{o})$ for a block D_k of type $\mathbf{A}_n$ or $\mathbf{D}_n$. For the three exceptional blocks of type $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$, the bounds are computed by brute force on the fly, and we do not state the results here. In fact, we do the same for $\mathbf{A}_n$, $n \leqslant 7$, and $\mathbf{D}_n$, $n \leqslant 8$, obtaining slightly better bounds in a few cases.

Consider a block $D_k \otimes \mathbb{Q}$ in decomposition (3.1) with D_k of type $\mathbf{A}_n$. We assume that $h|_k = \sum \eta_i e_i$, $i \in \mathcal{I}$ and, for each $\eta \in \{\eta_i\}$, let

$$\mathcal{I}(\eta) := \bigl\{i \bigm| \eta_i = \eta\bigr\} \subset \mathcal{I}, \quad \mathbf{H}(\eta) := \mathbb{Z}\mathcal{I}(\eta), \quad B(\eta) := \mathbf{H}(\eta) \otimes \mathbb{Q}.$$

The decomposition $D_k \otimes \mathbb{Q} \subset \bigoplus B(\eta)$ is used to find the count $c_k(\mathfrak{o})$ and estimate the bound $b_k(\mathfrak{o})$, both *via* (3.14), of a combinatorial orbit $\mathfrak{o}$. We reserve the notation $l \in \mathfrak{o}|_k$ for a "typical" element and denote by $l(\eta)$ its projection to $B(\eta)$.

We use repeatedly the observation that, if either $h|_k$ or l is of square 4, then necessarily $h|_k \cdot l = 2$, imposing certain restrictions to the other vector.

For references, we format the rest of this section as a sequence of numbered claims. Proofs are omitted as all necessary explanations are given in the text. In a few cases, the "block-by-block" bound can be improved by ruling out certain linear combinations; this improvement is shown *via* (old bound) $\mapsto$ (new bound)*.

Claim 3.29. Assume that $\mathfrak{o}|_k \subset \underline{m} \neq \underline{0}$, so that $l = [o]$, $|o| = m$.

(1) If $h|_k = [u]$, $u \subset \mathcal{I}$, then, by (3.9a), the two two constants $i := |u \cap o|$ and $h|_k \cdot l$ determine each other and the restriction $\mathfrak{o}|_k$. Applying (3.14) to the partition $\{B(\eta)\}$, in view of (3.15) and (3.9b) we get

$$c_k(\mathfrak{o}) = \mathcal{C}_{i,|u|}\mathcal{C}_{m-i,|\bar{u}|}, \quad b_k(\mathfrak{o}) \leqslant \min\bigl\{\mathcal{A}_{i,|u|}\mathcal{C}_{m-i,|\bar{u}|}, \mathcal{C}_{i,|u|}\mathcal{A}_{m-i,|\bar{u}|}\bigr\}.$$

(2) If $h = h|_k = \mathbf{1}_u - \mathbf{1}_v \in 4_\bullet$, then $u \subset o$ and $v \subset \bar{o}$, *cf.* (3.10), and we have

$$c_k(\mathfrak{o}) = \mathcal{C}_{m-2,n-3}, \quad b_k(\mathfrak{o}) \leqslant \mathcal{A}_{m-2,n-3}.$$

(3) If $h|_k = \mathbf{1}_u - \mathbf{1}_v \in 2_\circ$ is a root, then

$$c_k(\mathfrak{o}) = \mathcal{C}_{m-2,n-1}, \quad b_k(\mathfrak{o}) \leqslant \mathcal{A}_{m-2,n-1}, \quad \text{if } h|_k \cdot l = 0 \text{ and } u, v \subset o,$$
$$c_k(\mathfrak{o}) = \mathcal{C}_{m,n-1}, \quad b_k(\mathfrak{o}) \leqslant \mathcal{A}_{m,n-1}, \quad \text{if } h|_k \cdot l = 0 \text{ and } u, v \subset \bar{o},$$
$$c_k(\mathfrak{o}) = \mathcal{C}_{m-1,n-1}, \quad b_k(\mathfrak{o}) \leqslant \mathcal{A}_{m-1,n-1}, \quad \text{if } h|_k \cdot l = \pm 1.$$

Claim 3.30. Assume that $\mathfrak{o}|_k \subset 4_\bullet$, so that $l := \mathbf{1}_r - \mathbf{1}_s$, $|r| = |s| = 2$.

(1) If $h|_k = [u]$, $u \subset \mathcal{I}$, then $r \subset u$, $s \subset \bar{u}$, see (3.10), and, by (3.15) and (3.8b),

$$c_k(\mathfrak{o}) = \mathcal{C}_{2,|u|}\mathcal{C}_{2,|\bar{u}|}, \quad b_k(\mathfrak{o}) \leqslant \min\bigl\{\mathcal{A}_{2,|u|}\mathcal{C}_{2,|\bar{u}|}, \mathcal{C}_{2,|u|}\mathcal{A}_{2,|\bar{u}|}\bigr\}.$$

(2) If $h = h|_k = \mathbf{1}_u - \mathbf{1}_v \in 4_\bullet$, then

$$c_k(\mathfrak{o}) = \mathcal{C}_{2,n-3}, \quad b_k(\mathfrak{o}) \leqslant \mathcal{A}_{2,n-3}, \quad \text{if } r = u \text{ or } s = v,$$
$$c_k(\mathfrak{o}) = 8\mathcal{C}_{2,n-3}, \quad b_k(\mathfrak{o}) \leqslant 4(n-4), \quad \text{if } |s \cap u| = |r \cap v| = 1;$$

for the latter, $l(0)$ is a root in $\mathbf{A}_{n-4} \subset \mathbf{H}(0)$ and we use Lemma 3.18.

(3) If $h|_k = \mathbf{1}_u - \mathbf{1}_v \in 2_\circ$, then $u \subset r$, $v \subset s$ and, similar to case (2),

$$c_k(\mathfrak{o}) = 2\mathcal{C}_{2,n-1}, \quad b_k(\mathfrak{o}) \leqslant n-2.$$

Claim 3.31. Assume that $\mathfrak{o}|_k \subset 2_\circ$, so that $l := \mathbf{1}_r - \mathbf{1}_s$, $|r| = |s| = 1$. Assume also that $h|_k \cdot l \neq 0$, as otherwise Lemma 3.18 applies to (the components of) $h|_k^\perp$.

(1) If $h|_k = [u]$, $u \subset \mathcal{I}$, then $r \subset u$, $s \subset \bar{u}$ (or *vice versa*), see (3.10); hence, by (3.15), (3.8), and (3.23),

$$c_k(\mathfrak{o}) = |u||\bar{u}|, \quad b_k(\mathfrak{o}) \leqslant \min\bigl\{|u|, |\bar{u}|\bigr\}.$$

(2) If $h = h|_k = \mathbf{1}_u - \mathbf{1}_v \in 4_\bullet$, then $r \subset u$, $s \subset v$ and

$$c_k(\mathfrak{o}) = 4, \quad b_k(\mathfrak{o}) \leqslant 2 \mapsto 1^*.$$

Indeed, any pair of distinct vectors in an admissible set $\mathfrak{C}_k \subset \mathfrak{o}|_k$ is of the form l, $h - l$, and then the root $l \in N$ is in $\operatorname{span}_0 \mathfrak{C}_k \oplus \mathbb{Q}h$, see (3.17).

(3) If $h|_k = \mathbf{1}_u - \mathbf{1}_v \in 2_\circ$, then, by (3.23),

$$c_k(\mathfrak{o}) = n-1, \quad b_k(\mathfrak{o}) = 1, \quad \text{if } h|_k \cdot l = \pm 1 \text{ (two orbits each)},$$
$$c_k(\mathfrak{o}) = 1, \quad b_k(\mathfrak{o}) = 1, \quad \text{if } h|_k \cdot l = \pm 2.$$

3.7. **Blocks of type $\mathbf{D}_n$.** Consider a block $D_k \otimes \mathbb{Q}$ in decomposition (3.1) with D_k of type $\mathbf{D}_n$ and proceed as in §3.6, using the notation therein. Since the group $O(\mathbf{D}_n)$ contains the reflection against any of e_i, in the expression $h|_k = \sum \eta_i e_i$ we can and always do assume that all $\eta_i \geqslant 0$. Then, the description of combinatorial orbits is still quite "combinatorial" in its nature.

Claim 3.32. Assume that $\mathfrak{o}|_k \subset \underline{2}$. Then, from (3.15) and (3.23), we have

$$c_k(\mathfrak{o}) = 2\bigl|\mathcal{I}(0)\bigr|, \quad b_k(\mathfrak{o}) \leqslant 2, \quad \text{if } h|_k \cdot l = 0,$$
$$c_k(\mathfrak{o}) = \bigl|\mathcal{I}(\eta)\bigr|, \quad b_k(\mathfrak{o}) = 1, \quad \text{if } \eta := \bigl|h|_k \cdot l\bigr| > 0.$$

Claim 3.33. Assume that $\mathfrak{o}|_k \subset \underline{1}$ or $\underline{3}$, so that $l = [o]$, $o \subset \mathcal{I}$. Then $h|_k \notin 4^2_\bullet$.

(1) If $h|_k = 0$ or $h|_k \in \underline{2}$ or $h|_k \in 4_\bullet$, *i.e.*, $h|_k = \mathbf{1}_u$, $|u| \in \{0, 1, 4\}$, then

$$c_k(\mathfrak{o}) = \mathcal{P}_{n-|u|}, \quad b_k(\mathfrak{o}) \leqslant \mathcal{D}_{n-|u|}.$$

(2) If $h|_k = [\mathcal{I}] \in \underline{1}$ or $\underline{3}$, then $|o|$ is determined by $h|_k \cdot l = \text{const}$ and

$$c_k(\mathfrak{o}) = \mathcal{C}_{|o|,n}, \quad b_k(\mathfrak{o}) \leqslant \mathcal{A}_{|o|,n}.$$

(3) If $h|_k = \mathbf{1}_u \in 2_\circ$, then

$$c_k(\mathfrak{o}) = 2\mathcal{P}_{n-2}, \quad b_k(\mathfrak{o}) \leqslant 2\mathcal{D}_{n-2}, \quad \text{if } h|_k \cdot l = 0,$$
$$c_k(\mathfrak{o}) = \mathcal{P}_{n-2}, \quad b_k(\mathfrak{o}) \leqslant \mathcal{D}_{n-2}, \quad \text{if } h|_k \cdot l = \pm 1.$$

Claim 3.34. Assume that $\mathfrak{o}|_k \subset 4^2_\bullet$. Then $h|_k = \mathbf{1}_u$, $|u| \in \{1, 2, 4\}$, and

$$c_k(\mathfrak{o}) = |u|, \quad b_k(\mathfrak{o}) \leqslant |u| \mapsto 1^*.$$

Indeed, if $l_1 \neq l_2$, then $\frac{1}{2}(l_1 + l_2)$ would be a root.

Claim 3.35. Assume that $\mathfrak{o}|_k \subset 4_\bullet$, so that $l = \sum_{i \in o} \pm e_i$, $|o| = 4$. Then $h|_k \notin \underline{0}, \underline{2}$.

(1) If $h|_k = [\mathcal{I}] \in \underline{1}$ or $\underline{3}$, then, by (3.15) and (3.11),

$$c_k(\mathfrak{o}) = \mathcal{C}_{4,n}, \quad b_k(\mathfrak{o}) \leqslant \mathcal{A}_{4,n}.$$

(2) If $h|_k = 2e_i \in 4^2_\bullet$, then, by the definition of $\mathcal{T}_*$,

$$c_k(\mathfrak{o}) = 8\mathcal{C}_{3,n-1}, \quad b_k(\mathfrak{o}) \leqslant \mathcal{T}_{n-1}.$$

(3) If $h|_k = \mathbf{1}_u \in 4_\bullet$, $|u| = 4$, then there are two orbits $\mathfrak{o}|_k$:

$$c_k(\mathfrak{o}) = 4, \quad b_k(\mathfrak{o}) \leqslant 4 \mapsto 1^*, \quad \text{if } o = u,$$
$$c_k(\mathfrak{o}) = 4\mathcal{C}_{2,4}\mathcal{C}_{2,n-4}, \quad b_k(\mathfrak{o}) \leqslant \mathcal{C}_{2,4}(n-4), \quad \text{if } |o \cap u| = 2.$$

In the former case, the sign pattern of each vector $l \in \mathfrak{o}|_k = \mathfrak{o}$ is $+++-$ (up to permutation); hence, $\frac{1}{2}(l_1 + l_2)$ would be a root whenever $l_1 \neq l_2$. In the latter case, Lemma 3.18 is applied to $\mathbf{H}(0)$.

(4) If $h|_k = \mathbf{1}_u \in 2_\circ$, $|u| = 2$, then $u \subset o$ and, by Lemma 3.18,

$$c_k(\mathfrak{o}) = 4\mathcal{C}_{2,n-2}, \quad b_k(\mathfrak{o}) \leqslant n - 2.$$

Claim 3.36. Assume that $\mathfrak{o}|_k \subset 2_\circ$, so that $l = \sum_{i \in o} \pm e_i$, $|o| = 2$. Assume also that $h|_k \cdot l \neq 0$, as otherwise Lemma 3.18 applies to (the components of) $h|_k^\perp$.

(1) If $h|_k = e_i \in \underline{2}$, then $i \in o$ and, by (3.15) and (3.11),

$$c_k(\mathfrak{o}) = 2(n-1), \quad b_k(\mathfrak{o}) \leqslant 2.$$

(2) If $h|_k = [\mathcal{I}] \in \underline{1}$ or $\underline{3}$, then $h|_k \cdot l = \pm 2 = \text{const}$ and, by (3.15),

$$c_k(\mathfrak{o}) = \mathcal{C}_{2,n}, \quad b_k(\mathfrak{o}) \leqslant \mathcal{A}_{2,n}.$$

(3) if $h|_k = 2e_i \in 4^2_\bullet$, then $i \in o$ and, similar to Claim 3.31(2),

$$c_k(\mathfrak{o}) = 2(n-1), \quad b_k(\mathfrak{o}) \leqslant 2 \mapsto 1^*.$$

(4) If $h|_k = \mathbf{1}_u \in 4_\bullet$, $|u| = 4$, then $o \subset u$ and, similar to Claim 3.31(2),

$$c_k(\mathfrak{o}) = \mathcal{C}_{2,4}, \quad b_k(\mathfrak{o}) \leqslant \mathcal{A}_{2,4} = 2 \mapsto 1^*.$$

(5) If $h|_k = \mathbf{1}_u \in 2_\circ$, $|u| = 2$, then there are four orbits:

$$c_k(\mathfrak{o}) = 4(n-2), \quad b_k(\mathfrak{o}) \leqslant 4 \mapsto 3^*, \quad \text{if } h|_k \cdot l = \pm 1,$$
$$c_k(\mathfrak{o}) = 1, \quad b_k(\mathfrak{o}) = 1, \quad \text{if } h|_k \cdot l = \pm 2.$$

In the formar case, letting $o = \{1, 2\}$, a 4-element admissible set must be of the form $l_{1,2} = e_1 \pm e_3$, $l_{3,4} = e_2 \pm e_4$, and $\frac{1}{2}(l_1 + l_2 - l_3 - l_4)$ is a root.

3.8. **Special D-type blocks.** Consider a block decomposition $N \subset B_1 \oplus B_2$ with $B_1 = D_k \otimes \mathbb{Q}$, see (3.1), and D_k of type $\mathbf{D}_n$. This **D**-type block B_1 is called *special* (with respect to a combinatorial orbit $\mathfrak{o}$) if either

- $\mathfrak{o}|_1 \subset \underline{2}$ and $\mathfrak{o}|_1 \cdot h|_1 = 0$, see Claim 3.32, or
- $\mathfrak{o}|_1 \subset 2_\circ$, $h|_1 \in \underline{2}$, and $\mathfrak{o}|_1 \cdot h|_1 \neq 0$, see Claim 3.36(1).

In both cases, $b_1(\mathfrak{o}) \leqslant 2$, so that (3.14) would typically result in the rough estimate $b(\mathfrak{o}) \leqslant 2c_2(\mathfrak{o})$. Often, this can be improved by means of the following lemma.

Lemma 3.37. *If the block B_1 in a block decomposition $N \subset B_1 \oplus B_2$ is special with respect to a combinatorial orbit $\mathfrak{o}$, then $b(\mathfrak{o}) \leqslant c_2(\mathfrak{o}) + b_2(\mathfrak{o})$.*

Proof. Let $\mathfrak{C} \subset \mathfrak{o}$ be an admissible set. For each $l_2 \in \mathfrak{o}|_2$, there are at most two vectors $l_1 + l_2 \in \mathfrak{C}$ and, if there are two, they are of the form $\pm e_i + \text{const}$, so that their difference is $2e_i$ for some generator $e_i \in B_1$. If two distinct vectors $2e_i \neq 2e_j$ could be obtained in this way, then their half-sum would be a root in $\operatorname{span} \mathfrak{C}$. Hence, at most one vector $2e_i$ can be obtained, and this can be done starting from at most $b_2(\mathfrak{o})$ vectors $l_2 \in \mathfrak{o}|_2$, so that we arrive at

$$b(\mathfrak{o}) \leqslant 2 \cdot b_2(\mathfrak{o}) + 1 \cdot \big(c_2(\mathfrak{o}) - b_2(\mathfrak{o})\big) = c_2(\mathfrak{o}) + b_2(\mathfrak{o}). \qquad \square$$

Lemma 3.37 can be iterated: if in a decomposition $N \subset B_0 \oplus B_1 \oplus \ldots \oplus B_n$ all blocks but B_0 are special, then

$$b(\mathfrak{o}) \leqslant b_0(\mathfrak{o}) + \sum_{k=1}^{n-1} \prod_{i=0}^{k} c_i(\mathfrak{o}). \tag{3.38}$$

The best estimate is obtained if the special blocks B_k, $k = 1, \ldots, n$, are ordered by the increasing of $c_k(\mathfrak{o})$.

4. Niemeier lattices with many roots

The goal of this section is the following statement, ruling out 18 out of the 24 Niemeier lattices, *viz.* those with many roots.

Theorem 4.1 (see §4.2)**.** *Let $N := N(D)$ be a Niemeier lattice generated over $\mathbb{Q}$ by a root lattice D* other than

$$6\mathbf{D}_4,\ 6\mathbf{A}_4,\ 8\mathbf{A}_3,\ 12\mathbf{A}_2,\ \text{or}\ 24\mathbf{A}_1.$$

Then, $|\mathfrak{C}| < 720$ for any 4-polarization $h \in N$ and any admissible set $\mathfrak{C} \subset \mathfrak{O}_h$.

4.1. **Homogeneous chains of admissible sets.** Consider a 4-polarized Niemeier lattice $N \ni h$. For a subset $\mathcal{S} \subset \mathfrak{O}_h$ and subgroup $G \subset O_h(N)$, denote

$$\operatorname{stab}(\mathcal{S}; G) := \text{the setwise stabilizer of } \mathcal{S} \text{ in } G.$$

If $G = O_h(N)$, the group is omitted from the notation, abbreviating it to $\operatorname{stab}\mathcal{S}$.

Fix a pair $\mathcal{Q} \subset \mathcal{S} \subset \mathfrak{O}_h$ of subsets, non necessarily admissible or saturated, let O be the orbit of $\mathcal{Q}$ under $G := \operatorname{stab}\mathcal{S}$, and assume that the pair is *homogeneous* in the sense that the *multiplicity*

$$m := m(l) := \#\big\{\mathcal{Q}' \in O \bigm| l \in \mathcal{Q}'\big\} = \text{const}, \quad l \in \mathcal{S}, \tag{4.2}$$

is constant throughout $\mathcal{S}$. (*E.g.*, this is obviously the case if either G is transitive on $\mathcal{S}$ or all elements $\mathcal{Q}' \in O$ are pairwise disjoint and $\bigcup \mathcal{Q}' = \mathcal{S}$.) Denote

$$s := |\mathcal{S}|, \quad q := |\mathcal{Q}|, \quad o := |O|, \quad \text{so that} \quad qo = sm.$$

The following obvious lemma on group actions is crucial for our computation.

Lemma 4.3. *In the notation above, for any subset $\mathfrak{C} \subset \mathcal{S}$ of cardinality $n := |\mathfrak{C}|$, there is an element $\mathcal{Q}' \subset O$ such that $|\mathfrak{C} \cap \mathcal{Q}'| \geqslant \lceil n \cdot q/s \rceil$.* ◁

Recall that our ultimate goal is finding all large admissible/geometric subsets $\mathfrak{C} \subset \mathfrak{O}_h$, and this task reduces to the computation of the sets of the form

$$\mathcal{B}_n(\mathcal{S}) := \big\{\mathcal{S}\text{-saturated admissible sets } \mathfrak{C} \subset \mathcal{S} \;\big|\; |\mathfrak{C}| \geqslant n\big\}/\operatorname{stab}\mathcal{S}. \tag{4.4}$$

The latter is computed by brute force, adding conics one-by-one to increase the rank by one unit at a time. (Certainly, we make use of the symmetry groups as much as possible, but these technicalities are discussed in [12, §1.4].) Experimentally, the number of (orbits of) admissible sets $\mathfrak{C} \subset \mathcal{S}$ is relatively small if $\mathfrak{C}$ is either small or close to the maximum, whereas it may grow quite large in the middle of the range. Lemma 4.3 lets us break this computation into several steps and avoid the above "middle": when computing $\mathcal{B}_n(\mathcal{S})$, we can start with $\mathcal{B}_{\lceil np/s \rceil}(\mathcal{Q})$.

Warning 4.5. It may happen that the group $\operatorname{stab}(\mathcal{Q}; G)$ is smaller than $\operatorname{stab}\mathcal{Q}$. In this case, we have two options:

(1) break $(\operatorname{stab}\mathcal{Q})$-orbits $[\mathfrak{C}] \in \mathcal{B}_*(\mathcal{Q})$ into $\operatorname{stab}(\mathcal{Q}; G)$-orbits before continuing the computation, or
(2) extend $\mathfrak{C}$ not to $\mathcal{S}$ only, but to all elements of the $(\operatorname{stab}\mathfrak{C})$-orbit of $\mathcal{S}$.

We use both approaches, *cf.* `select` in [12, §4.1].

Often, a longer *homogeneous chain* $\mathcal{Q}_1 \subset \mathcal{Q}_2 \subset \ldots \subset \mathcal{Q}_N := \mathcal{S}$ needs to be used to speed up the computation. Typically, $\mathcal{Q}_k$ is the union of several elements of the $(\operatorname{stab}\mathcal{Q}_{k+1})$-orbit of $\mathcal{Q}_{k-1}$. Details are explained in [12] case by case.

4.2. **Proof of Theorem 4.1.** We consider Niemeier lattices one-by-one, using their description in [4, Table 16.1]. For each lattice N, we list the $O(N)$-orbits of square 4 vectors $h \in N$ and, for a representative h of each orbit, we list orbits and combinatorial orbits (see §3.1) of conics. Upon this, in most cases it remains to compute the partial counts and bounds using §3.6 and §3.7 and apply (3.14) to arrive at $b(\mathfrak{O}_h) < 720$. We refer to [12, §2, §3] for a full account of this computation and to §4.3, §4.4 below for several simple examples worked out in detail.

Occasionally, the bounds need to be reduced by Lemma 3.37, *cf.* §4.5 below.

In eleven cases, *viz.*

- one polarization for $N(2\mathbf{A}_7 \oplus 2\mathbf{D}_5)$, see [12, §3.1.1],
- two polarizations for $N(4\mathbf{A}_6)$, see [12, §3.2.1, §3.2.2], and
- all eight polarizations for $N(4\mathbf{A}_5 \oplus \mathbf{D}_4)$, see [12, §3.3],

the exact bounds $b(\mathfrak{o})$ need to be computed by brute force. This is done step-by-step as explained in §4.1; the precise choices making the computation reasonably fast are found in the relevant references above. □

Remark 4.6. Probably, *a posteriori* the computation in the last paragraph of the proof can also be "explained" in the spirit of Lemma 3.37. We leave these attempts to the reader since, in the subsequent sections, hard computer aided computation is still unavoidable.

4.3. **Example: the lattice $N(\mathbf{A}_{24})$.** The lattice is the index 5 extension of $\mathbf{A}_{24}$ by the discriminant classes that are multiples of $\underline{5} \in \mathbb{Z}/25 = \operatorname{discr}\mathbf{A}_{24}$, see §3.6. It is immediate that, up to $O(N)$, there are but two square 4 vectors $h \in N$.

4.3.1. *Case* 1: $h = \mathbf{1}_u - \mathbf{1}_v \in 4_\bullet$, $|u| = |v| = 2$. There are five combinatorial orbits, constituting three orbits.

(1) A pair of dual combinatorial orbits $\mathfrak{o} \subset \underline{20}$, $\mathfrak{o}^* \subset \underline{5}$: by Claim 3.29(2) and (3.23), we have $b(\mathfrak{o}) = b(\mathfrak{o}^*) \leqslant \mathcal{A}_{18,21} \leqslant 70$.
(2) A pair of dual combinatorial orbits $\mathfrak{o} \subset 4_\bullet$, with a typical vector l as in the first equation of Claim 3.30(2): we have $b(\mathfrak{o}) \leqslant \mathcal{A}_{2,21} \leqslant 10$ for each.
(3) A self-dual combinatorial orbit $\mathfrak{o} \subset 4_\bullet$, with a typical vector l as in the second equation of Claim 3.30(2): we have $b(\mathfrak{o}) \leqslant 4 \cdot 20 = 80$.

Summarizing, $b(\mathfrak{O}_h) \leqslant 2 \cdot 70 + 2 \cdot 10 + 80 = 240$.

4.3.2. *Case* 2: $h = [u] \in \underline{20}$, $|u| = 20$. There are but two dual orbits, consisting of a single combinatorial orbit each.

(1) $\mathfrak{o} \subset \underline{20}$: by Claim 3.29(1) with $i = 18$ and (3.23), we have
$$b_k(\mathfrak{o}) \leqslant \min\{\mathcal{A}_{18,20}\mathcal{C}_{2,5}, \mathcal{C}_{18,20}\mathcal{A}_{2,5}\} \leqslant \min\{100, 380\} = 100.$$
(2) $\mathfrak{o} \subset 4_\bullet$: by Claim 3.30(1), we get the same bound $b(\mathfrak{o}) \leqslant 100$.

Summarizing, $b(\mathfrak{O}_h) \leqslant 100 + 100 = 200$.

4.4. **Example: the lattice $N(2\mathbf{D}_{12})$.** The lattice is the index 4 extension of $2\mathbf{D}_{12}$ by the discriminant classes $\underline{1} \oplus \underline{2}$, $\underline{2} \oplus \underline{1}$, and $\underline{3} \oplus \underline{3}$, see §3.7. Up to $O(N)$, there are four square 4 vectors $h \in N$.

4.4.1. *Case* 1: $h|_1 = 2e_i \in 4^2_\bullet$ *and* $h|_2 = 0$. There are three combinatorial orbits, all self-dual, constituting three separate orbits.

(1) $\mathfrak{o}|_1 \subset 4_\bullet$ and $\mathfrak{o}|_2 = \{0\}$: by Claim 3.35(2) and (3.28), $b(\mathfrak{o}) \leqslant \mathcal{T}_{11} \leqslant 66$.
(2) $\mathfrak{o}|_1 \subset 2_\circ$ and $\mathfrak{o}|_2 \subset 2_\circ$: we have
$$c_1(\mathfrak{o}) = 22, \quad b_1(\mathfrak{o}) = 1 \quad \text{by Claim 3.36(3) and}$$
$$c_2(\mathfrak{o}) = 264, \quad b_2(\mathfrak{o}) \leqslant 12 \quad \text{by Lemma 3.18,}$$
resulting in $b(\mathfrak{o}) \leqslant 264$.
(3) $\mathfrak{o}|_1 \subset \underline{2}$ and $\mathfrak{o}|_2 \subset \underline{1}$: since $c_1(\mathfrak{o}) = b_1(\mathfrak{o}) = 1$ by Claim 3.32, from (3.24) and Claim 3.33(1) we have $b(\mathfrak{o}) = b_2(\mathfrak{o}) \leqslant \mathcal{D}_{12} \leqslant 128$.

Summarizing, $b(\mathfrak{O}_h) \leqslant 66 + 264 + 128 = 458$.

4.4.2. *Case* 2: $h|_1 = \mathbf{1}_u \in 2_\circ$ *and* $h|_2 = \mathbf{1}_v \in 2_\circ$, $|u| = |v| = 2$. There are six orbits.

(1) $\mathfrak{o}|_1 \subset \underline{1}$ and $\mathfrak{o}|_2 \subset \underline{2}$ (or *vice versa*, two combinatorial orbits): as $c_2(\mathfrak{o}) = 2$ by Claim 3.32, $b(\mathfrak{o}) \leqslant 2b_1(\mathfrak{o}) \leqslant 2\mathcal{D}_{10} \leqslant 64$ by Claim 3.33(3).
(2) $\mathfrak{o}|_k \subset 2_\circ$ and $\mathfrak{o}|_k \cdot h|_k = 1$ for $k = 1, 2$: we have $c_k(\mathfrak{o}) = 40$ and $b_k(\mathfrak{o}) \leqslant 3$ by Claim 3.36(5); hence, $b(\mathfrak{o}) \leqslant 120$ by (3.14).
(3) $\mathfrak{o}|_1 \in 4_\bullet$ and $\mathfrak{o}|_2 = \{0\}$ (or *vice versa*, two combinatorial orbits): in view of Claim 3.35(4), we have $b(\mathfrak{o}) = b_1(\mathfrak{o}) \leqslant 10$.
(4) The dual of (3), with $\mathfrak{o}|_1 \in 2_\circ$ and $\mathfrak{o}|_2 = \{h|_2\}$ (or *vice versa*).
(5) $\mathfrak{o}|_1 \in 4^2_\bullet$ and $\mathfrak{o}|_2 = \{0\}$ (or *vice versa*, two combinatorial orbits): in view of Claim 3.34, we have $b(\mathfrak{o}) = b_1(\mathfrak{o}) \leqslant 1$.
(6) The dual of (5), with $\mathfrak{o}|_1 \in 2_\circ$ and $\mathfrak{o}|_2 = \{h|_2\}$ (or *vice versa*).

Summarizing, $b(\mathfrak{O}_h) \leqslant 2 \cdot 64 + 120 + 2 \cdot 2 \cdot 10 + 2 \cdot 2 \cdot 1 = 292$.

4.4.3. *Case* 3: $h|_1 = \mathbf{1}_u \in 4_\bullet$, $|u| = 4$, *and* $h|_2 = 0$. Each of the five orbits consists of a single combinatorial orbit.

(1) $\mathfrak{o}|_1 \subset 4_\bullet$ and $\mathfrak{o}|_2 = \{0\}$: we have $b(\mathfrak{o}) \leqslant \mathcal{C}_{2,4} \cdot 8 = 48$ by Claim 3.35(3).

(2) $\mathfrak{o}|_1 \subset \underline{1}$ and $\mathfrak{o}|_2 \subset \underline{2}$: we have

$$c_1(\mathfrak{o}) = 128, \quad b_1(\mathfrak{o}) \leqslant \mathcal{D}_8 \leqslant 8 \quad \text{by Claim 3.33(1) and}$$
$$c_2(\mathfrak{o}) = 24, \quad b_2(\mathfrak{o}) \leqslant 2 \quad \text{by Claim 3.32,}$$

resulting in $b(\mathfrak{o}) \leqslant \min\{128 \cdot 2, 24 \cdot 8\} = 256$. Since B_2 is a special $\mathbf{D}$-type block, Lemma 3.37 reduces this down to $b(\mathfrak{o}) \leqslant 128 + 8 = 136$.

(3) $\mathfrak{o}|_1 \subset 2_\circ$ and $\mathfrak{o}|_2 \subset 2_\circ$: we have

$$c_1(\mathfrak{o}) = 6, \quad b_1(\mathfrak{o}) \leqslant 1 \quad \text{by Claim 3.36(4) and}$$
$$c_2(\mathfrak{o}) = 264, \quad b_2(\mathfrak{o}) \leqslant 12 \quad \text{by Lemma 3.18,}$$

resulting in $b(\mathfrak{o}) \leqslant \min\{6 \cdot 12, 264 \cdot 1\} = 72$.

(4) $\mathfrak{o}|_1 \subset 4^2_\bullet$ and $\mathfrak{o}|_2 = \{0\}$: we have $b(\mathfrak{o}) = 1$ by Claim 3.34.

(5) the dual of (4), with $\mathfrak{o}|_1 \subset 4_\bullet$ as in the first equation of Claim 3.35(3).

Summarizing, $b(\mathfrak{O}_h) \leqslant 48 + 256 + 72 + 2 \cdot 1 = 314$, reducible down to 258.

4.4.4. *Case* 4: $h|_1 = e_i \in \underline{2}$ *and* $h|_2 = \mathbf{1}_{\mathcal{I}} \in \underline{1}$. There are six orbits, each consisting of a single combinatorial orbit; they split into three pairs of dual ones.

(1) $\mathfrak{o}|_1 = \{2h|_1\} \subset 4^2_\bullet$ and $\mathfrak{o}|_2 = \{0\}$: we have $b(\mathfrak{o}) = 1$ by Claim 3.34.

(2) the dual of (1), with $\mathfrak{o}|_1 = \{-h|_1\} \subset \underline{2}$ and $\mathfrak{o}|_2 = \{h|_2\} \subset \underline{1}$.

(3) $\mathfrak{o}|_1 \subset 2_\circ$ and $\mathfrak{o}|_2 \subset 2_\circ$: we have

$$c_1(\mathfrak{o}) = 22, \quad b_1(\mathfrak{o}) \leqslant 2 \quad \text{by Claim 3.36(1) and}$$
$$c_2(\mathfrak{o}) = 66, \quad b_2(\mathfrak{o}) \leqslant \mathcal{A}_{2,12} \leqslant 6 \quad \text{by Claim 3.36(2),}$$

resulting in $b(\mathfrak{o}) \leqslant \min\{22 \cdot 6, 66 \cdot 2\} = 132$. Since B_1 is a special $\mathbf{D}$-type block, Lemma 3.37 reduces this down to $b(\mathfrak{o}) \leqslant 66 + 6 = 72$.

(4) the dual of (3), with $\mathfrak{o}|_1 \subset \underline{2}$ and $\mathfrak{o}|_2 \subset \underline{1}$.

(5) $\mathfrak{o}|_1 \subset \underline{2}$ and $\mathfrak{o}|_2 \subset \underline{1}$: since $c_1(\mathfrak{o}) = b_1(\mathfrak{o}) = 1$ by Claim 3.32, using (3.23) and Claim 3.33(2) we arrive at $b(\mathfrak{o}) = b_2(\mathfrak{o}) \leqslant \mathcal{A}_{4,12} \leqslant 48$.

(6) the dual of (5), with $\mathfrak{o}|_1 = \{0\}$ and $\mathfrak{o}|_2 \subset 4_\bullet$.

Summarizing, $b(\mathfrak{O}_h) \leqslant 2 \cdot 1 + 2 \cdot 132 + 2 \cdot 48 = 362$, reducible down to 242.

4.5. **Example: the lattice** $N(4\mathbf{D}_6)$**.** The lattice is the index 16 extension of $4\mathbf{D}_6$ by the linear combinations of all even permutations of $\underline{0} \oplus \underline{1} \oplus \underline{2} \oplus \underline{3}$. Up to $O(N)$, there are five square 4 vectors $h \in N$. We consider one, $h|_1 \in 4^2_\bullet$ and $h|_k = 0$ for $k \geqslant 2$, *i.e.*, the only class where, unlike §4.4, Lemma 3.37 makes a difference.

All combinatorial orbits are self-dual and split into four orbits. Orbits (2) and (3) below consist of three combinatorial orbits each, obtained from the one shown by all cyclic permutations of the indices $2, 3, 4$.

(1) $\mathfrak{o}|_1 \subset 4_\bullet$, $\mathfrak{o}|_k = \{0\}$ for $k = 2, 3, 4$: by Claim 3.35(2), $b(\mathfrak{o}) = \mathcal{T}_5 = 8$.

(2) $\mathfrak{o}|_1 \subset 2_\circ$, $\mathfrak{o}|_2 \subset 2_\circ$, $\mathfrak{o}|_3 = \mathfrak{o}|_4 = \{0\}$: we have

$$c_1(\mathfrak{o}) = 10, \quad b_1(\mathfrak{o}) = 1 \quad \text{by Claim 3.36(3) and}$$
$$c_2(\mathfrak{o}) = 60, \quad b_2(\mathfrak{o}) \leqslant 6 \quad \text{by Lemma 3.18,}$$

resulting in $b(\mathfrak{o}) \leqslant \min\{10 \cdot 6, 60 \cdot 1\} = 60$.

(3) $\mathfrak{o}|_1 \subset \underline{2}$, $\mathfrak{o}|_1 \subset \underline{3}$, $\mathfrak{o}_3 = \{0\}$, $\mathfrak{o}|_4 \subset \underline{1}$: we have, for $k = 2, 4$,

$$c_1(\mathfrak{o}) = 1, \quad b_1(\mathfrak{o}) = 1 \quad \text{by Claim 3.32 and}$$
$$c_k(\mathfrak{o}) = 32, \quad b_k(\mathfrak{o}) \leqslant \mathcal{D}_6 = 3 \quad \text{by Claim 3.33(1),}$$

resulting in $b(\mathfrak{o}) \leqslant 1 \cdot 3 \cdot 32 = 96$.

(4) $\mathfrak{o}|_k \subset \underline{2}$ for $k = 1, \ldots, 4$. by Claim 3.32, we have $c_1(\mathfrak{o}) = b_1(\mathfrak{o}) = 1$ and, for $k = 2, 3, 4$,

$$c_k(\mathfrak{o}) = 12, \quad b_k(\mathfrak{o}) \leqslant 2,$$

resulting in $b(\mathfrak{o}) \leqslant 1 \cdot 2 \cdot 12 \cdot 12 = 288$. However, since the last three blocks are special, (3.38) reduces this down to $1 + 1 + 12 + 12^2 = 158$.

Summarizing, we arrive at $b(\mathfrak{O}_h) \leqslant 8 + 3 \cdot 60 + 3 \cdot 96 + 288 = 764$, which is not quite satisfactory. However, Lemma 3.37 and (3.38) reduce this bound down to 634.

5. Lattices with few roots: brute force

In this section, we consider the remaining Niemeier lattices rationally generated by roots. This time, even the exact bounds $b(\mathfrak{o})$ for all orbits result in $b(\mathfrak{O}_h) \geqslant 720$, and we need to deal with geometric rather than just admissible sets.

Theorem 5.1 (see §5.1)**.** *Let $N := N(D)$ be a Niemeier lattice with D one of*

$$6\mathbf{D}_4,\ 6\mathbf{A}_4,\ 8\mathbf{A}_3,\ 12\mathbf{A}_2,\ \text{or}\ 24\mathbf{A}_1.$$

Then, with five exceptions, viz. (5.4) *and* (5.8)–(5.11) *below, one has $|\mathfrak{C}| < 720$ for any 4-polarization $h \in N$ and any geometric subset $\mathfrak{C} \subset \mathfrak{O}_h$.*

5.1. **Proof of Theorem 5.1.** We start with computing the sharp bounds $b(\mathfrak{o})$ for all combinatorial orbits $\mathfrak{o}$. If the additive bound $b(\mathfrak{O}_h) := \sum_{\mathfrak{o}} b(\mathfrak{o})$ exceeds 718, we proceed by improving the similar bounds for (some) orbits $\bar{\mathfrak{o}}$ or unions $\mathcal{S}$ thereof.

Roughly, we try to prove that $\mathcal{B}_m(\mathcal{S}) = \varnothing$, see (4.4), implying a reduced bound $\bar{b}(\mathcal{S}) \leqslant m - 2$. Usually, the equality $\mathcal{B}_m(\mathcal{S}) = \varnothing$ above holds only modulo a number of sets of large rank, which are singled out and analyzed in the course of the proof (see mostly [12, §1.2 and §1.6]); it is these sets that give rise to the exceptions in Theorem 5.1 and, eventually, the quartics in Theorem 1.1.

The computation is done by the *puzzle assembly* [12, §1.5]: instead of adding individual conics one by one, we try to put together as many and as large "pieces" as possible to fit a large number of conics into a limited rank. The rôle of pieces of the puzzle is played by the elements of the pre-computed set $\mathcal{B}_0(\mathfrak{o})$, $\mathfrak{o} \subset \bar{\mathfrak{o}}$.

To ensure the convergence, we use a number of technical tricks:

- *clusters* $\mathfrak{o} \subset \mathfrak{c} \subset \bar{\mathfrak{o}}$ similar to homogeneous chains in §4.1, see [12, §1.7];
- computation *up to rank* 18, or even 17 or 16, see [12, §1.2]: if $\mathcal{B}_m(\mathcal{S})$ consists of but a few sets of large rank, these sets are analyzed by brute force;
- *rank pushing*, see [12, §1.6], *i.e.*, switching back to brute force for subsets $\mathfrak{C} \subset \mathcal{S}$ of rank close to $\operatorname{rk} \mathcal{S}$.

Further details are found in [12], where we also present the topmost layer of the code used for each pair $h \in N$. In the rest of this section, we only show the five exceptional sets and a few otherwise interesting examples: they appear in $N(8\mathbf{A}_3)$, see §5.2, and $N(24\mathbf{A}_1)$, see §5.3. □

Convention 5.2. In this and next section, a saturated subset $\mathfrak{C} \subset \mathfrak{O}_h$ is described by means of its saturation $S := \operatorname{sat} \mathfrak{C}$ which, in turn, is determined by the polarized sublattice

$$h \in \operatorname{ort}_h \mathfrak{C} := \left(h_S^\perp\right)_N^\perp \subset N.$$

The reason for considering $\operatorname{ort}_h \mathfrak{C}$ instead of $S_N^\perp$ is purely aesthetical: typically, it is generated by shorter vectors. This is particularly useful in §6 below, where, dealing with the Leech lattice Λ, it suffices to consider square 4 vectors only.

The original set $\mathfrak{C}$ is recovered as the set of vectors $l \in N$ such that

$$l^2 = 4, \quad 2l \cdot v = h \cdot v \quad \text{for each } v \in \operatorname{ort}_h \mathfrak{C}. \tag{5.3}$$

5.2. **The lattice $N(8\mathbf{A}_3)$.** The lattice N is the index 256 extension of $8\mathbf{A}_1$ by the discriminant classes

$$\underline{3} \oplus (\underline{2} \oplus \underline{0} \oplus \underline{0} \oplus \underline{1} \oplus \underline{0} \oplus \underline{1} \oplus \underline{1}),$$

where the parenthesized expression runs over all seven cyclic permutations of its arguments (see [4]). In particular, the kernel contains the class $\bigoplus_8 \underline{2}$.

There are four $O(N)$-orbits of square 4 vectors $h \in N$; only one of them, *viz.* the one represented by $h = h|_1 \in 4_\bullet$ in D_1, results in large geometric subsets.

For the only exceptional set $\mathfrak{C}$ of cardinality 728, the lattice $\operatorname{ort}_h \mathfrak{C}$ is generated by any three roots generating D_1 and two vectors $u = \bigoplus_k u|_k$, $v = \bigoplus_k v|_k$, where

$$u|_k, v|_k \in \underline{2}, \quad u|_1 = v|_1, \quad u|_1 \cdot h|_1 = 0, \quad u|_k \cdot v|_k = 0 \quad \text{for } k = 2, \ldots, 8. \tag{5.4}$$

In the notation of §3.3, assuming that $h|_1 = \mathbf{1}_\mathcal{I}$ and letting $r := \{1, 2\}$, $s := \{1, 3\}$, we can take $u|_1 = v|_1 = v|_k = [r]$ and $u|_k = [s]$ for $k = 2, \ldots, 8$.

5.3. **The lattice $N(24\mathbf{A}_1)$.** The lattice N is the index 4096 extension of $24\mathbf{A}_1$ generated by the discriminant classes $\bigoplus_{k \in o} \underline{1}_k$, where $o \subset \Omega$ is a codeword in the extended binary Golay code $\mathcal{G}$, see, *e.g.*, [4]. There are two $O(N)$-orbits of square 4 vectors h: either $h = \mathbf{1}_s$, $s \subset \Omega$, $|s| = 2$, or $h = \frac{1}{2}\mathbf{1}_o$, where $o \in \mathcal{G}$ is an octad. In the latter case, there are no exceptional or otherwise interesting geometric sets; we refer to [12, §6.2] for the complete list of the examples found.

Thus, denote $s := \{1, 2\}$ and assume that $h = \mathbf{1}_s$. Then, the conics are either $l := e_i \pm e_j$, $i \in s$, $j \in \Omega \smallsetminus s$, or

$$l := \frac{1}{2} \sum_{k \in o} \epsilon_k e_k, \quad \text{where } o \supset s \text{ is an octad, } \epsilon_1 = \epsilon_2 = 1, \ \epsilon_k = \pm 1 \text{ for } k > 2. \tag{5.5}$$

Among the examples found, nine are of special interest; see [12, §6.1] for the complete list in terms of the original codewords as in `sage`.

Convention 5.6. We represent the generators $v_n := \sum_k v_{n,k} e_k$ (the first always being h) of the lattice $\operatorname{ort}_h \mathfrak{C}$, see Convention 5.2, by pictographs, using the following notation for the coefficients $v_{n,k}$:

$$(\bullet) \mapsto 1, \quad (\circ) \mapsto -1, \quad (-) \mapsto \tfrac{1}{2}, \quad (=) \mapsto -\tfrac{1}{2}, \quad (+) \mapsto \tfrac{3}{2}.$$

Needless to say that, for each n, the set $\{k \in \Omega \,|\, v_{n,k} \neq 0 \bmod \mathbb{Z}\}$ is a codeword. For better transparency, we permute the index set Ω so that the subsets $\{1, \ldots, 8\}$ and $\{7 \ldots, 14\}$ are among the octads. Only the first 16 positions are shown; it is understood that the last character extends to the rest of the index set.

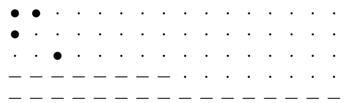
Given the first two generators, the configuration is easily recovered as the set of conics (5.5) satisfying condition (5.3) *for the last three generators v of $\operatorname{ort}_h \mathfrak{C}$*. For each configuration $\mathfrak{C}$ we show also the transcendental lattice(s) $T := NS(X)^\perp$ of the quartic(s) X where this configuration is realized, in the inline notation

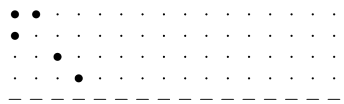
$$[a, b, c] \quad \text{stands for} \quad \mathbb{Z}u + \mathbb{Z}v, \quad u^2 = a, \ u \cdot v = b, \ v^2 = c, \tag{5.7}$$

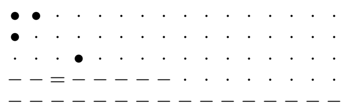
and the numbers $r + 2c$ of the connected components of the equiconical stratum (itemized according to T), in the form (r, c), where

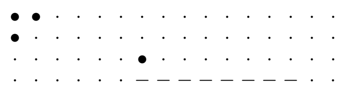
- r is the number of real components and
- c is the number of pairs of complex conjugate ones.

These data are computed using Nikulin [27] and, for (5.16), Miranda–Morrison [22], similar to §7.1 below.

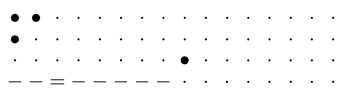
(5.8) $|\mathfrak{C}| = 800$:  $[4, 0, 40]$ $(1, 0)$

(5.9) $|\mathfrak{C}| = 800$:  $[4, 0, 40]$ $(1, 0)$

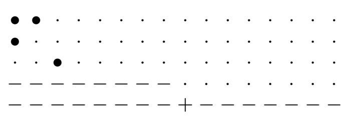
(5.10) $|\mathfrak{C}| = 736$:  $[12, 0, 16]$ $(1, 0)$

(5.11) $|\mathfrak{C}| = 728$:  $[14, 0, 14]$ $(1, 0)$

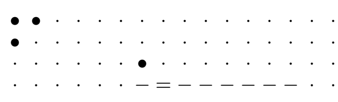
The first four sets are among the exceptions mentioned in Theorem 5.1.

(5.12) $|\mathfrak{C}| = 680$:  $[4, 2, 56]$ $(1, 0)$ $[16, 6, 16]$ $(1, 0)$

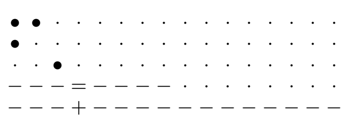
This graph is also realized with reducible conics, namely, by X''_{60} in [14]: it has 60 lines and $170 + 510 = 680$ conics, see Remark 7.4 below.

(5.13) $|\mathfrak{C}| = 660$:  $[4, 0, 60]$ $(1, 0)$

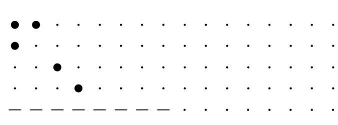
This is the same abstract $K3$-surface as the Barth–Bauer quartic with 664 conics, see [10]; the latter is also embeddable to $N(24\mathbf{A}_2)$, see [12, §6.1].

(5.14) $|\mathfrak{C}| = 656$:  $[2, 0, 122]$ $(1, 0)$ $[10, 4, 26]$ $(0, 1)$

This is the largest known configuration of real conics on a real quartic, see §7.2.

(5.15) $|\mathfrak{C}| = 640$:  $[4, 0, 72]$ $(1, 0)$

This is a Barth–Bauer $4^2\mathfrak{A}_4$-quartics, *cf.* Addendum 1.4, (5.16), and §7.3.

(5.16) $|\mathfrak{C}| = 608$:  $\mathrm{rk} = 19$ $(1, 0)$

This is the largest known configuration of Picard rank 19, see §7.3.

6. The Leech lattice

The only Niemeier lattice left to be considered is the root free Leech lattice Λ. Recall that there is a single $O(\Lambda)$-orbit of square 4 vectors $h \in \Lambda$, see, *e.g.*., [4].

Theorem 6.1 (see §6.2)**.** *Let $\Lambda \ni h$ be a 4-polarized Leech lattice. Then, with four exceptions* (6.3)–(6.6) *below, one has $|\mathfrak{C}| < 720$ for any geometric set $\mathfrak{C} \subset \mathfrak{O}_h$.*

6.1. **Iterated index 2 subgroups.** There is a single orbit $\bar{\mathfrak{o}} = \mathfrak{O}_h$, whereas, since Λ is root free, we have $R(\Lambda) = 1$ and all combinatorial orbits are singletons. These facts make the application of the puzzle assembly algorithm problematic. Besides, since there are no roots, almost any subset $\mathfrak{C} \subset \mathfrak{O}_h$ is admissible, see Definition 3.2, whereas it is the inadmissibility that rules out most intermediate sets in the other Niemeier lattices.

Therefore, we have to shift the paradigm and, instead of starting from $\varnothing$ and working upwards, we start from $\mathfrak{O}_h$ and go downwards, searching for large *geometric* subsets $\mathfrak{C} \subset \mathfrak{O}_h$. In fact, most of the time the principal criterion is $\operatorname{rk} \mathfrak{C} \leqslant 20$.

The following lemma has essentially appeared in [13]. We present its formal proof since it is to be followed literally by the new version of the algorithm.

Lemma 6.2. *Let $N \ni h$ be a 4-polarized Niemeier lattice. Then, for any saturated subset $\mathfrak{C} \subset \mathfrak{O}_h$, there is a chain*

$$\mathfrak{C} = \mathfrak{C}_n \subset \mathfrak{C}_{n-1} \subset \ldots \subset \mathfrak{C}_1 \subset \mathfrak{C}_0 = \mathfrak{O}_h$$

such that, for each $k = 1, \ldots, n$, one has $\mathfrak{C}_k = \mathfrak{O}_h \cap S_k$ for a certain polarized index 2 sublattice $S_k \subset \operatorname{span}_{\mathbb{Z}} \mathfrak{C}_{k-1}$, $S_k \ni h$.

Proof. Clearly, since $\operatorname{span}_{\mathbb{Z}} \mathfrak{O}_h$ is generated by conics, there is a sequence

$$\mathfrak{C} = \mathfrak{S}_m \subset \mathfrak{S}_{m-1} \subset \ldots \subset \mathfrak{S}_1 \subset \mathfrak{S}_0 = \mathfrak{O}_h$$

of *saturated* subsets such that $\operatorname{rk} \mathfrak{S}_k = \operatorname{rk} \mathfrak{S}_{k-1} - 1$ for each $k = 1, \ldots, m$. Fix a pair $\mathfrak{S}_{k+1} \subset \mathfrak{S}_k$ of consecutive sets, let $\mathfrak{S}_{k,0} := \mathfrak{S}_k$, and denote by

$$p_0 \colon \operatorname{span}_{\mathbb{Z}} \mathfrak{S}_{k,0} \to Q_0 := \operatorname{span}_{\mathbb{Z}} \mathfrak{S}_{k,0} / \operatorname{span}_{\mathbb{Z}} \mathfrak{S}_{k+1}$$

the quotient projection. The group $Q_0 \cong \mathbb{Z}$ is generated by (the images of) finitely many conics $l \in \mathfrak{S}_{k,0} \smallsetminus \mathfrak{S}_{k+1}$, and it is immediate that the cardinality of the set

$$\mathfrak{S}_{k,1} := \mathfrak{S}_{k,0} \cap p_0^{-1}(2Q_0) \supset \mathfrak{S}_{k+1}$$

is strictly less than $|\mathfrak{S}_{k,0}|$. Iterating this procedure, we arrive at a chain

$$\mathfrak{S}_k = \mathfrak{S}_{k,0} \supset \mathfrak{S}_{k,1} \supset \ldots \supset \mathfrak{S}_{k,r} = \mathfrak{S}_{k+1}$$

satisfying the "index 2 sublattice" condition; it is bound to terminate as all inclusions are proper and $|\mathfrak{S}_k \smallsetminus \mathfrak{S}_{k+1}| < \infty$. Replacing each pair $\mathfrak{S}_{k+1} \subset \mathfrak{S}_k$ of consecutive sets with such a chain, we obtain a sequence as in the statement. □

Note that we do not assert that whenever the index $[\operatorname{span}_{\mathbb{Z}} \mathfrak{C}_p : \operatorname{span}_{\mathbb{Z}} \mathfrak{C}_q]$ is finite it is a power of 2. Though, in our computation it was always the case: before the rank drops, we observed indices up to 32.

Lemma 6.2 is used to construct large geometric subsets of $\mathfrak{O}_h$. We start with $\mathfrak{O}_h$ (or, in some cases, with another large admissible saturated set) and construct chains

$$\mathfrak{O}_h =: \mathfrak{C}_0 \supset \mathfrak{C}_1 \supset \ldots$$

recursively: once a set $\mathfrak{C}_k$ has been constructed, we

(1) consider the projection $(\mathrm{mod}\ 2)\colon \operatorname{span}_{\mathbb{Z}} \mathfrak{C}_k \to V_k := (\operatorname{span}_{\mathbb{Z}} \mathfrak{C}_k) \otimes \mathbb{F}_2$;
(2) compute the $(\operatorname{stab} \mathfrak{C}_k)$-orbits on the annihilator $h^\perp \subset V_k^\vee$;
(3) for a representative v of each orbit, let $\mathfrak{C}_{k+1} := \mathfrak{C}_k \cap (\mathrm{mod}\ 2)^{-1}(v^\perp)$;
(4) discard the subsets $\mathfrak{C}_{k+1}$ of cardinality less than a preset threshold.

Unfortunately, this procedure, used in [8, 13], diverges at a rate unacceptable for the current problem. To make it more tame and reduce the overcounting, we follow the proof of Lemma 6.2 more literally. More precisely, at each step $\mathfrak{S}_k$, we

- keep track of the "original" saturated set $\bar{\mathfrak{C}}_k = \operatorname{sat} \mathfrak{C}_k$;
- discard $\mathfrak{C}_{k+1}$ if $\operatorname{rk} \mathfrak{C}_{k+1} = \operatorname{rk} \mathfrak{C}_k$ and $\operatorname{span}_{\mathbb{Z}} \bar{\mathfrak{C}}_k / \operatorname{span}_{\mathbb{Z}} \mathfrak{C}_{k+1}$ is not cyclic;
- discard $\mathfrak{C}_{k+1}$ if $\operatorname{rk} \mathfrak{C}_{k+1} < \operatorname{rk} \mathfrak{C}_k$ and $\mathfrak{C}_{k+1}$ is not saturated.

Besides, we perform the computation rank by rank: once all saturated sets of some rank r have been collected, before proceeding we leave a single representative of each $O_h(N)$-orbit only. In fact, since we are interested in the existence of large subsets only (not in a particular embedding to $\mathfrak{O}_h$) and the computation depends on $\operatorname{span}_{\mathbb{Z}} \bar{\mathfrak{C}}_k$, we go one step further and leave a single representative of each graph isomorphism class. Other technical details are discussed in [12, §7].

6.2. **Proof ot Theorem 6.1.** The lattice $\Lambda \subset \mathbf{H}_{24}\bigl(\frac{1}{8}\bigr)$ is generated by the square 4 vectors of the form

- $4 \cdot \mathbf{1}_u$, where $u \subset \mathcal{I}$ and $|u| = 2$,
- $2 \cdot \mathbf{1}_o$, where $o \subset \mathcal{I}$ is an octad of the Golay code, or
- $\mathbf{1}_{\mathcal{I}} - 4e_i$, where $i \in \mathcal{I}$,

as well as all vectors obtained from these by the simultaneous reversal of the sign at the elements of a codeword of the extended binary Golay code. This description does not reflect the full automorphism group $O(\Lambda)$: the latter is transitive on the square 4 vectors $h \in \Lambda$ (see, *e.g.*, [4]).

Thus, we fix a 4-polarization $h \in \Lambda$ and start with the h-even index 2 sublattice $\bar{\Lambda} := \operatorname{span}_{\mathbb{Z}} \mathfrak{O}_h \subset \Lambda$; it has larger orthogonal group $\bigl[O_h(\bar{\Lambda}) : O_h(\Lambda)\bigr] = 2$. Besides, since Λ is root free, the set $\mathfrak{O}_h$ itself and any subset thereof is admissible in $\bar{\Lambda}$. It remains to invoke Lemma 6.2 and list all (graph isomorphism classes of) subsets $\mathfrak{C} \subset \mathfrak{O}_h$ of rank $\operatorname{rk} \mathfrak{C} \leqslant 20$, see (3.4), and size $|\mathfrak{C}| \geqslant 720$. With the new improvements (see §6.1 and [12, §7.1–§7.5]) this takes less than two days (*vs.* two months in the version of [13]) and we arrive at nine sets $\mathfrak{C}$: one has $\operatorname{rk} \mathfrak{C} = 20$ and

$$|\mathfrak{C}| = 896, \underline{800}, 768, 760, 740, \underline{736}, 736, \underline{728}, \underline{720}.$$

Strictly speaking, found are graph isomorphism classes rather than $O_h(\Lambda)$-orbits; hence, we continue the analysis in terms of the lattice $\operatorname{span}_{\mathbb{Z}} \mathfrak{C}$, which is a graph invariant. For each set $\mathfrak{C}$, we use Nikulin's theory [27] to compute the genus of the lattice $\operatorname{ort}_h \mathfrak{C}$ (see Convention 5.2) and, referring to Gordon L. Nipp's tables [28, 25], we find that there is but a single *root free* representative $\operatorname{ort}_h \mathfrak{C}$. Furthermore, all classes in the genus of any proper finite index extension do have roots and, hence, cannot be embedded to Λ.

We conclude that the sublattice $\operatorname{span}_{\mathbb{Z}} \mathfrak{C} \subset \Lambda$ must be primitive and, hence, it can be used in Definition 3.3. Only the four underlined sets are geometric; the corresponding lattices $\operatorname{ort}_h \mathfrak{C}$ are shown below, both as the Gram matrix and as a quintuple of square 4 generators. (The remaining five sets are found in [12, (7.10)].) For the generators, we adopt Convention 5.6, except that we use the notation

$$(\bullet) \mapsto 4, \quad (\circ) \mapsto -4, \quad (+) \mapsto 2, \quad (-) \mapsto -2$$

for the coefficients, assume that $\{1,\ldots,8\}$ and $\{5,\ldots,12\}$ are octads, and cut the display at position 14, extending each vector by zeroes.

$$|\mathfrak{C}|=800\colon\quad\begin{bmatrix}4&0&0&0&2\\0&4&0&1&2\\0&0&4&1&2\\0&1&1&4&0\\2&2&2&0&4\end{bmatrix}\qquad\begin{array}{ccccccccccccc}\cdot&\cdot&\cdot&\cdot&\bullet&\cdot&\cdot&\circ&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\bullet&\cdot&\cdot&\bullet&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&+&+&-&+&-&+&+&+&\cdot\\+&+&+&+&+&+&+&+&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&+&+&-&-&+&+&+&+&\cdot\end{array}\tag{6.3}$$

$$|\mathfrak{C}|=736\colon\quad\begin{bmatrix}4&2&0&0&0\\2&4&0&0&2\\0&0&4&1&2\\0&0&1&4&2\\0&2&2&2&4\end{bmatrix}\qquad\begin{array}{ccccccccccccc}\circ&\cdot&\cdot&\cdot&\bullet&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\bullet&\bullet&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\bullet&\cdot&\bullet&\cdot&\cdot&\cdot&\cdot\\+&+&+&+&+&+&+&+&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&+&+&+&+&+&+&+&+&\cdot\end{array}\tag{6.4}$$

$$|\mathfrak{C}|=728\colon\quad\begin{bmatrix}4&2&2&2&2\\2&4&0&1&0\\2&0&4&0&1\\2&1&0&4&2\\2&0&1&2&4\end{bmatrix}\qquad\begin{array}{ccccccccccccc}\cdot&\cdot&\cdot&\cdot&\bullet&\bullet&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\+&+&+&+&+&+&+&+&\cdot&\cdot&\cdot&\cdot&\cdot\\+&-&-&-&+&+&-&+&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&+&+&+&-&-&+&+&+&\cdot\\\cdot&\cdot&\cdot&\cdot&+&+&-&-&+&+&+&+&\cdot\end{array}\tag{6.5}$$

$$|\mathfrak{C}|=720\colon\quad\begin{bmatrix}4&0&0&2&2\\0&4&0&0&2\\0&0&4&0&2\\2&0&0&4&1\\2&2&2&1&4\end{bmatrix}\qquad\begin{array}{ccccccccccccc}\bullet&\bullet&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\bullet&\bullet&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\bullet&\bullet&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\bullet&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\bullet&\cdot&\cdot&\cdot&\cdot\\+&+&+&+&+&+&+&+&\cdot&\cdot&\cdot&\cdot&\cdot\end{array}\tag{6.6}$$

Since only square 4 vectors are involved, it is not difficult to show, by brute force, that each of the four lattices $\operatorname{ort}_h\mathfrak{C}$ above admits a unique, up to $O_h(\Lambda)$, polarized isometry $(\operatorname{ort}_h\mathfrak{C}\ni h)\hookrightarrow(\Lambda\ni h)$, completing the proof of the uniqueness. □

7. Proofs of the main results

In this section we collect our previous findings and fill in the missing parts of the proofs of the principal results stated in the introduction.

7.1. **Proof of Theorem 1.1.** According to Proposition 3.5, the graph of conics of a smooth quartic is isomorphic to a saturated admissible set $\mathfrak{C}$ in a 4-polarized Niemeier lattice and, by Theorems 4.1, 5.1, and 6.1, up to the relevant orthogonal groups, there are but nine such sets of cardinality $|\mathfrak{C}|\geqslant 720$, *viz.*

$$(5.8)\cong(5.9)\cong(6.3),\quad(5.10)\cong(6.4),\quad(5.4)\cong(5.11)\cong(6.5),\quad(6.6);$$

here, the abstract graph isomorphisms are established using the `digraph` package in `GAP` [18]. For each of the nine sets $\mathfrak{C}$ we have $S:=\operatorname{span}\mathfrak{C}=\operatorname{span}_{\mathbb{Z}}\mathfrak{C}$ and, hence, the rest of the computation depends on the abstract graph isomorphism class only, leaving but four cases.

For each of the first three graphs, the modified lattice $(S\ni h)^\sharp$, see §2.2, admits a unique (up to the group $O^+(\mathbf{L})$ of auto-isometries of $\mathbf{L}$ preserving the orientation of maximal positive definite subspaces) primitive isometry $S^\sharp\hookrightarrow\mathbf{L}$, resulting in a single projective equivalence class of quartics (see, *e.g.*, Dolgachev [17]). On the other hand, $(S\ni h)^\sharp$ has no h-odd index 2 extensions; hence, the configuration is realized by line-free quartics only. For both assertions, the computation employing

Nikulin's discriminant forms [27] is considered quite straightforward nowadays and therefore is left to the reader. We have it implemented in `GAP`.

For (6.6), the situation is the opposite: there is no primitive isometry $S^\sharp \hookrightarrow \mathbf{L}$ but, up to $O_h(S^\sharp)$, there is a unique h-odd index 2 extension $S' \supset S^\sharp$, and the latter does admit a unique primitive isometry $S' \hookrightarrow \mathbf{L}$, resulting in a single quartic X. Upon computing lines and conics on X by means of (2.2), we identify it as Schur's quartic (as the only smooth quartic with 64 lines, see [14]). Alternatively, Schur's is the only quartic with the transcendental lattice $[8, 4, 8]$, see (5.7) and [6]. □

Remark 7.1. For each quartic X encountered in this paper, including the four in Theorem 1.1, the lattice $NS(X)$ is generated over $\mathbb{Z}$ by lines and conics; hence, the group $O_h(NS(X)) = \operatorname{Aut}(\operatorname{Fn}_* X)$ can easily be computed using the `digraph` package in `GAP` [18]. Then one can also compute the groups $\operatorname{Sym}_h X \subset \operatorname{Aut}_h X$ of (symplectic) projective automorphisms of X. In particular, $\operatorname{Sym}_h X = M_{20}$ for quartic (1) in Theorem 1.1 (first observed by X. Roulleau, private communication) and, due to the uniqueness [3, 10] of a quartic with a symplectic action of M_{20}, one can reuse the equation found in [23].

7.2. **Proof of Addendum 1.3.** As explained in [14] (in the context of lines, but the argument extends to rational curves of any degree), when estimating the maximal number of real conics on a real quartic it suffices to assume that *all* conics on a quartic X are real and that they generate $NS(X)$ over $\mathbb{Q}$. Furthermore, if $NS(X)$ is rationally generated by conics, then X admits a real structure with all conics real if and only if the transcendental lattice $T := NS(X)^\perp$ contains $\mathbb{Z}a$, $a^2 = 2$, or $\mathbf{U}(2)$ as a sublattice. Thus, Addendum 1.3 is an immediate consequence of an analysis of the examples found in the course of the proof; the quartic is (5.14). □

7.3. **Proof of Addendum 1.4.** The configuration of conics is (5.16) in $N(24\mathbf{A}_1)$, which is of rank 19. The transcendental lattice $T := NS(X)^\perp$ of a generic member $X \in \mathcal{X}$ is

$$T \cong [-4] \oplus [8] \oplus [8] \cong [4] \oplus [-8] \oplus [-8],$$

where $[s]$ stands for the rank 1 lattice $\mathbb{Z}a$, $a^2 = s$. The group $\operatorname{Sym}_h X \cong 4^2\mathfrak{A}_4$ is computed in terms of the Fano graph $\mathfrak{C} = \operatorname{Fn} X$, see Remark 7.1. The geometric oversets $\mathfrak{C}' \supset \mathfrak{C}$ are found by applying Lemma 6.2 (with all the improvements to the algorithm discussed in [12, §7.1–§7.5]) to all other examples found in the course of the proof for this particular polarization of $N(24\mathbf{A}_1)$; they are (5.8), (5.9), (5.10), and (5.15). Then, general theory of $K3$-surfaces implies that any geometric overset can be realized by a degeneration of quartics. □

Remark 7.2. Applying Lemma 6.2 to the other known examples, we could not find a rank 19 geometric set larger than (5.16) in Addendum 1.4.

Remark 7.3. We do not assert that the list of (graph isomorphism classes of) oversets in the proof of Addendum 1.4 is complete. Probably, the complete list of degenerations can be found by analyzing corank 1 abstract graph extensions of $\mathfrak{C}$ in (5.16), but we leave this exercise to the reader.

7.4. **Concluding remarks.** A great deal of open questions related to conics on quartic surfaces are left beyond the scope of this paper. Thus, we do not discuss fields of definition of positive characteristic or quartics with **A**–**D**–**E** singularities, which are still $K3$-surfaces. (We did try to apply an analogue of Lemma 6.2 to the set $\mathfrak{O}_h \subset N(24\mathbf{A}_1)$, but we could not find a configuration with more than a couple of hundreds of conics. Note that, in the presence of singularities, the total number of *all* conics, including those containing exceptional divisors as components, may be much larger than 800.)

It is not immediately clear whether the examples given by Addenda 1.3 and 1.4 are, indeed, maximal. Besides, even though Theorem 1.1 does give us the bound $N_4^*(2) = 720$, it is not clear how the maximal number of *irreducible* conics may be affected by the presence of lines or exceptional divisors. In all examples in the next remark, in the presence of *many* lines, most conics are reducible.

Remark 7.4. According to [10], the maximal number of *reducible* conics on a smooth quartic is 576, attained at Schur's classical quartic (4) in Theorem 1.1. As an example, we computed the configurations of conics on all smooth quartics with more than 48 lines (see [14, 16]). Only six of them have more than 600 conics, *viz.* precisely those with at least 56 lines:

X_{64}:	64 lines,	$144 + 576 = 720$ conics,	no planes, see (6.6),
X'_{60}:	60 lines,	$140 + 500 = 640$ conics,	no planes, see [12, §5.2],
X''_{60}:	60 lines,	$170 + 510 = 680$ conics,	10 planes, see (5.12),
X_{56}:	56 lines,	$184 + 440 = 624$ conics,	16 planes, see [12, §6.2],
Y_{56}:	56 lines,	$188 + 448 = 636$ conics,	20 planes,
Q_{56}:	56 lines,	$208 + 448 = 656$ conics,	24 planes, see [12, §6.2].

Here, by a *plane* we mean a plane section split into two irreducible conics: formally, these conics are undetectable in terms of lines only.

Remarkably, most configurations appear on our list of examples, substantiating the suggestion that it must be close to complete, even though some gaps do not look quite convincing.

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