CONICS ON SMOOTH QUARTICS: THE COMPUTATION

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This text is a companion to Alex Degtyarev, *Conics on smooth quartic surfaces*: we outline the technical details of the computation and present more complete results thereof. References to the main text are prefixed with $**$. We mostly keep the notation and terminology of $**$.

The principal goal of these notes is outlining a few tricks and recording the parameters used so that the computation can be reproduced.

1. Algorithms

1.1. **The notation.** Essentially, we discuss various ways of computing the sets $\mathcal{B}_n(\mathcal{S})$ introduced in $**$(4.4). However, for our practical purposes, we modify and refine this purely theoretical notion.

We stick to $**$Definition 3.2 and $**$Definition 3.3 of, respectively, *admissible* and *geometric* sets. An admissible set set $\mathfrak{C}$ is called *exceptional* if $|\mathfrak{C}| \geqslant 720$. Exceptional, as well as some other (see §1.2 below) sets $\mathfrak{C}$ are detected in the course of computation, their oversets $\mathfrak{C}' \supset \mathfrak{C}$ are analyzed by other means, upon which $\mathfrak{C}$ is excluded from the further consideration. Thus, the assertion

$$\mathcal{B}^*_m(\mathcal{S}) = \varnothing$$

(where the superscript is explained in §1.2 below) means that $|\mathfrak{C}| < m$ for any admissible/geometric subset $\mathfrak{C} \subset \mathcal{S}$ *with the exception of a number of sets that have already been analyzed.* (Here and below, the choice between admissible and geometric depends on the problem at hand: we always insist that $\operatorname{rk}\mathfrak{C} \leqslant 20$, see $**$(3.4), but we often ignore the other conditions in $**$Definition 3.3.)

Needless to say that it is this "further analysis" that gives rise to the exceptional sets mentioned in $**$Theorem 5.1 and $**$Theorem 6.1 and, eventually, to the four quartics in $**$Theorem 1.1.

Therefore, in addition to the additive bounds $b(\mathcal{S})$ in $**$§3.2, we use the *reduced bounds* $\bar{b}(\mathcal{S})$:

$$\bar{b}(\mathcal{S}) < m \quad \text{if and only if } \mathcal{B}^*_m(\mathcal{S}) = \varnothing.$$

Often, the direct computation of $\mathcal{B}^*_m(\mathcal{S})$ is not feasible and we break it into several steps. Thus, for an admissible set $\mathfrak{C}$ and a set $\mathcal{S} \subset \mathfrak{D}$, we let $\mathcal{S}' := \mathcal{S} \cup \sup \mathfrak{C}$ and (*cf.* also Convention 1.5)

$$\mathcal{B}^*_m(\mathfrak{C}; \mathcal{S}) := \bigl\{\text{admissible/geometric sets } \mathfrak{C} \subset \mathfrak{C}' \subset \mathcal{S}' \bigm| |\mathfrak{C}'| \geqslant m\bigr\}/\!\sim. \tag{1.1}$$

(Here and below, the quotient is modulo the suitable stabilizer.) In terms of (1.1) we can define a number of other constructions, which mainly reflect the way of the computation rather than the final result. Thus, given pairwise disjoint sets $\mathcal{S}_1, \mathcal{S}_2, \ldots \subset \mathfrak{D}$ and integers $0 \leqslant m_1 \leqslant m_2 \leqslant \ldots$, we define recursively

$$\mathcal{B}^*_{m_1,m_2,\ldots}(\mathcal{S}_1; \mathcal{S}_2; \ldots) := \bigcup \mathcal{B}^*_{m_2,\ldots}(\mathfrak{C}; \mathcal{S}_2, \ldots)/\!\sim, \quad \mathfrak{C} \in \mathcal{B}^*_{m_1}(\mathcal{S}_1), \text{ and} \tag{1.2}$$

$$\mathcal{B}^*_m(\mathcal{S}_1, \mathcal{S}_2, \ldots) := \bigcup \mathcal{B}^*_m(\mathfrak{C}; \mathcal{S}_2, \ldots)/\!\sim, \qquad \mathfrak{C} \in \mathcal{B}^*_0(\mathcal{S}_1). \tag{1.3}$$

As a variation of (1.3), for a *homogeneous chain* $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \ldots \subset \mathcal{S}_n$ (see $**$§4.1), we have

$$\mathcal{B}^*_m(\mathcal{S}_1 < \mathcal{S}_2 < \ldots) := \mathcal{B}^*_m(\mathcal{S}_1, \mathcal{S}_2 \smallsetminus \mathcal{S}_1, \ldots). \tag{1.4}$$

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Note, though, that in this latter case, instead of starting from scratch $\mathcal{B}^*_0$ in (1.3), by **Lemma 4.3 we can start from $\mathcal{B}^*_{m_{i-1}}(\mathcal{S}_{i-1})$ when computing the next set $\mathcal{B}^*_{m_i}(\mathcal{S}_i)$, where

$$m_n := m, \qquad m_{i-1} := \kappa_i \left\lceil m_i \frac{|\mathcal{S}_{i-1}|}{\kappa_i|\mathcal{S}_i|}\right\rceil \quad \text{for } i = n \text{ down to } 2; \quad \kappa_i \in \{1, 2\};$$

here, $\kappa_i = 2$ iff $\mathcal{S}_{i-1}$ is self-dual: then, so is any $\mathcal{S}_{i-1}$-saturated subset $\mathfrak{C} \subset \mathcal{S}_{i-1}$ and $|\mathfrak{C}|$ is even.

Thanks to the punctuation, we can combine these constructs and describe a long computation and its result by means of a single equation like

$$\mathcal{B}^*_{m,n}(\mathcal{P}, \mathcal{Q} < \mathcal{R}; \mathcal{S}_1, \mathcal{S}_2) = \varnothing.$$

The deciphering is left to the reader.

Convention 1.5. Clearly, most algorithms boil down to the following problem, *cf.* (1.1):

$$\text{compute } \mathcal{B}^*_n(\mathfrak{C}; \mathcal{S} \smallsetminus \mathcal{Q}) \text{ for each set } \mathfrak{C} \in \mathcal{B}^*_m(\mathcal{Q})$$

or iterations thereof. To reduce the overcounting, we assume that the fixed set $\mathfrak{C}$ no longer changes. That is, we consider only

$$\text{extensions } \mathfrak{C} \subset \mathfrak{C}' \subset \mathcal{S} \text{ such that } \operatorname{sat}_{\mathcal{Q}} \mathfrak{C}' = \mathfrak{C};$$

otherwise $\mathfrak{C}'$ would appear as an extension of the other set $\operatorname{sat}_{\mathcal{Q}} \mathfrak{C}' \in \mathcal{B}^*_m(\mathcal{Q})$.

With the exception of §1.4 below, we assume all sets $\mathcal{S}_i$ self dual; hence, so are saturated subsets $\mathfrak{C} \subset \bigcup \mathcal{S}_i$. Furthermore, often $\mathcal{S}_i$ are unions of orbits. We introduce the shortcut

$$\bar{\mathfrak{o}}^+_n := \bar{\mathfrak{o}}_n \text{ if } \bar{\mathfrak{o}}^*_n = \bar{\mathfrak{o}}_n \quad \text{or} \quad \bar{\mathfrak{o}}^+_n := \bar{\mathfrak{o}}_n \cup \bar{\mathfrak{o}}^*_n \text{ otherwise.}$$

Only one of each pair of dual orbits is used in the computation, but the goals $m, \ldots$ in $\mathcal{B}^*_{m,\ldots}$ take both into account. Due to the duality, all cardinalities are even. Hence, the sets (1.3) are merely a means of estimating the reduced bounds as follows:

$$\mathcal{B}^*_m(\mathcal{S}_1, \ldots) = \varnothing \quad \text{implies that} \quad \bar{b}(\mathcal{S}_1 \cup \ldots) \leqslant m - 2. \tag{1.6}$$

On the other hand, the sets (1.2) may be much easier to compute if m_1 is large enough. Typically, these sets are used in the form of the following obvious statement:

$$\text{if} \quad \mathcal{B}^*_{m,m+n-2}(\mathcal{S}; \mathcal{T}) = \mathcal{B}^*_{n,m+n-2}(\mathcal{T}; \mathcal{S}) = \varnothing, \quad \text{then} \quad \bar{b}(\mathcal{S}, \mathcal{T}) \leqslant m + n - 4. \tag{1.7}$$

More sophisticated implications are explained on a case-by-case basis, *cf.* the end of §4.5.3.

1.2. **Computation *up to rank* r.** One has $\operatorname{rk} \mathfrak{C} \leqslant 20$ for any geometric set $\mathfrak{C} \subset \mathfrak{O}_h$, see **(3.4), and this inequality plays a crucial rôle in our computation. For a subset $\mathcal{S} \subset \mathfrak{O}_h$, define

$$\mathcal{B}^r_m(\mathcal{S}) := \left\{ \mathfrak{C} \in \mathcal{B}_m(\mathcal{S}) \;\middle|\; \operatorname{rk} \mathfrak{C} \leqslant 20 - r \right\}$$

and similar counterparts for the other sets (1.2)–(1.4). We use roman numerals for r, thus speaking about $\mathcal{B}^0 = \mathcal{B}, \mathcal{B}', \mathcal{B}'', \mathcal{B}''', \mathcal{B}^{\text{iv}}$. The reason is that, on the one hand,

- often $\mathcal{B}_m(\mathcal{S}) \neq \varnothing$, preventing us from making conclusion (1.6), (1.7); on the other hand,
- if $\operatorname{rk} \mathfrak{C}$ is large, it is relatively easy to list all saturated admissible oversets $\mathfrak{C} \subset \mathfrak{C}' \subset \mathfrak{O}$ of rank $\operatorname{rk} \mathfrak{C} < \operatorname{rk} \mathfrak{C}' \leqslant 20$ (*cf.* §1.3) and select the exceptional ones; others can be ignored.

More precisely, we can easily *push* the rank by up to to units by listing all $(\operatorname{stab} \mathfrak{C})$-orbits of lines or planes in $N/\operatorname{span} \mathfrak{C}$ generated by conics (*cf.* §1.3 below) and considering them one-by-one. We can also apply, at least partially, Convention 1.5 and inhibit the appearance of new conics from a certain subset $\mathcal{Q} \subset \mathcal{S}$ for which the intersection $\mathfrak{C}' \cap \mathcal{Q} = \mathfrak{C} \cap \mathcal{Q}$ has already been fixed.

To save memory, in all algorithms we periodically apply this trick and discard all sets of rank $\operatorname{rk} \mathfrak{C} \geqslant 18$, even before checking whether they would survive the subsequent steps; in other words, by default we always compute at most $\mathcal{B}''$. If, upon the computation, we find that $\mathcal{B}''$ consists of reasonably few sets $\mathfrak{C}$ of rank $\operatorname{rk} \mathfrak{C} = 16, 17$, we push the rank in two increments, first up to 18, and then up to 20. This works reasonably well for rank $\operatorname{rk}\mathfrak{C} = 17$, but it is quite time consuming for $\operatorname{rk}\mathfrak{C} = 16$; we leave it as the last resort, only if very few such sets need to be ruled out.

In other words, in all equations below $\mathcal{B}''$ is the default whereas the appearance of $\mathcal{B}'''$ or $\mathcal{B}^{\mathrm{iv}}$ indicates the fact that $\mathcal{B}'' \neq \varnothing$ but the few sets found have been analysed and discarded.

1.3. **Linear algebra *vs.* arithmetic.** In this paper, we consider *primitive* sublattices of N only; hence, when establishing the admissibility of a set $\mathfrak{C} \subset \mathfrak{O}$ or computing its saturation, we can work over $\mathbb{Q}$ rather than $\mathbb{Z}$. (See also the refined combinatorial bounds in **§3.5.) In fact, given a set $\mathfrak{C} \subset \mathfrak{O}$, we compute and store the matrix of the projection

$$p\colon N \to (N/\operatorname{span}\mathfrak{C}) \otimes \mathbb{Q},$$

which can be applied (on demand) to the roots $r \in N$ or other conics $l \in \mathfrak{O}$. The non-zero results are normalized and also stored; we denote the normalization by $\bar{p}(l)$.

This approach is faster than working with integral matrices and it reduces the number of essentially distinct sets $\mathfrak{C} \subset \mathfrak{O}$ to be considered, *viz.*, (relatively) saturated sets only.

It also makes it easy to push the rank (*cf.* §1.2 and §1.6) by up to two units. Indeed, assume that we are computing a certain set $\mathcal{B}^*_n(\mathcal{S})$, typically in the presence of a certain "fixed" subset $\mathcal{Q} \subset \mathcal{S}$, *cf.* Convention 1.5. Then, the corank 1 extensions of an intermediate subset $\mathfrak{C} \subset \mathcal{S}$ are essentially found together with $\operatorname{sat}_{\mathcal{S}}\mathfrak{C}$: they are the elements of the set

$$\bar{P}_{\mathfrak{C}} := \bigl\{\bar{p}(l) \neq 0 \bigm| l \in \mathcal{S}\bigr\}$$

of normalized projections that are *not* of the form

$$\bar{p}(r) \text{ for a root } r \in N \quad\text{or}\quad \bar{p}(l) \text{ for } l \in \mathcal{Q}$$

(or otherwise prohibited, *cf.* (1.13) below). The sizes of these extensions are seen immediately from the data already stored and small sets can be discarded even before any further analysis.

The corank 2 extensions are slightly more involved but still doable: we need to consider pairs of valid projections $p_1, p_2 \in \bar{P}_{\mathfrak{C}}$ and analyze the possible linear dependencies.

It is worth mentioning that each step of the brute force algorithm §1.4 is essentially pushing the rank by one unit; thus, *de facto* it is performed at the end of the previous step, when computing the saturation.

Remark 1.8. Technically, the extra data such as the projection p or the set $\bar{P}_{\mathfrak{C}}$ (computed on demand conic by conic) are stored in a separate record linked to $\mathfrak{C}$ and can be reused. For example, $\mathcal{B}_m(\mathcal{Q})$ is a list of $\mathcal{Q}$-saturated sets which are to be used as seeds in the computation of $\mathcal{B}_n(\mathcal{S})$ for an overset (often more than one) $\mathcal{S} \supset \mathcal{Q}$. More precisely, the seeds are the saturations $\operatorname{sat}_{\mathcal{S}}\mathfrak{C}$, $\mathfrak{C} \in \mathcal{B}_m(\mathcal{Q})$. We use the data stored to compute $\operatorname{sat}_{\mathcal{S}}\mathfrak{C}$ and link to it *the same* data record.

1.4. **Brute force.** Whenever necessary, we compute the sets $\mathcal{B}_m(\mathfrak{o})$ and estimate the bounds $b(\mathfrak{o})$ for individual combinatorial orbits. For each orbit $\bar{\mathfrak{o}}_n$, at most one combinatorial orbit $\mathfrak{o} \subset \bar{\mathfrak{o}}_n$ is considered, *viz.* the one presented in the tables; if necessary, *e.g.*, in §1.5 below, the set $\mathcal{B}_m(\mathfrak{o})$ is translated to other combinatorial orbits $\mathfrak{o}' \subset \bar{\mathfrak{o}}$ *via* the group $O_h(N)$.

This computation is explained, in the form of a sequence of records

n `:= rec(...)`,

at the beginning of the subsection dedicated to the particular polarization $h \in N$; if an index n is not mentioned, the orbit $\bar{\mathfrak{o}}_n$ is not considered. An empty record

n `:= rec()`,

means computing from scratch the whole set $\mathcal{B}_0(\mathfrak{o})$. Otherwise, the algorithm is adjusted by the following parameters:

`fixed :=` $\pm k$ (other parameters make no sense unless this one is set),

`min   :=` m (ignored unless `fixed` is set),

`chain :=` c (ignored unless `fixed` and `min` are set),

whose meaning is explained below.

If `fixed := ` k is set, we fix a vector $a \in \mathfrak{o}|_{|k|}$ and compute the set $\mathcal{B}_m(\mathfrak{o}_k^= < \mathfrak{o})$ (in the notation of §1.1), where

$$\mathfrak{o}_k^= := \begin{cases} \{l \in \mathfrak{o} \mid l|_k = a\} & \text{if } k > 0, \\ \{l \in \mathfrak{o} \mid l|_{-k} = a \text{ or } l|_{-k} = h|_{-k} - a\} & \text{if } k < 0. \end{cases}$$

For this, we also need `min := ` m. If m is not set, we merely compute $\mathcal{B}_0(\mathfrak{o}_k^=)$ and invoke **Lemma 4.3 to draw a conclusion about $b(\mathfrak{o})$, see item † in Convention 2.3.

If `chain := ` c is also set, we insert one (if c is a positive integer) or more (if $c = [c_1, \ldots, c_n]$ is a list) extra sets

$$\mathfrak{o}_k^= =: \mathfrak{c}_0 \subset \mathfrak{c}_1 \subset \ldots \subset \mathfrak{c}_n \subset \mathfrak{c}_{n+1} := \mathfrak{o},$$

where, for each $i = 1, \ldots, n$, the set $\mathfrak{c}_i$ is a union of c_i elements of the $(\operatorname{stab} \mathfrak{c}_{n+1})$-orbit of $\mathfrak{c}_{i-1}$ such that both $\mathfrak{c}_{i-1} \subset \mathfrak{c}_i$ and $\mathfrak{c}_i \subset \mathfrak{c}_{n+1}$ are homogeneous (see **§4.1). These sets are computed automatically and, if more than one is found, one is chosen randomly, with a preference to those of smaller rank. (If none were found, `Error();` would be raised.)

In §4–§6, after finding the bound $b(\mathfrak{o})$, we may also need the actual set $\mathcal{B}_m(\mathfrak{o})$ to be used in the puzzle assembly algorithm (see §1.5 below), and that may require a lower goal `min := ` m (*cf.* Remark 1.10 below). These lower goals are shown as `GAP` comments:

`# min := ` m.

The computation itself is straightforward brute force. When computing a set $\mathcal{B}_m(\mathcal{S})$, with $\mathcal{S} = \mathfrak{o}$, $\mathfrak{o}_k^=$, or $\mathfrak{c}_i$, we consider $\mathcal{S}$-saturated subsets $\mathfrak{C} \subset \mathcal{S}$ and increase the rank step-by-step, by one unit at a time, as explained in §1.3. All new sets are sorted to avoid repetitions, *i.e.*, a single representative of each $(\operatorname{stab} \mathcal{S})$-orbit present is retained. We start with checking that $\mathcal{S}$ itself is not admissible (as it never is) and, hence,

$$\operatorname{rk} \mathfrak{C} \leqslant r_{\max} := \min\{20, \operatorname{rk} \mathcal{S} - 1\} \tag{1.9}$$

for each geometric set $\mathfrak{C} \subset \mathcal{S}$. For subsets $\mathfrak{C}$ of rank $\operatorname{rk} \mathfrak{C} \geqslant r_{\max} - 2$, we push the rank by up to two units at once, also as explained in §1.3, and, before sorting, retain only those oversets $\mathfrak{C}' \supset \mathfrak{C}$ that satisfy the goal $|\mathfrak{C}'| \geqslant m$.

1.5. **Puzzle assembly.** Let $\mathcal{S}$ be a union of several combinatorial orbits, all within the same orbit $\bar{\mathfrak{o}}$. Then, instead of the brute forse §1.4, the sets $\mathcal{B}_m^*(\mathcal{S})$, $\mathcal{B}_m^*(\mathfrak{C}; \mathcal{S})$, *etc.* are computed by the *puzzle assembly*: intuitively, we try to put together as many and as large "pieces" as possible in order to fit a large number of conics into a limited rank.

The rôle of the "pieces" of the puzzle is played by the elements of $\mathcal{B}_0(\mathfrak{o})$, $\mathfrak{o} \subset \bar{\mathfrak{o}}$. Clearly, it suffices to compute this set for one of the combinatorial orbits only (we use the one shown in the tables); the others are of the form $\mathcal{B}_0(\mathfrak{o}) \cdot g$, $g \in O_h(N)$.

Remark 1.10. The computation of the full set $\mathcal{B}_0(\mathfrak{o})$ may be expensive, and offen it suffices to know a smaller set $\mathcal{B}_n(\mathfrak{o})$. Starting from a heuristic estimate, the precise value of n (indicated below *via* the `GAP` comments `# min := ` n, *cf.* §1.4) is determined experimentally: at each step, the algorithm computes the minimal size of the pieces required to continue (see item (1) in the algorithm below) and, if a certain size is not available, an `Error();` is raised.

Let G be the relevant stabilizer, *i.e.*, $G := \operatorname{stab} \mathcal{S}$, or $G := \operatorname{stab}(\mathcal{S}, \operatorname{stab} \mathfrak{C})$, *etc.*

We start with the given set $\mathfrak{C}$ (possibly $\mathfrak{C} = \varnothing$) and extend it by a sequence

$$\mathfrak{C} =: \mathfrak{C}_0 \subset \mathfrak{C}_1 \subset \ldots \subset \mathfrak{C}_n, \quad \mathfrak{S}_k := \mathfrak{C}_k \smallsetminus \mathfrak{C}_{k-1} \subset \mathfrak{o}_k \text{ for } k \geqslant 1, \tag{1.11}$$

where $(\mathfrak{o}_1^+, \ldots \mathfrak{o}_n^+)$ is a rearrangement of the (pairs of dual) combinatorial orbits constituting $\mathcal{S}$. (*De facto*, usually this process aborts prematurely *via* §1.6 below.) In order to keep the intermediate sets $\mathfrak{C}_k$ as large as possible and reduce the overcounting, we always assume that

$$|\mathfrak{S}_1| \geqslant |\mathfrak{S}_2| \geqslant \ldots \geqslant |\mathfrak{S}_n|, \tag{1.12}$$

and that is why the combinatorial orbits $\mathfrak{o}_i \subset \mathcal{S}$ may have to be rearranged, see item (3) below.

Let $\bar{\mathfrak{C}}_k := \operatorname{sat}_{\mathcal{S}} \mathfrak{C}_k$. In addition to Convention 1.5 concerning the initial set $\mathfrak{C}$, if any, we also keep track of the combinatorial orbits already used and assume the intersections fixed, *i.e.*, at each step k we check that

$$|\bar{\mathfrak{C}}_k \cap \mathfrak{o}_i| = |\mathfrak{S}_i| \text{ for } i \leqslant k, \qquad |\bar{\mathfrak{C}}_k \cap \mathfrak{o}_i| \leqslant |\mathfrak{S}_k| \text{ for } i > k, \text{ see (1.12)}. \tag{1.13}$$

Thus, let $\kappa := 2$ (if the combinatorial orbits in $\bar{\mathfrak{o}}$ are self-dual) or $\kappa := 1$ (otherwise), and let $2n/\kappa$ be the number of combinatorial orbits constituting $\mathcal{S}$. Then step k of the algorithm proceeds as follows. (At this step, a set $\mathfrak{C}_{k-1}$ and, hence, the sequence $(\mathfrak{o}_1, \ldots, \mathfrak{o}_{k-1})$ are fixed whereas the notation $\mathfrak{o}_i$, $i \geqslant k$ merely refers to all remaining combinatorial orbits.)

(1) Compute the size bounds

$$s_{\max} := |\mathfrak{S}_{k-1}| \text{ (if } k > 1), \qquad s_{\min} := \kappa \left\lceil \frac{m - |\mathfrak{C}| - \kappa \sum_{i<k} |\mathfrak{S}_i|}{2(n-k-1)} \right\rceil, \text{ see (1.12)}. \tag{1.14}$$

It is at this point that we check the completeness of $\mathcal{B}_n(\mathfrak{o})$, see Remark 1.10.

(2) If there are combinatorial orbits $\mathfrak{o}_i$, $i \geqslant k$, with $|\bar{\mathfrak{C}}_{k-1} \cap \mathfrak{o}_i| > s_{\max}$, abort; if there are those with $|\bar{\mathfrak{C}}_{k-1} \cap \mathfrak{o}_i| = s_{\max}$, add *all of them* to the list (1.11) and proceed to the corresponding subsequent step, retaining all auxiliary data computed in §1.3, *cf.* Remark 1.8.

(3) Let $G_k := \operatorname{stab}(\bar{\mathfrak{C}}_{k-1}, G)$ and take for $\mathfrak{o}_k$ a single representative $\mathfrak{o}_k \subset \mathfrak{o}_k^+$ of each G_k-orbit on the set $\{\mathfrak{o}_k^+, \ldots, \mathfrak{o}_n^+\}$. Proceed separately for each $\mathfrak{o}_k$ chosen.

(4) Consider all sets (not just orbits!) $\mathfrak{S} \in \mathcal{B}_0(\mathfrak{o}_k)$ satisfying

$$s_{\min} \leqslant |\mathfrak{S}| < s_{\max}, \qquad \mathfrak{S} \supset \bar{\mathfrak{C}}_{k-1} \cap \mathfrak{o}_k;$$

pick for $\mathfrak{S}_k$ a single representative of each $\operatorname{stab}(\mathfrak{o}_k, G_k)$-orbit of such sets $\mathfrak{S}$.

(5) Let $\mathfrak{C}_k := \mathfrak{C}_{k-1} \cup \mathfrak{S}_k$, check the admissibility, compute $\bar{\mathfrak{C}}_k$, verify (1.13), Convention 1.5, (1.19) below, *etc.* Optionally, we can also check if $\bar{\mathfrak{C}}_k$ is (sub-)geometric.

(6) If $\operatorname{rk} \mathfrak{C}_k \geqslant r_{\max} - 2$, analyze it immediately (see §1.6 below) and discard; otherwise, add $\mathfrak{C}_k$ to the master list *unless* the latter already has a set G-isomorphic to $\mathfrak{S}_k$.

A further analysis is performed upon the completion of each partial *pattern* $|\mathfrak{S}_1| \geqslant \ldots \geqslant |\mathfrak{S}_k|$:

- analyze all sets $\mathfrak{S}_k$ of rank $\operatorname{rk} \mathfrak{S}_k \geqslant r_{\max} - 2 - \texttt{push}$, see §1.6 below;
- if sufficiently many (by default, at least 128) sets of rank $\operatorname{rk} \mathfrak{C}_k \geqslant 18$ have been collected, analyze and discard these sets as explained in §1.2.

Remark 1.15. Here and in §1.4, each steps concludes in *sorting*, *i.e.*, retaining, for the subsequent steps, a single representative of each orbit of the relevant symmetry group. When comparing two sets $\mathfrak{C}'$, $\mathfrak{C}''$, we compare some "obvious" invariants first, such as ranks, sizes, sorted lists of the valencies of the vertices, *etc.*, upon which we invoke `RepresentativeAction`, which may be slow. Probably, better performance could be achieved by using the canonical labelling from the `digraph` package, *cf.* (7.4) below, but this has not been implemented yet.

1.6. **Rank pushing.** When computing $\mathcal{B}^*_m(\mathcal{S})$, we always keep in mind the maximal rank (1.9). By default, we seamlessly *push the rank* by up to two units: when a subset $\mathfrak{C} \subset \mathcal{S}$ of rank

$$\operatorname{rk} \mathfrak{C} \geqslant r_{\max} - 2$$

is discovered, we immediately analyze its corank 1 or 2 extensions $\mathfrak{C} \subset \mathfrak{C}' \subset \mathcal{S}$ (see §1.3) and exclude $\mathfrak{C}$ and the extensions found from the consideration, storing those of size $|\mathfrak{C}'| \geqslant m$ in the output.

Occasionally, we set the `push` parameter of the `data` object (see §4.1 below) to 1 or 2. In this case, after each "mini-step" of the algorithm, upon the preliminary sorting, the rank is pushed by up to, respectively, 3 or 4 units; this is done in two increments, similar to §1.2.

In some cases, *cf.* §5.1 below, in order to avoid the computation of the full set $\mathcal{B}_0(\mathfrak{o})$, we set `push :=` 3 (or even 4), pushing the rank by up to 6 units: for each set $\mathfrak{C}$ of rank $\operatorname{rk} \mathfrak{C} = r_{\max} - 3$ (or $r_{\max} - 4$ if `push` $= 4$) we switch to the brute force algorithm §1.4, extending $\mathfrak{C}$ by individual conics $l \in \mathcal{S}$ rather than (translates of) subsets of $\mathfrak{o}$. Experimentally, relatively few $(\operatorname{stab} \mathcal{S})$-orbits of such sets are encountered but, on the other hand, the computation of $\mathcal{B}_0(\mathfrak{o})$ is quite expensive.

Remark 1.16. When computing $\mathcal{B}^*_*(\mathfrak{C};\mathcal{S})$ for $\mathfrak{C} \in \mathcal{B}^*_*(\mathcal{Q})$, the circumstances may change and $\mathfrak{C}$ itself may immediately be "eligible" for the rank pushing in $\mathcal{S}$. In this case, we skip the puzzle assembly algorithm for $\mathfrak{C}$, merely pushing its rank.

1.7. **Clusters.** Often, we have to break an orbit $\bar{\mathfrak{o}}$ into *clusters*,

$$\bar{\mathfrak{o}} = \bigcup\nolimits_k \mathfrak{c}_k,$$

where each cluster $\mathfrak{c}_k$ is a union of several combinatorial orbits so that

- $\{\mathfrak{c}_k\}$ is the $O_h(N)$-orbit of (any of) $\mathfrak{c}_k$.

Note that the pair $\mathfrak{c}_k \subset \bar{\mathfrak{o}}$ is automatically homogeneous (see $**$§4.1).

Clusters are usually defined "geometrically", in terms of support. To this end, we represent the Niemeier lattice in question as in $**$(3.1) (*cf.* also Convention 2.2 below), with an index set Ω, and define the *support* of a vector $v \in N$ as

$$\operatorname{supp} v := \bigl\{k \in \Omega \bigm| v|_k \neq 0\bigr\}.$$

The support $\operatorname{supp}\mathfrak{o}$ of a combinatorial orbit $\mathfrak{o}$ is the common support of its conics $l \in \mathfrak{o}$.

We use the shortcut $\bar{\Omega} := \Omega \smallsetminus \operatorname{supp}\hbar$.

Remark 1.17. In the tables below, the support $\operatorname{supp} h$ is the set of columns *other* than $[\![0]\!]_0$; for the combinatorial orbits $\mathfrak{o}$ shown, $\operatorname{supp}\mathfrak{o}$ is the set of columns *other* than $[0]_0$.

A "straightforward" application of clusters (typically used if there are but two or three clusters) is a computation of

$$\mathcal{B}^*_*(\mathfrak{c}_1 < \mathfrak{c}_1 \cup \mathfrak{c}_2 < \mathfrak{c}_1 \cup \mathfrak{c}_2 \cup \mathfrak{c}_3 < \ldots),$$

see (1.4), with each step

$$\mathcal{B}^*_*(\mathfrak{C}_{k-1}, \mathfrak{c}_1 \cup \ldots \cup \mathfrak{c}_k) \tag{1.18}$$

done by the puzzle assembly §1.5. In this case, similar to (1.12), for a set $\mathfrak{C}$ to be constructed we always assume that

$$|\mathfrak{C} \cup \mathfrak{c}_1| \geqslant |\mathfrak{C} \cap \mathfrak{c}_2| \geqslant |\mathfrak{C} \cup \mathfrak{c}_3| \geqslant \ldots. \tag{1.19}$$

Then, at each step (1.18), we can impose both lower *and upper* bounds similar to (1.14); occasionally, we can further reduce the overcounting by checking an analogue of (1.13).

Remark 1.20. The drawback of (1.19) is the fact that, in (1.18), the order of the subsequent clusters $\mathfrak{c}_k, \mathfrak{c}_{k+1}, \ldots$ may depend on $\mathfrak{C}_{k-1}$, as they may not be in the same $(\operatorname{stab}\mathfrak{C}_{k-1})$-orbit. It is here that we have to use the function `@select` from §4.1, *cf.*, *e.g.*, §4.5.4.

It is not uncommon to consider several nested systems of clusters $\{\mathfrak{c}_i\}$, $\{\mathfrak{c}'_k\}$, $\{\mathfrak{c}''_m\}$, *etc.* so that

- each $\mathfrak{c}_i$ is a union of some of $\mathfrak{o} \subset \bar{\mathfrak{o}}$, each $\mathfrak{c}'_k$ is a union of some of $\mathfrak{c}_i$, *etc.*, and
- the chain $\mathfrak{c}_i \subset \mathfrak{c}'_k \subset \mathfrak{c}''_m \subset \ldots \subset \bar{\mathfrak{o}}$ is homogeneous (see $**$§4.1).

A typical application is a computation of the form

$$\mathcal{B}^*_*(\mathfrak{c}_1 < \mathfrak{c}'_k < \mathfrak{c}''_m < \ldots),$$

see (1.4) and the explanation/assumprions thereafter, and we use *pipes* as explained in §4.2. As in Remark 1.20, the particular indices $k, m, \ldots$ may depend on the intermediate sets encountered, and we use `@select` from §4.1, *cf.*, *e.g.*, $\mathfrak{c}_o \subset \mathfrak{c}'_{o,o'} := \mathfrak{c}_o \cup \mathfrak{c}_{o'}$ in §4.4.3.

Remark 1.21. We try to define clusters so that, in the sequence

$$\operatorname{rk}\mathfrak{o} < \operatorname{rk}\mathfrak{c}_i < \operatorname{rk}\mathfrak{c}'_k < \operatorname{rk}\mathfrak{c}''_m < \ldots < \operatorname{rk}\bar{\mathfrak{o}}$$

each increment is by just a few units. This way, tweaking the `push` parameter, we can reduce most of the computation to the rank pushing §1.6 during or even before (see Remark 1.16) the puzzle assembly algorithm §1.5.

2. Sufficient combinatorial bounds

For 15 of the 24 Niemeier lattices, the bound $b(\mathfrak{O}) < 720$ follows immediately from the rough combinatorial estimates found in ∗∗§3 (see also the examples worked out in ∗∗§4.3 through ∗∗§4.5). This computation is presented in the tables below. The meaning of the columns is as follows:

(1) index (2) vectors $h|_k$ or $l|_k$ (3) $n(\bar{\mathfrak{o}})$ (4) count (5) bound,

where

(1) is the index (see Convention 2.1) of the class of $h \in N$ in the row starting a block (the *h-row*; in bold) and, for each h, the indices of the $O_h(N)$-orbits in the other rows (*l-rows*);
(2) are the components $h|_k \in D_k^\vee$ or $l|_k \in D_k^\vee$ in the notation of Convention 2.2;
(3) is the number $n(\bar{\mathfrak{o}})$ of combinatorial orbits in orbit $\bar{\mathfrak{o}}$; it is marked with a * if $\bar{\mathfrak{o}}$ is self-dual, and with a ** if each combinatorial orbit $\mathfrak{o} \subset \bar{\mathfrak{o}}$ is self-dual;
(4) is the count $c(\mathfrak{O})$ (in an h-row) or $c(\mathfrak{o})$, *per combinatorial orbit* (in an l-row);
(5) is an estimate on the bound $b(\mathfrak{O})$ (in an h-row) or $b(\mathfrak{o})$ (in an l-row), see Convention 2.3.

Convention 2.1. For references, we assign an index to each automorphism class of 4-polarizations $h \in N$; a description of these classes (the h-row) starts each block of the table and their indices are shown in bold. The remaining rows of each block (the l-rows) correspond to the $O_h(N)$-orbits $\bar{\mathfrak{o}}_1, \bar{\mathfrak{o}}_2, \ldots$, also indexed in a certain way, and show a "typical" combinatorial orbit $\mathfrak{o} \subset \bar{\mathfrak{o}}_n$. If an orbit $\bar{\mathfrak{o}}_n$ is not self-dual (no * or ** in the third column), it is understood that $\bar{\mathfrak{o}}_n^* = \bar{\mathfrak{o}}_{n+1}$ and the latter orbit is omitted from the tables. This fact explains the gaps in the indexing.

Convention 2.2. The second column splits into several subcolumns, one for each block D_k in the decomposition

$$N := N(D) = N\big(\textstyle\bigoplus_k D_k\big), \quad k \in \Omega,$$

see ∗∗(3.1). In the l-rows, we show a representative l of one of the combinatorial orbits constituting the given orbit, in terms of the invariants described in ∗∗§3.4.

For the components $h|_k \in D_k^\vee$ of h we use the notation $[\![h|_k^2]\!]_d$, whereas for those of a conic l we use $\big[l|_k \cdot h|_k\big]_d$. The subscript d is either 0, $\circ$, or $\bullet$ if the argument ($h|_k$ or $l|_k$) is an integral vector of square 0, 2, or 4, respectively, *cf.* ∗∗§3.3, or a positive integer if the argument is a shortest vector in the discriminant class $\underline{d}$, in the notation of ∗∗§3.3 again.

For short vectors, these data mostly determine the pair $(h|_k, l|_k)$ up to $R(D_k)$, and the occasional superscripts are mainly artifacts left over from other papers where longer vectors h were considered. A notable exception is the pair $\overset{2}{\bullet}$ representing a square 4 vector $\pm 2e_i \in \mathbf{D}_n$ (as opposed to $\mathbf{1_u}$, $|u| = 4$), see the description of $\mathbf{D}_n$ in ∗∗§3.3.

Convention 2.3. Shown in the fifth column is an estimate on $b(\mathfrak{o})$ *per s single combinatorial orbit*. The original estimates are given by ∗∗(3.14). Occasionally, this naïve estimate can be reduced; this fact is indicated *via*

(old bound) → (new bound)(reason),

with a reference to one of the following reasons:

- ∗ the bound has been reduced by ∗∗Lemma 3.37;
- ✓ the sharp bound has been obtained by brute force §1.4;
- † a partial computation has been performed, without the goal `min := ` m set, see §1.4;
- ‡ a full computation as in §1.4 has been performed, but the `min := ` m goal has not been achieved, so that the sharp bound is still not known.

The bounds $b(\mathfrak{o})$ known to be sharp are underlined.

Needless to say that the tables were compiled by `GAP` as one of the outcomes of the algorithm; in particular, that is why we do not have full control over the superscripts in Convention 2.2.

2.1. **The lattice** $N(\mathbf{D}_{24})$**.** There are 2 classes; $b(\mathfrak{O}) \leqslant 337$.

1:	$\llbracket 4\rrbracket^2_\bullet$		14168	337
1:	$[2]^-_\bullet$	1^{**}	14168	337
2:	$\llbracket 4\rrbracket_\bullet$		4568	122
1:	$[2]^+_\bullet$	1^{**}	4560	120
2:	$[2]^2_\bullet$	1	4	1

2.2. **The lattice** $N(\mathbf{D}_{16} \oplus \mathbf{E}_8)$**.** There are 5 classes; $b(\mathfrak{O}) \leqslant 370$.

1:	$\llbracket 4\rrbracket^2_\bullet$	$\llbracket 0\rrbracket_0$		10840	370
1:	$[2]^-_\bullet$	$[0]_0$	1^{**}	3640	130
2:	$[2]_\circ$	$[0]_\circ$	1^{**}	7200	240
2:	$\llbracket 4\rrbracket_1$	$\llbracket 0\rrbracket_0$		3640	280
1:	$[2]_1$	$[0]_0$	1	1820	140
3:	$\llbracket 4\rrbracket_\bullet$	$\llbracket 0\rrbracket_0$		5080	250
1:	$[2]_1$	$[0]_0$	1^{**}	2048	128
2:	$[2]^+_\bullet$	$[0]_0$	1^{**}	1584	72
3:	$[2]_\circ$	$[0]_\circ$	1^{**}	1440	48
4:	$[2]^2_\bullet$	$[0]_0$	1	4	1
4:	$\llbracket 0\rrbracket_0$	$\llbracket 4\rrbracket_\bullet$		7000	240
1:	$[0]_\circ$	$[2]_\circ$	1^{**}	6720	224
2:	$[0]_0$	$[2]_\bullet$	1^{**}	280	16
5:	$\llbracket 2\rrbracket_\circ$	$\llbracket 2\rrbracket_\circ$		4120	212
1:	$[1]_\circ$	$[1]_\circ$	1^{**}	3136	168
2:	$[2]_\bullet$	$[0]_0$	1	364	14
4:	$[2]^2_\bullet$	$[0]_0$	1	2	1
6:	$[2]_\circ$	$[0]_\circ$	1	126	7

2.3. **The lattice** $N(3\mathbf{E}_8)$**.** There are 2 classes; $b(\mathfrak{O}) \leqslant 240$.

1:	$[\![4]\!]_\bullet$	$[\![0]\!]_0$	$[\![0]\!]_0$		7000	240
1:	$[2]_\bullet$	$[0]_0$	$[0]_0$	1^{**}	280	16
2:	$[2]_\circ$	$[0]_\circ$	$[0]_0$	2^{**}	3360	112
2:	$[\![2]\!]_\circ$	$[\![2]\!]_\circ$	$[\![0]\!]_0$		4120	212
1:	$[2]_\circ$	$[0]_0$	$[0]_\circ$	2^{*}	240	8
2:	$[1]_\circ$	$[1]_\circ$	$[0]_0$	1^{**}	3136	168
3:	$[2]_\bullet$	$[0]_0$	$[0]_0$	2	126	7

2.4. **The lattice** $N(\mathbf{A}_{24})$**.** There are 2 classes; $b(\mathfrak{O}) \leqslant 240$.

1:	$[\![4]\!]_\bullet$		4760	240
1:	$[2]_{20}$	2^{*}	1330	70
2:	$[2]_\bullet^+$	2^{*}	210	10
3:	$[2]_\bullet^-$	1^{**}	1680	80
2:	$[\![4]\!]_{20}$		3800	200
1:	$[2]_{20}$	1	1900	100

2.5. **The lattice** $N(2\mathbf{D}_{12})$**.** There are 4 classes; $b(\mathfrak{O}) \leqslant 458$.

1:	$[\![4]\!]_\bullet^2$	$[\![0]\!]_0$		9176	458
1:	$[2]_\bullet^-$	$[0]_0$	1^{**}	1320	66
2:	$[2]_\circ$	$[0]_\circ$	1^{**}	5808	264
3:	$[2]_2$	$[0]_1$	1^{**}	2048	128
2:	$[\![2]\!]_\circ$	$[\![2]\!]_\circ$		4376	292
1:	$[1]_1$	$[1]_2$	2^{**}	1024	64
2:	$[1]_\circ$	$[1]_\circ$	1^{**}	1600	120
3:	$[2]_\bullet$	$[0]_0$	2	180	10
5:	$[2]_\bullet^2$	$[0]_0$	2	2	1
3:	$[\![4]\!]_\bullet$	$[\![0]\!]_0$		5336	$314 \to 258$
1:	$[2]_\bullet^+$	$[0]_0$	1^{**}	672	48
2:	$[2]_1$	$[0]_2$	1^{**}	3072	$192 \to 136$ *
3:	$[2]_\circ$	$[0]_\circ$	1^{**}	1584	72
4:	$[2]_\bullet^2$	$[0]_0$	1	4	1

4:	$[\![1]\!]_2$	$[\![3]\!]_1$		3896	$362 \to 242$
1:	$[2]^2_\bullet$	$[0]_0$	1	1	1
3:	$[1]_\circ$	$[1]_\circ$	1	1452	$132 \to 72\,*$
5:	$[1]_2$	$[1]_1$	1	495	48

2.6. **The lattice** $N(\mathbf{A}_{17} \oplus \mathbf{E}_7)$**.** There are 5 classes; $b(\mathfrak{O}) \leqslant 372$.

1:	$[\![0]\!]_0$	$[\![4]\!]_\bullet$		6424	372
1:	$[0]_{15}$	$[2]_1$	2^*	1632	96
2:	$[0]_\circ$	$[2]_\circ$	1^{**}	3060	170
3:	$[0]_0$	$[2]^+_\bullet$	1^{**}	80	8
4:	$[0]_0$	$[2]^-_\bullet$	1^{**}	20	2
2:	$[\![4]\!]_\bullet$	$[\![0]\!]_0$		4984	360
1:	$[2]_{12}$	$[0]_0$	2^*	1001	91
2:	$[2]^+_\bullet$	$[0]_0$	2^*	91	7
3:	$[2]^-_\bullet$	$[0]_0$	1^{**}	728	52
4:	$[2]_{15}$	$[0]_1$	2^*	784	42
5:	$[2]_\circ$	$[0]_\circ$	1^{**}	504	28
3:	$[\![4]\!]_{12}$	$[\![0]\!]_0$		4024	332
1:	$[2]_6$	$[0]_0$	1^{**}	924	92
2:	$[2]_{15}$	$[0]_1$	1^{**}	1120	60
3:	$[2]_{12}$	$[0]_0$	1	990	90
4:	$[\![2]\!]_\circ$	$[\![2]\!]_\circ$		4504	298
1:	$[1]_{15}$	$[1]_1$	2^*	1440	96
2:	$[1]^+_\circ$	$[1]_\circ$	2^*	512	32
3:	$[2]_\bullet$	$[0]_0$	1	240	15
5:	$[2]_\circ$	$[0]_\circ$	1	60	6
5:	$[\![\frac{5}{2}]\!]_{15}$	$[\![\frac{3}{2}]\!]_1$		4024	280
1:	$[2]_{12}$	$[0]_0$	1	455	35
3:	$[2]_\bullet$	$[0]_0$	1	315	21
5:	$[\frac{5}{2}]_{15}$	$[\frac{\bar{1}}{2}]_1$	1	27	3
7:	$[\frac{3}{2}]_{15}$	$[\frac{1}{2}]_1$	1	1215	81

2.7. **The lattice $N(\mathbf{D}_{10} \oplus 2\mathbf{E}_7)$.** There are 7 classes; $b(\mathfrak{O}) \leqslant 462$.

1:	$[\![4]\!]^2_\bullet$	$[\![0]\!]_0$	$[\![0]\!]_0$		8344	462
1:	$[2]^-_\bullet$	$[0]_0$	$[0]_0$	1^{**}	672	42
2:	$[2]_\circ$	$[0]_\circ$	$[0]_0$	2^{**}	2268	126
3:	$[2]_2$	$[0]_1$	$[0]_1$	1^{**}	3136	168
2:	$[\![\frac{5}{2}]\!]_3$	$[\![0]\!]_0$	$[\![\frac{3}{2}]\!]_1$		4024	378
1:	$[2]_\bullet$	$[0]_0$	$[0]_0$	1	210	21
3:	$[\frac{5}{2}]_3$	$[0]_0$	$[\frac{\bar{1}}{2}]_1$	1	27	3
5:	$[2]_1$	$[0]_1$	$[0]_0$	1	560	30
7:	$[\frac{3}{2}]_3$	$[0]_0$	$[\frac{1}{2}]_1$	1	1215	135
3:	$[\![0]\!]_0$	$[\![4]\!]_\bullet$	$[\![0]\!]_0$		6424	$364 \to 362$
1:	$[0]_1$	$[2]_1$	$[0]_0$	1^{**}	1024	64
2:	$[0]_\circ$	$[2]_\circ$	$[0]_0$	1^{**}	1800	100
3:	$[0]_2$	$[2]_1$	$[0]_1$	1^{**}	2240	$120 \to 118\,*$
4:	$[0]_0$	$[2]^+_\bullet$	$[0]_0$	1^{**}	80	8
5:	$[0]_0$	$[2]^-_\bullet$	$[0]_0$	1^{**}	20	2
6:	$[0]_0$	$[2]_\circ$	$[0]_\circ$	1^{**}	1260	70
4:	$[\![1]\!]_2$	$[\![\frac{3}{2}]\!]_1$	$[\![\frac{3}{2}]\!]_1$		4024	$424 \to 328$
1:	$[\frac{1}{2}]_3$	$[0]_0$	$[\frac{3}{2}]_1$	2^{*}	256	16
2:	$[2]^2_\bullet$	$[0]_0$	$[0]_0$	1	1	1
4:	$[1]_\circ$	$[1]_\circ$	$[0]_0$	2	486	$54 \to 30\,*$
6:	$[1]_2$	$[\frac{3}{2}]_1$	$[\frac{\bar{1}}{2}]_1$	2	27	3
8:	$[1]_2$	$[\frac{1}{2}]_1$	$[\frac{1}{2}]_1$	1	729	81
5:	$[\![4]\!]_\bullet$	$[\![0]\!]_0$	$[\![0]\!]_0$		5464	314
1:	$[2]^+_\bullet$	$[0]_0$	$[0]_0$	1^{**}	360	36
2:	$[2]_3$	$[0]_0$	$[0]_1$	2^{**}	1792	96
3:	$[2]_\circ$	$[0]_\circ$	$[0]_0$	2^{**}	756	42
4:	$[2]^2_\bullet$	$[0]_0$	$[0]_0$	1	4	1
6:	$[\![0]\!]_0$	$[\![2]\!]_\circ$	$[\![2]\!]_\circ$		4504	$428 \to 308$
1:	$[0]_\circ$	$[2]_\circ$	$[0]_0$	2^{*}	180	10
2:	$[0]_2$	$[1]_1$	$[1]_1$	1^{**}	2880	$288 \to 168\,*$
3:	$[0]_0$	$[1]_\circ$	$[1]_\circ$	1^{**}	1024	96
4:	$[0]_0$	$[2]_\bullet$	$[0]_0$	2	60	6
7:	$[\![2]\!]_\circ$	$[\![2]\!]_\circ$	$[\![0]\!]_0$		4504	308
1:	$[1]_1$	$[1]_1$	$[0]_0$	1^{**}	1536	96

2:	$[1]_\circ$	$[1]_\circ$	$[0]_0$	1^{**}	1024	96
3:	$[1]_2$	$[1]_1$	$[0]_1$	1^{**}	1344	72
4:	$[2]_\bullet$	$[0]_0$	$[0]_0$	1	112	8
6:	$[2]_\bullet^2$	$[0]_0$	$[0]_0$	1	2	1
8:	$[2]_\circ$	$[0]_\circ$	$[0]_0$	1	60	6
10:	$[2]_\circ$	$[0]_0$	$[0]_\circ$	1	126	7

2.8. **The lattice** $N(\mathbf{A}_{15} \oplus \mathbf{D}_9)$**.** There are 7 classes; $b(\mathfrak{O}) \leqslant 550$.

1:	$\llbracket 0\rrbracket_0$	$\llbracket 4\rrbracket_\bullet^2$		7928	550
1:	$[0]_{12}$	$[2]_2$	2^*	1820	140
2:	$[0]_\circ$	$[2]_\circ$	1^{**}	3840	240
3:	$[0]_0$	$[2]_\bullet^-$	1^{**}	448	30
2:	$\llbracket 4\rrbracket_8$	$\llbracket 0\rrbracket_0$		4088	$504 \to 380$
1:	$[2]_{12}$	$[0]_2$	2^{**}	1260	$140 \to 78$ $*$
2:	$[2]_8$	$[0]_0$	1	784	112
3:	$\llbracket 0\rrbracket_0$	$\llbracket 4\rrbracket_\bullet$		5528	378
1:	$[0]_\circ$	$[2]_\circ$	1^{**}	1440	90
2:	$[0]_{14}$	$[2]_3$	2^*	1920	128
3:	$[0]_0$	$[2]_\bullet^+$	1^{**}	240	30
4:	$[0]_0$	$[2]_\bullet^2$	1	4	1
4:	$\llbracket 4\rrbracket_\bullet$	$\llbracket 0\rrbracket_0$		5048	$432 \to 360$
1:	$[2]_8$	$[0]_0$	1^{**}	924	92
2:	$[2]_\bullet^+$	$[0]_0$	2^*	66	6
3:	$[2]_\bullet^-$	$[0]_0$	1^{**}	528	44
4:	$[2]_{12}$	$[0]_2$	2^*	1188	$108 \to 72$ $*$
5:	$[2]_\circ$	$[0]_\circ$	1^{**}	576	36
6:	$[2]_{14}$	$[0]_3$	2^*	256	16
5:	$\llbracket 2\rrbracket_\circ$	$\llbracket 2\rrbracket_\circ$		4568	338
1:	$[1]_{12}$	$[1]_2$	2^*	728	56
2:	$[1]_\circ^+$	$[1]_\circ$	2^*	392	28
3:	$[1]_{14}$	$[1]_3$	2^*	896	64
4:	$[2]_\bullet$	$[0]_0$	1	182	13
6:	$[2]_\circ$	$[0]_\circ^+$	1	84	7
8:	$[2]_\circ$	$[0]_\circ^-$	1	2	1
6:	$\llbracket 3\rrbracket_{12}$	$\llbracket 1\rrbracket_2$		4088	$346 \to 322$
1:	$[\frac{3}{2}]_{14}$	$[\frac{1}{2}]_3$	1^{**}	768	48

2:	$[2]_8$	$[0]_0$	1	495	48	
4:	$[2]_\bullet$	$[0]_0$	1	396	36	
6:	$[3]_{12}$	$[-1]_2$	1	1	1	
8:	$[2]_{12}$	$[0]_2$	1	768	64 $\to$	52 *
7:	$\llbracket \frac{7}{4} \rrbracket_{14}$	$\llbracket \frac{9}{4} \rrbracket_3$		4088	310	
1:	$[2]_\bullet$	$[0]_0$	1	91	7	
3:	$[\frac{3}{2}]_{12}$	$[\frac{1}{2}]_2$	1	819	63	
5:	$[1]_\circ$	$[1]_\circ$	1	1008	72	
7:	$[\frac{7}{4}]_{14}$	$[\frac{1}{4}]_3$	1	126	13	

2.9. **The lattice** $N(2\mathbf{A}_{12})$**.** There are 4 classes; $b(\mathfrak{O}) \leqslant 436$.

1:	$\llbracket 2 \rrbracket_\circ$	$\llbracket 2 \rrbracket_\circ$		4664	436
1:	$[1]_8$	$[1]_1$	4^*	330	33
2:	$[1]_{10}$	$[1]_{11}$	4^*	605	55
3:	$[1]_\circ^+$	$[1]_\circ^+$	4^*	121	11
4:	$[2]_\bullet$	$[0]_0$	2	110	10
2:	$\llbracket 4 \rrbracket_\bullet$	$\llbracket 0 \rrbracket_0$		5144	416
1:	$[2]_\bullet^+$	$[0]_0$	2^*	36	4
2:	$[2]_\bullet^-$	$[0]_0$	1^{**}	288	32
3:	$[2]_8$	$[0]_1$	2^*	1092	84
4:	$[2]_{10}$	$[0]_{11}$	2^*	702	54
5:	$[2]_\circ$	$[0]_\circ$	1^{**}	624	48
6:	$[2]_{11}$	$[0]_3$	2^*	286	26
3:	$\llbracket \frac{22}{13} \rrbracket_{11}$	$\llbracket \frac{30}{13} \rrbracket_3$		4184	398
1:	$[\frac{11}{13}]_{12}$	$[\frac{15}{13}]_8$	1^{**}	504	52
2:	$[2]_\bullet$	$[0]_0$	1	55	5
4:	$[\frac{16}{13}]_8$	$[\frac{10}{13}]_1$	1	495	48
6:	$[\frac{20}{13}]_{10}$	$[\frac{6}{13}]_{11}$	1	495	45
8:	$[1]_\circ$	$[1]_\circ$	1	660	60
10:	$[\frac{22}{13}]_{11}$	$[\frac{4}{13}]_3$	1	135	15
4:	$\llbracket \frac{12}{13} \rrbracket_{12}$	$\llbracket \frac{40}{13} \rrbracket_8$		4184	392
1:	$[\frac{10}{13}]_{10}$	$[\frac{16}{13}]_{11}$	1	660	60
3:	$[1]_\circ$	$[1]_\circ$	1	480	40
5:	$[\frac{11}{13}]_{11}$	$[\frac{15}{13}]_3$	1	672	56
7:	$[\frac{12}{13}]_{12}$	$[\frac{14}{13}]_8$	1	280	40

2.10. **The lattice** $N(\mathbf{A}_{11} \oplus \mathbf{D}_7 \oplus \mathbf{E}_6)$**.** There are 12 classes; $b(\mathfrak{O}) \leqslant 632$.

1:	$\llbracket 0 \rrbracket_0$	$\llbracket 4 \rrbracket_\bullet^2$	$\llbracket 0 \rrbracket_0$		7096	632	
1:	$[0]_6$	$[2]_2$	$[0]_0$	1^{**}	924	92	
2:	$[0]_\circ$	$[2]_\circ$	$[0]_0$	1^{**}	1584	132	
3:	$[0]_{10}$	$[2]_2$	$[0]_1$	2^*	1782	162	
4:	$[0]_0$	$[2]_\bullet^-$	$[0]_0$	1^{**}	160	12	
5:	$[0]_0$	$[2]_\circ$	$[0]_\circ$	1^{**}	864	72	
2:	$\llbracket 0 \rrbracket_0$	$\llbracket 0 \rrbracket_0$	$\llbracket 4 \rrbracket_\bullet$		6136	534	$\to 510$
1:	$[0]_8$	$[0]_0$	$[2]_2$	2^*	495	48	
2:	$[0]_\circ$	$[0]_0$	$[2]_\circ$	1^{**}	1056	88	
3:	$[0]_{10}$	$[0]_2$	$[2]_1$	2^*	924	84	$\to$ 72 $*$
4:	$[0]_{11}$	$[0]_3$	$[2]_2$	2^*	768	60	
5:	$[0]_0$	$[0]_\circ$	$[2]_\circ$	1^{**}	672	56	
6:	$[0]_0$	$[0]_0$	$[2]_\bullet^+$	2^*	1	1	
7:	$[0]_0$	$[0]_0$	$[2]_\bullet^-$	1^{**}	32	4	
3:	$\llbracket 2 \rrbracket_\circ$	$\llbracket 2 \rrbracket_\circ$	$\llbracket 0 \rrbracket_0$		4696	498	
1:	$[1]_6$	$[1]_2$	$[0]_0$	1^{**}	504	52	
2:	$[1]_9$	$[1]_1$	$[0]_0$	2^*	720	80	
3:	$[1]_\circ^+$	$[1]_\circ$	$[0]_0$	2^*	200	20	
4:	$[1]_{10}$	$[1]_2$	$[0]_1$	2^*	540	54	
5:	$[1]_{11}$	$[1]_3$	$[0]_2$	2^*	432	48	
6:	$[2]_\bullet$	$[0]_0$	$[0]_0$	1	90	9	
8:	$[2]_\circ$	$[0]_\circ^+$	$[0]_0$	1	40	5	
10:	$[2]_\circ$	$[0]_\circ^-$	$[0]_0$	1	2	1	
12:	$[2]_\circ$	$[0]_0$	$[0]_\circ$	1	72	6	
4:	$\llbracket 0 \rrbracket_0$	$\llbracket 4 \rrbracket_\bullet$	$\llbracket 0 \rrbracket_0$		5656	490	
1:	$[0]_9$	$[2]_1$	$[0]_0$	2^*	880	80	
2:	$[0]_\circ$	$[2]_\circ$	$[0]_0$	1^{**}	792	66	
3:	$[0]_{11}$	$[2]_3$	$[0]_2$	2^*	1296	108	
4:	$[0]_0$	$[2]_\bullet^+$	$[0]_0$	1^{**}	72	10	
5:	$[0]_0$	$[2]_\circ$	$[0]_\circ$	1^{**}	432	36	
6:	$[0]_0$	$[2]_\bullet^2$	$[0]_0$	1	4	1	
5:	$\llbracket 3 \rrbracket_6$	$\llbracket 1 \rrbracket_2$	$\llbracket 0 \rrbracket_0$		4216	536	$\to 476$
1:	$[\frac{3}{2}]_9$	$[\frac{1}{2}]_1$	$[0]_0$	2^{**}	640	60	
2:	$[2]_\bullet$	$[0]_0$	$[0]_0$	1	225	45	
4:	$[3]_6$	$[-1]_2$	$[0]_0$	1	1	1	

6:	$[2]_6$	$[0]_2$	$[0]_0$	1	432	$72 \to 42$ *
8:	$[2]_8$	$[0]_0$	$[0]_2$	2	405	45
6:	$[\![4]\!]_\bullet$	$[\![0]\!]_0$	$[\![0]\!]_0$		5176	$532 \to 474$
1:	$[2]_\bullet^+$	$[0]_0$	$[0]_0$	2^*	28	4
2:	$[2]_\bullet^-$	$[0]_0$	$[0]_0$	1^{**}	224	28
3:	$[2]_6$	$[0]_2$	$[0]_0$	1^{**}	980	$112 \to 78$ *
4:	$[2]_8$	$[0]_0$	$[0]_2$	2^*	756	84
5:	$[2]_9$	$[0]_1$	$[0]_0$	2^*	512	40
6:	$[2]_\circ$	$[0]_\circ$	$[0]_0$	1^{**}	336	28
7:	$[2]_\circ$	$[0]_0$	$[0]_\circ$	1^{**}	288	24
8:	$[2]_{10}$	$[0]_2$	$[0]_1$	2^*	378	$42 \to 30$ *
7:	$[\![\frac{9}{4}]\!]_9$	$[\![\frac{7}{4}]\!]_1$	$[\![0]\!]_0$		4216	466
1:	$[2]_\bullet$	$[0]_0$	$[0]_0$	1	108	12
3:	$[\frac{3}{2}]_6$	$[\frac{1}{2}]_2$	$[0]_0$	1	588	63
5:	$[2]_8$	$[0]_0$	$[0]_2$	1	243	27
7:	$[\frac{9}{4}]_9$	$[\frac{\bar{1}}{4}]_1$	$[0]_0$	1	35	5
9:	$[\frac{5}{4}]_9$	$[\frac{3}{4}]_1$	$[0]_0$	1	567	63
11:	$[\frac{3}{2}]_{10}$	$[\frac{1}{2}]_2$	$[0]_1$	1	567	63
8:	$[\![0]\!]_0$	$[\![2]\!]_\circ$	$[\![2]\!]_\circ$		4696	440
1:	$[0]_{10}$	$[1]_2$	$[1]_1$	2^*	792	72
2:	$[0]_{11}$	$[1]_3$	$[1]_2$	2^*	1152	96
3:	$[0]_0$	$[1]_\circ$	$[1]_\circ$	1^{**}	400	60
4:	$[0]_\circ$	$[2]_\circ$	$[0]_0$	1	132	11
6:	$[0]_0$	$[2]_\bullet$	$[0]_0$	1	40	5
8:	$[0]_0$	$[2]_\bullet^2$	$[0]_0$	1	2	1
10:	$[0]_0$	$[2]_\circ$	$[0]_\circ$	1	30	5
9:	$[\![\frac{8}{3}]\!]_8$	$[\![0]\!]_0$	$[\![\frac{4}{3}]\!]_2$		4216	$532 \to 436$
1:	$[\frac{4}{3}]_4$	$[0]_0$	$[\frac{2}{3}]_1$	1^{**}	700	80
2:	$[\frac{4}{3}]_{10}$	$[0]_2$	$[\frac{2}{3}]_1$	1^{**}	840	$120 \to 72$ *
3:	$[2]_\bullet$	$[0]_0$	$[0]_0$	1	168	24
5:	$[2]_6$	$[0]_2$	$[0]_0$	1	392	$56 \to 32$ *
7:	$[\frac{8}{3}]_8$	$[0]_0$	$[\frac{\bar{2}}{3}]_2$	1	10	2
9:	$[\frac{5}{3}]_8$	$[0]_0$	$[\frac{1}{3}]_2$	1	512	64
11:	$[2]_9$	$[0]_1$	$[0]_0$	1	256	20
10:	$[\![\frac{5}{3}]\!]_{10}$	$[\![1]\!]_2$	$[\![\frac{4}{3}]\!]_1$		4216	$438 \to 422$
1:	$[\frac{5}{6}]_{11}$	$[\frac{1}{2}]_3$	$[\frac{2}{3}]_2$	1^{**}	640	60

2:	$[2]_\bullet$	$[0]_0$	$[0]_0$	1	45	5	
4:	$[1]_6$	$[1]_2$	$[0]_0$	1	210	21	
6:	$[\frac{4}{3}]_8$	$[0]_0$	$[\frac{2}{3}]_2$	1	450	50	
8:	$[\frac{3}{2}]_9$	$[\frac{1}{2}]_1$	$[0]_0$	1	320	30	
10:	$[1]_\circ$	$[1]_\circ$	$[0]_0$	1	240	$24 \to$	22 $*$
12:	$[1]_\circ$	$[0]_0$	$[1]_\circ$	1	320	32	
14:	$[\frac{5}{3}]_{10}$	$[1]_2$	$[\frac{\bar{2}}{3}]_1$	1	10	2	
16:	$[\frac{5}{3}]_{10}$	$[0]_2$	$[\frac{1}{3}]_1$	1	192	$24 \to$	18 $*$
18:	$[\frac{5}{3}]_{10}$	$[-1]_2$	$[\frac{4}{3}]_1$	1	1	1	
11:	$\llbracket 2\rrbracket_\circ$	$\llbracket 0\rrbracket_0$	$\llbracket 2\rrbracket_\circ$		4696	$454 \to$	418
1:	$[1]_8$	$[0]_0$	$[1]_2$	2^*	720	72	
2:	$[1]_\circ^+$	$[0]_0$	$[1]_\circ$	2^*	200	20	
3:	$[1]_{10}$	$[0]_2$	$[1]_1$	2^*	840	$84 \to$	66 $*$
4:	$[1]_{11}$	$[0]_3$	$[1]_2$	2^*	384	30	
5:	$[2]_\bullet$	$[0]_0$	$[0]_0$	1	90	9	
7:	$[2]_\circ$	$[0]_\circ$	$[0]_0$	1	84	7	
9:	$[2]_\circ$	$[0]_0$	$[0]_\circ$	1	30	5	
12:	$\llbracket\frac{11}{12}\rrbracket_{11}$	$\llbracket\frac{7}{4}\rrbracket_3$	$\llbracket\frac{4}{3}\rrbracket_2$		4216	414	
1:	$[\frac{2}{3}]_8$	$[0]_0$	$[\frac{4}{3}]_2$	1	165	16	
3:	$[\frac{3}{4}]_9$	$[\frac{5}{4}]_1$	$[0]_0$	1	385	35	
5:	$[1]_\circ$	$[1]_\circ$	$[0]_0$	1	231	21	
7:	$[1]_\circ$	$[0]_0$	$[1]_\circ$	1	176	16	
9:	$[\frac{5}{6}]_{10}$	$[\frac{1}{2}]_2$	$[\frac{2}{3}]_1$	1	770	70	
11:	$[\frac{11}{12}]_{11}$	$[\frac{7}{4}]_3$	$[\frac{\bar{2}}{3}]_2$	1	10	2	
13:	$[\frac{11}{12}]_{11}$	$[\frac{3}{4}]_3$	$[\frac{1}{3}]_2$	1	336	42	
15:	$[\frac{11}{12}]_{11}$	$[\frac{\bar{1}}{4}]_3$	$[\frac{4}{3}]_2$	1	35	5	

2.11. **The lattice** $N(4\mathbf{E}_6)$**.** There are 3 classes; $b(\mathfrak{O}) \leqslant 636$.

1:	$\llbracket 4\rrbracket_\bullet$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$		6136	636
1:	$[2]_\bullet^+$	$[0]_0$	$[0]_0$	$[0]_0$	2^*	1	1
2:	$[2]_\bullet^-$	$[0]_0$	$[0]_0$	$[0]_0$	1^{**}	32	4
3:	$[2]_\circ$	$[0]_\circ$	$[0]_0$	$[0]_0$	3^{**}	576	48
4:	$[2]_2$	$[0]_2$	$[0]_1$	$[0]_0$	6^*	729	81
2:	$\llbracket\frac{4}{3}\rrbracket_2$	$\llbracket\frac{4}{3}\rrbracket_2$	$\llbracket\frac{4}{3}\rrbracket_1$	$\llbracket 0\rrbracket_0$		4216	584
1:	$[\frac{4}{3}]_2$	$[\frac{2}{3}]_1$	$[0]_0$	$[0]_2$	6^*	270	30
2:	$[\frac{2}{3}]_1$	$[\frac{2}{3}]_1$	$[\frac{2}{3}]_2$	$[0]_0$	1^{**}	1000	200

3:	$[2]_\bullet$	$[0]_0$	$[0]_0$	$[0]_0$	3	10	2
5:	$[1]_\circ$	$[1]_\circ$	$[0]_0$	$[0]_0$	3	256	32
3:	$[\![2]\!]_\circ$	$[\![2]\!]_\circ$	$[\![0]\!]_0$	$[\![0]\!]_0$		4696	536
1:	$[2]_\circ$	$[0]_0$	$[0]_\circ$	$[0]_0$	4^*	72	6
2:	$[1]_\circ$	$[1]_\circ$	$[0]_0$	$[0]_0$	1^{**}	400	60
3:	$[1]_2$	$[1]_2$	$[0]_1$	$[0]_0$	4^*	972	108
4:	$[2]_\bullet$	$[0]_0$	$[0]_0$	$[0]_0$	2	30	5

2.12. **The lattice** $N(2\mathbf{A}_9 \oplus \mathbf{D}_6)$**.** There are 9 classes; $b(\mathfrak{O}) \leqslant 668$.

1:	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![4]\!]_\bullet^2$		6680	668
1:	$[0]_7$	$[0]_9$	$[2]_2$	4^*	1200	120
2:	$[0]_\circ$	$[0]_0$	$[2]_\circ$	2^{**}	900	90
3:	$[0]_0$	$[0]_0$	$[2]_\bullet^-$	1^{**}	80	8
2:	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![4]\!]_\bullet$		5720	578
1:	$[0]_5$	$[0]_0$	$[2]_1$	2^{**}	504	52
2:	$[0]_\circ$	$[0]_0$	$[2]_\circ$	2^{**}	540	54
3:	$[0]_8$	$[0]_1$	$[2]_3$	4^*	900	90
4:	$[0]_0$	$[0]_0$	$[2]_\bullet^+$	2^{**}	12	2
5:	$[0]_0$	$[0]_0$	$[2]_\bullet^2$	1	4	1
3:	$[\![\tfrac{5}{2}]\!]_5$	$[\![0]\!]_0$	$[\![\tfrac{3}{2}]\!]_1$		4280	536
1:	$[2]_\bullet$	$[0]_0$	$[0]_0$	1	100	20
3:	$[\tfrac{5}{2}]_5$	$[0]_0$	$[\tfrac{\bar{1}}{2}]_1$	1	15	3
5:	$[\tfrac{3}{2}]_5$	$[0]_0$	$[\tfrac{1}{2}]_1$	1	375	75
7:	$[2]_6$	$[0]_2$	$[0]_0$	2	225	25
9:	$[\tfrac{3}{2}]_7$	$[0]_9$	$[\tfrac{1}{2}]_2$	2	600	60
4:	$[\![4]\!]_\bullet$	$[\![0]\!]_0$	$[\![0]\!]_0$		5240	$542 \to 530$
1:	$[2]_\bullet^+$	$[0]_0$	$[0]_0$	2^*	15	3
2:	$[2]_\bullet^-$	$[0]_0$	$[0]_0$	1^{**}	120	20
3:	$[2]_5$	$[0]_0$	$[0]_1$	1^{**}	640	60
4:	$[2]_6$	$[0]_2$	$[0]_0$	2^*	675	75
5:	$[2]_7$	$[0]_9$	$[0]_2$	2^*	720	$72 \to 66$ *
6:	$[2]_\circ$	$[0]_\circ$	$[0]_0$	1^{**}	360	36
7:	$[2]_\circ$	$[0]_0$	$[0]_\circ$	1^{**}	240	24
8:	$[2]_8$	$[0]_6$	$[0]_0$	2^*	210	21
9:	$[2]_8$	$[0]_1$	$[0]_3$	2^*	320	30

5:	$[\![2]\!]_\circ$	$[\![0]\!]_0$	$[\![2]\!]_\circ$		4760	506
1:	$[1]_5$	$[0]_0$	$[1]_1$	1^{**}	560	64
2:	$[1]_7$	$[0]_9$	$[1]_2$	2^*	560	56
3:	$[1]_\circ^+$	$[0]_0$	$[1]_\circ$	2^*	128	16
4:	$[1]_8$	$[0]_1$	$[1]_3$	2^*	640	64
5:	$[1]_9$	$[0]_3$	$[1]_2$	2^*	240	24
6:	$[1]_9$	$[0]_8$	$[1]_1$	2^*	360	40
7:	$[2]_\bullet$	$[0]_0$	$[0]_0$	1	56	7
9:	$[2]_\circ$	$[0]_\circ$	$[0]_0$	1	90	9
11:	$[2]_\circ$	$[0]_0$	$[0]_\circ^+$	1	24	4
13:	$[2]_\circ$	$[0]_0$	$[0]_\circ^-$	1	2	1
6:	$[\![\frac{8}{5}]\!]_8$	$[\![\frac{12}{5}]\!]_6$	$[\![0]\!]_0$		4280	$548 \to 498$
1:	$[\frac{4}{5}]_4$	$[\frac{6}{5}]_8$	$[0]_0$	1^{**}	420	48
2:	$[\frac{4}{5}]_9$	$[\frac{6}{5}]_3$	$[0]_2$	1^{**}	480	$72 \to 46 *$
3:	$[\frac{4}{5}]_9$	$[\frac{6}{5}]_8$	$[0]_1$	1^{**}	384	36
4:	$[2]_\bullet$	$[0]_0$	$[0]_0$	1	28	4
6:	$[\frac{6}{5}]_6$	$[\frac{4}{5}]_2$	$[0]_0$	1	420	60
8:	$[\frac{7}{5}]_7$	$[\frac{3}{5}]_9$	$[0]_2$	1	384	$48 \to 36 *$
10:	$[1]_\circ$	$[1]_\circ$	$[0]_0$	1	384	48
12:	$[\frac{8}{5}]_8$	$[\frac{2}{5}]_6$	$[0]_0$	1	90	18
14:	$[\frac{8}{5}]_8$	$[\frac{2}{5}]_1$	$[0]_3$	1	192	18
7:	$[\![\frac{9}{10}]\!]_9$	$[\![\frac{21}{10}]\!]_3$	$[\![1]\!]_2$		4280	$502 \to 490$
1:	$[\frac{3}{5}]_6$	$[\frac{7}{5}]_2$	$[0]_0$	1	252	27
3:	$[\frac{7}{10}]_7$	$[\frac{3}{10}]_9$	$[1]_2$	1	252	28
5:	$[1]_\circ$	$[1]_\circ$	$[0]_0$	1	189	21
7:	$[1]_\circ$	$[0]_0$	$[1]_\circ$	1	90	10
9:	$[\frac{4}{5}]_8$	$[\frac{6}{5}]_6$	$[0]_0$	1	315	35
11:	$[\frac{4}{5}]_8$	$[\frac{7}{10}]_1$	$[\frac{1}{2}]_3$	1	432	48
13:	$[\frac{9}{10}]_9$	$[\frac{21}{10}]_3$	$[-1]_2$	1	1	1
15:	$[\frac{9}{10}]_9$	$[\frac{11}{10}]_3$	$[0]_2$	1	210	$30 \to 24 *$
17:	$[\frac{9}{10}]_9$	$[\frac{1}{10}]_3$	$[1]_2$	1	63	9
19:	$[\frac{9}{10}]_9$	$[\frac{3}{5}]_8$	$[\frac{1}{2}]_1$	1	336	42
8:	$[\![2]\!]_\circ$	$[\![2]\!]_\circ$	$[\![0]\!]_0$		4760	$552 \to 488$
1:	$[1]_6$	$[1]_2$	$[0]_0$	4^*	448	48
2:	$[1]_7$	$[1]_9$	$[0]_2$	4^*	336	$48 \to 32 *$
3:	$[2]_\circ$	$[0]_0$	$[0]_\circ$	2^*	60	6

4:	$[1]^{+}_{\circ}$	$[1]^{+}_{\circ}$	$[0]_0$	4^*	64	8
5:	$[1]_8$	$[1]_1$	$[0]_3$	4^*	256	24
6:	$[2]_{\bullet}$	$[0]_0$	$[0]_0$	2	56	7
9:	$\llbracket\frac{9}{10}\rrbracket_9$	$\llbracket\frac{8}{5}\rrbracket_8$	$\llbracket\frac{3}{2}\rrbracket_1$		4280	482
1:	$[\frac{1}{2}]_5$	$[0]_0$	$[\frac{3}{2}]_1$	1	126	13
3:	$[\frac{7}{10}]_7$	$[\frac{4}{5}]_9$	$[\frac{1}{2}]_2$	1	432	48
5:	$[1]_{\circ}$	$[1]_{\circ}$	$[0]_0$	1	144	16
7:	$[1]_{\circ}$	$[0]_0$	$[1]_{\circ}$	1	135	15
9:	$[\frac{4}{5}]_8$	$[\frac{6}{5}]_6$	$[0]_0$	1	252	28
11:	$[\frac{4}{5}]_8$	$[\frac{1}{5}]_1$	$[1]_3$	1	432	48
13:	$[\frac{9}{10}]_9$	$[\frac{3}{5}]_3$	$[\frac{1}{2}]_2$	1	336	36
15:	$[\frac{9}{10}]_9$	$[\frac{8}{5}]_8$	$[\frac{\bar{1}}{2}]_1$	1	15	3
17:	$[\frac{9}{10}]_9$	$[\frac{3}{5}]_8$	$[\frac{1}{2}]_1$	1	240	30
19:	$[\frac{9}{10}]_9$	$[\frac{2}{5}]_8$	$[\frac{3}{2}]_1$	1	28	4

2.13. **The lattice $N(3\mathbf{A}_8)$.** There are 5 classes; $b(\mathfrak{O}) \leqslant 588$.

1:	$\llbracket 4\rrbracket_{\bullet}$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$		5272	588
1:	$[2]^{+}_{\bullet}$	$[0]_0$	$[0]_0$	2^*	10	2
2:	$[2]^{-}_{\bullet}$	$[0]_0$	$[0]_0$	1^{**}	80	16
3:	$[2]_5$	$[0]_8$	$[0]_8$	2^*	810	90
4:	$[2]_6$	$[0]_3$	$[0]_0$	4^*	420	45
5:	$[2]_{\circ}$	$[0]_{\circ}$	$[0]_0$	2^{**}	288	32
6:	$[2]_7$	$[0]_7$	$[0]_1$	4^*	324	36
2:	$\llbracket\frac{8}{9}\rrbracket_8$	$\llbracket\frac{14}{9}\rrbracket_2$	$\llbracket\frac{14}{9}\rrbracket_2$		4312	576
1:	$[\frac{4}{9}]_4$	$[\frac{7}{9}]_1$	$[\frac{7}{9}]_1$	1^{**}	280	32
2:	$[1]_{\circ}$	$[1]_{\circ}$	$[0]_0$	2	112	14
4:	$[\frac{2}{3}]_6$	$[\frac{4}{3}]_3$	$[0]_0$	2	196	28
6:	$[\frac{7}{9}]_7$	$[\frac{4}{9}]_7$	$[\frac{7}{9}]_1$	2	336	42
8:	$[\frac{8}{9}]_8$	$[\frac{8}{9}]_5$	$[\frac{2}{9}]_8$	2	245	35
10:	$[\frac{8}{9}]_8$	$[\frac{14}{9}]_2$	$[\frac{\bar{4}}{9}]_2$	2	21	3
12:	$[\frac{8}{9}]_8$	$[\frac{5}{9}]_2$	$[\frac{5}{9}]_2$	1	196	28
3:	$\llbracket 2\rrbracket_{\circ}$	$\llbracket 2\rrbracket_{\circ}$	$\llbracket 0\rrbracket_0$		4792	570
1:	$[1]_5$	$[1]_8$	$[0]_8$	4^*	315	35
2:	$[2]_{\circ}$	$[0]_0$	$[0]_{\circ}$	2^*	72	8
3:	$[1]_6$	$[1]_3$	$[0]_0$	2^*	441	63
4:	$[1]^{+}_{\circ}$	$[1]^{+}_{\circ}$	$[0]_0$	2^*	49	7

5:	$[1]^+_\circ$	$[1]^-_\circ$	$[0]_0$	2^*	49	7
6:	$[1]_7$	$[1]_7$	$[0]_1$	2^*	441	49
7:	$[1]_7$	$[1]_1$	$[0]_7$	4^*	252	28
8:	$[1]_8$	$[1]_8$	$[0]_5$	2^*	126	13
9:	$[2]_\bullet$	$[0]_0$	$[0]_0$	2	42	6
4:	$\llbracket\frac{8}{9}\rrbracket_8$	$\llbracket\frac{8}{9}\rrbracket_8$	$\llbracket\frac{20}{9}\rrbracket_5$		4312	552
1:	$[\frac{7}{9}]_7$	$[\frac{1}{9}]_1$	$[\frac{10}{9}]_7$	2^*	384	48
2:	$[\frac{5}{9}]_5$	$[\frac{8}{9}]_8$	$[\frac{5}{9}]_8$	2	224	24
4:	$[1]_\circ$	$[1]_\circ$	$[0]_0$	1	64	8
6:	$[1]_\circ$	$[0]_0$	$[1]_\circ$	2	160	20
8:	$[\frac{2}{3}]_6$	$[0]_0$	$[\frac{4}{3}]_3$	2	280	40
10:	$[\frac{7}{9}]_7$	$[\frac{7}{9}]_7$	$[\frac{4}{9}]_1$	1	320	40
12:	$[\frac{8}{9}]_8$	$[\frac{8}{9}]_8$	$[\frac{2}{9}]_5$	1	60	12
5:	$\llbracket 2\rrbracket_6$	$\llbracket 2\rrbracket_3$	$\llbracket 0\rrbracket_0$		4312	546
1:	$[\frac{5}{3}]_5$	$[\frac{1}{3}]_8$	$[0]_8$	2^*	324	36
2:	$[2]_6$	$[0]_0$	$[0]_3$	2^*	84	9
3:	$[1]_3$	$[1]_6$	$[0]_0$	1^{**}	400	60
4:	$[\frac{4}{3}]_7$	$[\frac{2}{3}]_1$	$[0]_7$	2^*	324	36
5:	$[2]_\bullet$	$[0]_0$	$[0]_0$	2	45	9
7:	$[\frac{4}{3}]_4$	$[\frac{2}{3}]_1$	$[0]_1$	2	405	45
9:	$[1]_6$	$[1]_3$	$[0]_0$	1	324	54

2.14. **The lattice** $N(3\mathbf{D}_8)$**.** There are 5 classes; $b(\mathfrak{O}) \leqslant 498$.

1:	$\llbracket 4\rrbracket^2_\bullet$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$		7512	498
1:	$[2]^-_\bullet$	$[0]_0$	$[0]_0$	1^{**}	280	18
2:	$[2]_\circ$	$[0]_\circ$	$[0]_0$	2^{**}	1568	112
3:	$[2]_2$	$[0]_1$	$[0]_2$	2^{**}	2048	128
2:	$\llbracket 2\rrbracket_3$	$\llbracket 2\rrbracket_3$	$\llbracket 0\rrbracket_0$		4152	$528 \to 416$
1:	$[2]_3$	$[0]_0$	$[0]_3$	2^*	128	8
2:	$[\frac{3}{2}]_1$	$[\frac{1}{2}]_2$	$[0]_2$	2^*	1024	$128 \to 72$ *
3:	$[2]_\bullet$	$[0]_0$	$[0]_0$	2	70	8
5:	$[1]_3$	$[1]_3$	$[0]_0$	1	784	112
3:	$\llbracket 2\rrbracket_\circ$	$\llbracket 2\rrbracket_\circ$	$\llbracket 0\rrbracket_0$		4632	$436 \to 384$
1:	$[2]_\circ$	$[0]_0$	$[0]_\circ$	2^*	112	8
2:	$[1]_3$	$[1]_3$	$[0]_0$	1^{**}	1024	96
3:	$[1]_1$	$[1]_2$	$[0]_2$	2^{**}	1024	$96 \to 70$ *

4:	$[1]_\circ$	$[1]_\circ$	$[0]_0$	1^{**}	576	72
5:	$[1]_2$	$[1]_2$	$[0]_1$	1^{**}	512	32
6:	$[2]_\bullet$	$[0]_0$	$[0]_0$	2	60	6
8:	$[2]^2_\bullet$	$[0]_0$	$[0]_0$	2	2	1
4:	$\llbracket 4\rrbracket_\bullet$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$		5592	$498 \to 380$
1:	$[2]^+_\bullet$	$[0]_0$	$[0]_0$	1^{**}	144	16
2:	$[2]_3$	$[0]_3$	$[0]_0$	2^{**}	1024	64
3:	$[2]_1$	$[0]_2$	$[0]_2$	1^{**}	2048	$256 \to 138\,*$
4:	$[2]_\circ$	$[0]_\circ$	$[0]_0$	2^{**}	672	48
5:	$[2]^2_\bullet$	$[0]_0$	$[0]_0$	1	4	1
5:	$\llbracket 1\rrbracket_2$	$\llbracket 1\rrbracket_2$	$\llbracket 2\rrbracket_1$		4152	$460 \to 340$
1:	$[2]^2_\bullet$	$[0]_0$	$[0]_0$	2	1	1
3:	$[1]_\circ$	$[1]_\circ$	$[0]_0$	1	196	$28 \to 16\,*$
5:	$[1]_\circ$	$[0]_0$	$[1]_\circ$	2	392	$56 \to 32\,*$
7:	$[\frac12]_3$	$[0]_0$	$[\frac32]_3$	2	512	40
9:	$[1]_2$	$[1]_2$	$[0]_1$	1	70	8

2.15. **The lattice $N(4\mathbf{D}_6)$.** There are 5 classes; $b(\mathfrak{O}) \leqslant 634$.

1:	$\llbracket 4\rrbracket^2_\bullet$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$		6680	$764 \to 634$
1:	$[2]^-_\bullet$	$[0]_0$	$[0]_0$	$[0]_0$	1^{**}	80	8
2:	$[2]_\circ$	$[0]_\circ$	$[0]_0$	$[0]_0$	3^{**}	600	60
3:	$[2]_2$	$[0]_3$	$[0]_0$	$[0]_1$	3^{**}	1024	96
4:	$[2]_2$	$[0]_2$	$[0]_2$	$[0]_2$	1^{**}	1728	$288 \to 158\,*$
2:	$\llbracket 1\rrbracket_2$	$\llbracket 1\rrbracket_2$	$\llbracket 1\rrbracket_2$	$\llbracket 1\rrbracket_2$		4280	$632 \to 536$
1:	$[\frac12]_3$	$[\frac12]_1$	$[0]_0$	$[1]_2$	12^{*}	256	32
2:	$[2]^2_\bullet$	$[0]_0$	$[0]_0$	$[0]_0$	4	1	1
4:	$[1]_\circ$	$[1]_\circ$	$[0]_0$	$[0]_0$	6	100	$20 \to 12\,*$
3:	$\llbracket 4\rrbracket_\bullet$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$		5720	$546 \to 534$
1:	$[2]^+_\bullet$	$[0]_0$	$[0]_0$	$[0]_0$	2^{**}	12	2
2:	$[2]_\circ$	$[0]_\circ$	$[0]_0$	$[0]_0$	3^{**}	360	36
3:	$[2]_3$	$[0]_1$	$[0]_0$	$[0]_2$	6^{**}	768	$72 \to 70\,*$
4:	$[2]^2_\bullet$	$[0]_0$	$[0]_0$	$[0]_0$	1	4	1
4:	$\llbracket \frac32\rrbracket_3$	$\llbracket 1\rrbracket_2$	$\llbracket \frac32\rrbracket_1$	$\llbracket 0\rrbracket_0$		4280	$632 \to 512$
1:	$[\frac32]_3$	$[\frac12]_1$	$[0]_0$	$[0]_2$	2^{*}	192	$24 \to 18\,*$
2:	$[1]_1$	$[\frac12]_3$	$[\frac12]_2$	$[0]_0$	2^{*}	576	72
3:	$[2]_\bullet$	$[0]_0$	$[0]_0$	$[0]_0$	2	15	3

5:	$[1]_\circ$	$[1]_\circ$	$[0]_0$	$[0]_0$	2	150	$30 \to 18$ *
7:	$[1]_\circ$	$[0]_0$	$[1]_\circ$	$[0]_0$	1	225	45
9:	$[\frac{3}{2}]_3$	$[-1]_2$	$[\frac{3}{2}]_1$	$[0]_0$	1	1	1
11:	$[\frac{3}{2}]_3$	$[0]_0$	$[\frac{1}{2}]_2$	$[0]_1$	2	192	18
13:	$[1]_1$	$[0]_0$	$[1]_3$	$[0]_2$	1	432	$72 \to 42$ *
5:	$[\![2]\!]_\circ$	$[\![2]\!]_\circ$	$[\![0]\!]_0$	$[\![0]\!]_0$		4760	$636 \to 498$
1:	$[2]_\circ$	$[0]_0$	$[0]_\circ$	$[0]_0$	4^*	60	6
2:	$[1]_\circ$	$[1]_\circ$	$[0]_0$	$[0]_0$	1^{**}	256	48
3:	$[1]_3$	$[1]_1$	$[0]_0$	$[0]_2$	2^{**}	768	$128 \to 80$ *
4:	$[1]_3$	$[1]_2$	$[0]_1$	$[0]_0$	4^{**}	512	48
5:	$[1]_2$	$[1]_2$	$[0]_2$	$[0]_2$	1^{**}	576	$96 \to 54$ *
6:	$[2]_\bullet$	$[0]_0$	$[0]_0$	$[0]_0$	2	24	4
8:	$[2]_\bullet^2$	$[0]_0$	$[0]_0$	$[0]_0$	2	2	1

3. Sufficient estimates on combinatorial orbits

For three lattices, the naïve combinatorial estimates of **§3 do not suffice to establish the inequality $b(\mathfrak{O}_h) < 720$, and we need to compute (some of) the bounds $b(\mathfrak{o})$ by brute force §1.4. After each table, for each offending vector h, we indicate the choice of the parameters (in the notation of §1.4) used in the computation. These parameters are encoded in the function

(3.1) `4_`⟨*lattice name*⟩`_`h`_auto()` defined in the file `"4RS.txt"`,

h referring to the index of the pair $N \ni h$ in the tables.

Together with the previous section, these data constitute the proof of **Theorem 4.1.

3.1. The lattice $N(2\mathbf{A}_7 \oplus 2\mathbf{D}_5)$. There are 12 classes; $b(\mathfrak{O}) \leqslant 702$.

1:	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![4]\!]^2_\bullet$	$[\![0]\!]_0$		6264	$796 \to 702$
1:	$[0]_4$	$[0]_0$	$[2]_2$	$[0]_2$	2^{**}	700	$80 \to \underline{64}$ ✓
2:	$[0]_\circ$	$[0]_0$	$[2]_\circ$	$[0]_0$	2^{**}	448	$56 \to \underline{44}$ ✓
3:	$[0]_6$	$[0]_6$	$[2]_2$	$[0]_0$	2^{*}	784	112
4:	$[0]_7$	$[0]_1$	$[2]_2$	$[0]_3$	2^{*}	1024	$128 \to 113$ ‡
5:	$[0]_0$	$[0]_0$	$[2]^-_\bullet$	$[0]_0$	1^{**}	32	4
6:	$[0]_0$	$[0]_0$	$[2]_\circ$	$[0]_\circ$	1^{**}	320	$40 \to \underline{32}$ ✓
2:	$[\![2]\!]_4$	$[\![0]\!]_0$	$[\![1]\!]_2$	$[\![1]\!]_2$		4344	$740 \to 680$
1:	$[1]_6$	$[0]_6$	$[1]_2$	$[0]_0$	4^{*}	168	24
2:	$[1]_6$	$[0]_0$	$[\frac{1}{2}]_1$	$[\frac{1}{2}]_1$	2^{**}	384	96
3:	$[2]_\bullet$	$[0]_0$	$[0]_0$	$[0]_0$	1	36	8
5:	$[2]_4$	$[0]_4$	$[0]_0$	$[0]_0$	1	70	8
7:	$[2]_4$	$[0]_0$	$[1]_2$	$[-1]_2$	2	1	1
9:	$[2]_4$	$[0]_0$	$[0]_2$	$[0]_2$	1	64	$16 \to 10$ *
11:	$[1]_4$	$[0]_0$	$[1]_2$	$[0]_2$	2	128	$32 \to 20$ *
13:	$[\frac{3}{2}]_5$	$[0]_7$	$[0]_0$	$[\frac{1}{2}]_3$	4	256	32
3:	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![4]\!]_\bullet$	$[\![0]\!]_0$		5784	$692 \to 676$
1:	$[0]_\circ$	$[0]_0$	$[2]_\circ$	$[0]_0$	2^{**}	336	42
2:	$[0]_5$	$[0]_1$	$[2]_3$	$[0]_0$	4^{*}	448	48
3:	$[0]_6$	$[0]_0$	$[2]_1$	$[0]_1$	4^{*}	448	56
4:	$[0]_7$	$[0]_7$	$[2]_3$	$[0]_2$	2^{*}	640	$80 \to 72$ *
5:	$[0]_0$	$[0]_0$	$[2]_\circ$	$[0]_\circ$	1^{**}	240	30
6:	$[0]_0$	$[0]_0$	$[2]_\bullet$	$[0]_0$	1	4	1
4:	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![2]\!]_\circ$	$[\![2]\!]_\circ$		4824	656
1:	$[0]_4$	$[0]_0$	$[1]_2$	$[1]_2$	2^{**}	280	32
2:	$[0]_\circ$	$[0]_0$	$[2]_\circ$	$[0]_0$	4^{*}	56	7
3:	$[0]_6$	$[0]_0$	$[1]_1$	$[1]_1$	4^{*}	448	64
4:	$[0]_7$	$[0]_7$	$[1]_3$	$[1]_2$	4^{*}	512	64

5:	$[0]_0$	$[0]_0$	$[1]_\circ$	$[1]_\circ$	1^{**}	144	36
6:	$[0]_0$	$[0]_0$	$[2]_\bullet$	$[0]_0$	2	12	3
8:	$[0]_0$	$[0]_0$	$[2]_\bullet^2$	$[0]_0$	2	2	1
5:	$[\![4]\!]_\bullet$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![0]\!]_0$		5304	730 $\to$ 646
1:	$[2]_\bullet^+$	$[0]_0$	$[0]_0$	$[0]_0$	2^*	6	2
2:	$[2]_\bullet^-$	$[0]_0$	$[0]_0$	$[0]_0$	1^{**}	48	10
3:	$[2]_4$	$[0]_4$	$[0]_0$	$[0]_0$	1^{**}	420	48
4:	$[2]_4$	$[0]_0$	$[0]_2$	$[0]_2$	1^{**}	600	120 $\to$ 68 $*$
5:	$[2]_\circ$	$[0]_\circ$	$[0]_0$	$[0]_0$	1^{**}	224	28
6:	$[2]_\circ$	$[0]_0$	$[0]_\circ$	$[0]_0$	2^{**}	160	20
7:	$[2]_5$	$[0]_7$	$[0]_0$	$[0]_3$	4^*	512	64
8:	$[2]_6$	$[0]_6$	$[0]_2$	$[0]_0$	4^*	280	40 $\to$ 32 $*$
9:	$[2]_6$	$[0]_0$	$[0]_1$	$[0]_1$	2^*	256	32
6:	$[\![\frac{3}{2}]\!]_6$	$[\![0]\!]_0$	$[\![\frac{5}{4}]\!]_1$	$[\![\frac{5}{4}]\!]_1$		4344	644
1:	$[\frac{3}{4}]_3$	$[0]_7$	$[\frac{5}{4}]_1$	$[0]_0$	2^*	160	20
2:	$[\frac{3}{4}]_7$	$[0]_5$	$[0]_0$	$[\frac{5}{4}]_1$	2^*	112	12
3:	$[\frac{3}{4}]_7$	$[0]_7$	$[\frac{3}{4}]_3$	$[\frac{1}{2}]_2$	2^*	400	50
4:	$[2]_\bullet$	$[0]_0$	$[0]_0$	$[0]_0$	1	15	3
6:	$[1]_4$	$[0]_0$	$[\frac{1}{2}]_2$	$[\frac{1}{2}]_2$	1	375	75
8:	$[1]_\circ$	$[0]_0$	$[1]_\circ$	$[0]_0$	2	120	20
10:	$[\frac{5}{4}]_5$	$[0]_7$	$[0]_0$	$[\frac{3}{4}]_3$	2	240	30
12:	$[\frac{3}{2}]_6$	$[0]_6$	$[\frac{1}{2}]_2$	$[0]_0$	2	140	20
14:	$[\frac{3}{2}]_6$	$[0]_0$	$[\frac{5}{4}]_1$	$[\frac{\bar{3}}{4}]_1$	2	5	1
16:	$[\frac{3}{2}]_6$	$[0]_0$	$[\frac{1}{4}]_1$	$[\frac{1}{4}]_1$	1	100	20
7:	$[\![\frac{3}{2}]\!]_6$	$[\![\frac{3}{2}]\!]_6$	$[\![1]\!]_2$	$[\![0]\!]_0$		4344	720 $\to$ 640
1:	$[\frac{5}{4}]_5$	$[\frac{1}{4}]_1$	$[\frac{1}{2}]_3$	$[0]_0$	2^*	288	48
2:	$[\frac{3}{4}]_3$	$[\frac{3}{4}]_7$	$[\frac{1}{2}]_1$	$[0]_0$	2^{**}	320	48
3:	$[\frac{3}{2}]_6$	$[0]_0$	$[\frac{1}{2}]_1$	$[0]_1$	2^*	128	16
4:	$[\frac{3}{4}]_7$	$[\frac{3}{4}]_7$	$[\frac{1}{2}]_3$	$[0]_2$	1^{**}	320	64 $\to$ 40 $*$
5:	$[2]_\bullet$	$[0]_0$	$[0]_0$	$[0]_0$	2	15	3
7:	$[1]_4$	$[1]_4$	$[0]_0$	$[0]_0$	1	225	45
9:	$[1]_4$	$[0]_0$	$[1]_2$	$[0]_2$	2	150	30 $\to$ 18 $*$
11:	$[1]_\circ$	$[1]_\circ$	$[0]_0$	$[0]_0$	1	144	24
13:	$[1]_\circ$	$[0]_0$	$[1]_\circ$	$[0]_0$	2	96	16 $\to$ 14 $*$
15:	$[\frac{5}{4}]_5$	$[\frac{3}{4}]_7$	$[0]_0$	$[0]_3$	2	192	24
17:	$[\frac{3}{2}]_6$	$[\frac{3}{2}]_6$	$[-1]_2$	$[0]_0$	1	1	1

8:	$\llbracket 2\rrbracket_4$	$\llbracket 2\rrbracket_4$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$		4344	$744 \to 632$
1:	$[2]_4$	$[0]_0$	$[0]_2$	$[0]_2$	2^*	100	$20 \to 12\,*$
2:	$[\frac{3}{2}]_5$	$[\frac{1}{2}]_7$	$[0]_0$	$[0]_3$	8^*	256	32
3:	$[1]_6$	$[1]_6$	$[0]_2$	$[0]_0$	4^{**}	360	$72 \to 48\,*$
4:	$[2]_\bullet$	$[0]_0$	$[0]_0$	$[0]_0$	2	36	8
6:	$[1]_4$	$[1]_4$	$[0]_0$	$[0]_0$	1	256	64
9:	$\llbracket \frac{7}{8}\rrbracket_7$	$\llbracket \frac{7}{8}\rrbracket_7$	$\llbracket \frac{5}{4}\rrbracket_3$	$\llbracket 1\rrbracket_2$		4344	$632 \to 624$
1:	$[1]_\circ$	$[1]_\circ$	$[0]_0$	$[0]_0$	1	49	7
3:	$[1]_\circ$	$[0]_0$	$[1]_\circ$	$[0]_0$	2	70	10
5:	$[1]_\circ$	$[0]_0$	$[0]_0$	$[1]_\circ$	2	56	8
7:	$[\frac{1}{2}]_4$	$[0]_0$	$[\frac{1}{2}]_2$	$[1]_2$	2	175	25
9:	$[\frac{5}{8}]_5$	$[\frac{7}{8}]_7$	$[0]_0$	$[\frac{1}{2}]_3$	2	168	24
11:	$[\frac{5}{8}]_5$	$[\frac{1}{8}]_1$	$[\frac{5}{4}]_3$	$[0]_0$	2	147	21
13:	$[\frac{3}{4}]_6$	$[\frac{3}{4}]_6$	$[\frac{1}{2}]_2$	$[0]_0$	1	245	35
15:	$[\frac{3}{4}]_6$	$[0]_0$	$[\frac{3}{4}]_1$	$[\frac{1}{2}]_1$	2	280	40
17:	$[\frac{7}{8}]_7$	$[\frac{7}{8}]_7$	$[\frac{5}{4}]_3$	$[-1]_2$	1	1	1
19:	$[\frac{7}{8}]_7$	$[\frac{7}{8}]_7$	$[\frac{1}{4}]_3$	$[0]_2$	1	80	$16 \to 12\,*$
21:	$[\frac{7}{8}]_7$	$[\frac{7}{8}]_7$	$[\frac{\bar{3}}{4}]_3$	$[1]_2$	1	5	1
10:	$\llbracket 2\rrbracket_\circ$	$\llbracket 0\rrbracket_0$	$\llbracket 2\rrbracket_\circ$	$\llbracket 0\rrbracket_0$		4824	$630 \to 608$
1:	$[1]_4$	$[0]_0$	$[1]_2$	$[0]_2$	1^{**}	400	$60 \to 46\,*$
2:	$[1]_\circ^+$	$[0]_0$	$[1]_\circ$	$[0]_0$	2^*	72	12
3:	$[1]_5$	$[0]_1$	$[1]_3$	$[0]_0$	2^*	480	60
4:	$[1]_6$	$[0]_6$	$[1]_2$	$[0]_0$	2^*	336	48
5:	$[1]_6$	$[0]_0$	$[1]_1$	$[0]_1$	2^*	384	48
6:	$[1]_7$	$[0]_3$	$[1]_1$	$[0]_0$	2^*	224	24
7:	$[1]_7$	$[0]_7$	$[1]_3$	$[0]_2$	2^*	320	$40 \to 36\,*$
8:	$[1]_7$	$[0]_1$	$[1]_2$	$[0]_3$	2^*	256	32
9:	$[2]_\bullet$	$[0]_0$	$[0]_0$	$[0]_0$	1	30	5
11:	$[2]_\circ$	$[0]_\circ$	$[0]_0$	$[0]_0$	1	56	7
13:	$[2]_\circ$	$[0]_0$	$[0]_\circ^+$	$[0]_0$	1	12	3
15:	$[2]_\circ$	$[0]_0$	$[0]_\circ^-$	$[0]_0$	1	2	1
17:	$[2]_\circ$	$[0]_0$	$[0]_0$	$[0]_\circ$	1	40	5
11:	$\llbracket 2\rrbracket_\circ$	$\llbracket 2\rrbracket_\circ$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$		4824	$684 \to 604$
1:	$[2]_\circ$	$[0]_0$	$[0]_\circ$	$[0]_0$	4^*	40	5
2:	$[1]_4$	$[1]_4$	$[0]_0$	$[0]_0$	1^{**}	400	60
3:	$[1]_\circ^+$	$[1]_\circ^+$	$[0]_0$	$[0]_0$	4^*	36	6
4:	$[1]_5$	$[1]_7$	$[0]_0$	$[0]_3$	8^*	240	30

5:	$[1]_6$	$[1]_6$	$[0]_2$	$[0]_0$	4^*	360	$60 \to 42\,*$
6:	$[1]_7$	$[1]_7$	$[0]_3$	$[0]_2$	4^*	160	$20 \to 18\,*$
7:	$[2]_\bullet$	$[0]_0$	$[0]_0$	$[0]_0$	2	30	5
12:	$[\![\frac{7}{8}]\!]_7$	$[\![\frac{15}{8}]\!]_5$	$[\![0]\!]_0$	$[\![\frac{5}{4}]\!]_1$		4344	$652 \to 600$
1:	$[1]_\circ$	$[1]_\circ$	$[0]_0$	$[0]_0$	1	105	15
3:	$[1]_\circ$	$[0]_0$	$[0]_0$	$[1]_\circ$	1	70	10
5:	$[\frac{1}{2}]_4$	$[\frac{3}{2}]_4$	$[0]_0$	$[0]_0$	1	175	25
7:	$[\frac{5}{8}]_5$	$[\frac{5}{8}]_7$	$[0]_0$	$[\frac{3}{4}]_3$	1	315	45
9:	$[\frac{3}{4}]_6$	$[\frac{5}{4}]_6$	$[0]_2$	$[0]_0$	1	210	$30 \to 24\,*$
11:	$[\frac{3}{4}]_6$	$[\frac{3}{4}]_2$	$[0]_0$	$[\frac{1}{2}]_2$	1	350	50
13:	$[\frac{3}{4}]_6$	$[0]_0$	$[0]_1$	$[\frac{5}{4}]_1$	1	112	14
15:	$[\frac{7}{8}]_7$	$[\frac{15}{8}]_5$	$[0]_0$	$[\frac{\bar{3}}{4}]_1$	1	5	1
17:	$[\frac{7}{8}]_7$	$[\frac{9}{8}]_3$	$[0]_1$	$[0]_0$	1	160	20
19:	$[\frac{7}{8}]_7$	$[\frac{7}{8}]_5$	$[0]_0$	$[\frac{1}{4}]_1$	1	150	30
21:	$[\frac{7}{8}]_7$	$[\frac{\bar{1}}{8}]_5$	$[0]_0$	$[\frac{5}{4}]_1$	1	30	6
23:	$[\frac{7}{8}]_7$	$[\frac{5}{8}]_7$	$[0]_3$	$[\frac{1}{2}]_2$	1	240	30
25:	$[\frac{7}{8}]_7$	$[\frac{3}{8}]_1$	$[0]_2$	$[\frac{3}{4}]_3$	1	250	$50 \to 30\,*$

3.1.1. *Vector* 1.

```
1  := rec(fixed :=  4, min := 64,  chain := 3),
2  := rec(fixed := -3, min := 44,  chain := 2),
4  := rec(fixed :=  4, min := 114, chain := 2),
6  := rec(fixed := -3, min := 32),
```

3.2. **The lattice $N(4\mathbf{A}_6)$.** There are 4 classes; $b(\mathfrak{O}) \leqslant 704$.

1:	$[\![4]\!]_\bullet$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![0]\!]_0$		5336	$766 \to 628$
1:	$[2]_\bullet^+$	$[0]_0$	$[0]_0$	$[0]_0$	2^*	3	$\underline{1}$
2:	$[2]_\bullet^-$	$[0]_0$	$[0]_0$	$[0]_0$	1^{**}	24	$\underline{6}$
3:	$[2]_\circ$	$[0]_\circ$	$[0]_0$	$[0]_0$	3^{**}	168	$24 \to \underline{22}$ ✓
4:	$[2]_4$	$[0]_2$	$[0]_6$	$[0]_0$	6^*	441	$63 \to \underline{49}$ ✓
5:	$[2]_5$	$[0]_4$	$[0]_0$	$[0]_6$	6^*	245	$35 \to \underline{27}$ ✓
6:	$[2]_5$	$[0]_1$	$[0]_1$	$[0]_1$	2^*	343	49
2:	$[\![\frac{6}{7}]\!]_6$	$[\![\frac{6}{7}]\!]_6$	$[\![\frac{6}{7}]\!]_1$	$[\![\frac{10}{7}]\!]_5$		4376	$748 \to 604$
1:	$[1]_\circ$	$[1]_\circ$	$[0]_0$	$[0]_0$	3	36	$\underline{6}$
3:	$[1]_\circ$	$[0]_0$	$[0]_0$	$[1]_\circ$	3	60	$\underline{10}$
5:	$[\frac{4}{7}]_4$	$[\frac{6}{7}]_6$	$[0]_0$	$[\frac{4}{7}]_2$	3	150	$30 \to \underline{20}$ ✓
7:	$[\frac{4}{7}]_4$	$[0]_0$	$[\frac{5}{7}]_2$	$[\frac{5}{7}]_6$	3	180	$30 \to \underline{24}$ ✓
9:	$[\frac{3}{7}]_3$	$[\frac{5}{7}]_5$	$[\frac{6}{7}]_1$	$[0]_0$	3	120	$18 \to \underline{16}$ ✓
11:	$[\frac{5}{7}]_5$	$[\frac{1}{7}]_1$	$[\frac{6}{7}]_1$	$[\frac{2}{7}]_1$	3	180	$30 \to \underline{24}$ ✓
13:	$[\frac{6}{7}]_6$	$[\frac{6}{7}]_6$	$[\frac{6}{7}]_1$	$[\frac{\bar{4}}{7}]_5$	1	10	$\underline{2}$
3:	$[\![2]\!]_\circ$	$[\![2]\!]_\circ$	$[\![0]\!]_0$	$[\![0]\!]_0$		4856	704
1:	$[1]_\circ^+$	$[1]_\circ^+$	$[0]_0$	$[0]_0$	4^*	25	5
2:	$[1]_4$	$[1]_2$	$[0]_6$	$[0]_0$	4^*	350	50
3:	$[1]_4$	$[1]_6$	$[0]_0$	$[0]_2$	4^*	210	30
4:	$[1]_5$	$[1]_6$	$[0]_4$	$[0]_0$	4^*	175	25
5:	$[1]_5$	$[1]_1$	$[0]_1$	$[0]_1$	4^*	245	35
6:	$[1]_6$	$[1]_6$	$[0]_1$	$[0]_5$	4^*	147	21
7:	$[2]_\bullet$	$[0]_0$	$[0]_0$	$[0]_0$	2	20	4
9:	$[2]_\circ$	$[0]_0$	$[0]_\circ$	$[0]_0$	2	42	6
4:	$[\![\frac{6}{7}]\!]_6$	$[\![\frac{12}{7}]\!]_3$	$[\![\frac{10}{7}]\!]_2$	$[\![0]\!]_0$		4376	688
1:	$[\frac{3}{7}]_3$	$[\frac{6}{7}]_5$	$[\frac{5}{7}]_1$	$[0]_0$	1^{**}	240	36
2:	$[1]_\circ$	$[1]_\circ$	$[0]_0$	$[0]_0$	1	72	12
4:	$[1]_\circ$	$[0]_0$	$[1]_\circ$	$[0]_0$	1	60	10
6:	$[\frac{4}{7}]_4$	$[\frac{8}{7}]_2$	$[\frac{2}{7}]_6$	$[0]_0$	1	225	45
8:	$[\frac{4}{7}]_4$	$[0]_0$	$[\frac{10}{7}]_2$	$[0]_6$	1	105	15
10:	$[\frac{5}{7}]_5$	$[\frac{9}{7}]_4$	$[0]_0$	$[0]_6$	1	168	24
12:	$[\frac{5}{7}]_5$	$[\frac{4}{7}]_1$	$[\frac{5}{7}]_1$	$[0]_1$	1	252	36
14:	$[\frac{5}{7}]_5$	$[\frac{3}{7}]_6$	$[\frac{6}{7}]_4$	$[0]_0$	1	240	40
16:	$[\frac{6}{7}]_6$	$[\frac{12}{7}]_3$	$[\frac{\bar{4}}{7}]_2$	$[0]_0$	1	10	2
18:	$[\frac{6}{7}]_6$	$[\frac{5}{7}]_3$	$[\frac{3}{7}]_2$	$[0]_0$	1	120	24

20:	$\left[\frac{6}{7}\right]_6$	$\left[\frac{\bar{2}}{7}\right]_3$	$\left[\frac{10}{7}\right]_2$	$\left[0\right]_0$	1	18	4
22:	$\left[\frac{6}{7}\right]_6$	$\left[\frac{8}{7}\right]_2$	$\left[0\right]_0$	$\left[0\right]_3$	1	105	15
24:	$\left[\frac{6}{7}\right]_6$	$\left[\frac{6}{7}\right]_5$	$\left[\frac{2}{7}\right]_6$	$\left[0\right]_1$	1	210	30
26:	$\left[\frac{6}{7}\right]_6$	$\left[\frac{4}{7}\right]_1$	$\left[\frac{4}{7}\right]_5$	$\left[0\right]_6$	1	210	30
28:	$\left[\frac{6}{7}\right]_6$	$\left[\frac{3}{7}\right]_6$	$\left[\frac{5}{7}\right]_1$	$\left[0\right]_5$	1	168	24
30:	$\left[\frac{6}{7}\right]_6$	$\left[0\right]_0$	$\left[\frac{8}{7}\right]_3$	$\left[0\right]_2$	1	105	15

3.2.1. *Vector 1.*

```
1  := rec(),
2  := rec(),
3  := rec(),
4  := rec(fixed := 3,  min := 49, chain := [2, 2]),
5  := rec(fixed := 4,  min := 27, chain := [3, 2]),
```

3.2.2. *Vector 2.*

```
1  := rec(),
3  := rec(),
5  := rec(fixed := 4,  min := 20, chain := 3),
7  := rec(fixed := 1,  min := 24, chain := [3, 3]),
9  := rec(fixed := 1,  min := 16, chain := [3, 3]),
11 := rec(fixed := 4,  min := 24, chain := [2, 2]),
13 := rec(),
```

3.3. **The lattice** $N(4\mathbf{A}_5 \oplus \mathbf{D}_4)$**.** There are 8 classes; $b(\mathfrak{O}) \leqslant 716$.

1:	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$	$\llbracket 4\rrbracket_\bullet^3$		5848	$962 \to 706$
1:	$[0]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	$[2]_\circ$	4^{**}	180	$30 \to \underline{24}$ ✓
2:	$[0]_3$	$[0]_0$	$[0]_0$	$[0]_3$	$[2]_3$	2^{**}	400	$60 \to \underline{48}$ ✓
3:	$[0]_4$	$[0]_1$	$[0]_5$	$[0]_0$	$[2]_3$	8^*	540	$90 \to \underline{64}$ ✓
4:	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[2]_\bullet^2$	2^*	4	$\underline{1}$
2:	$\llbracket 4\rrbracket_\bullet$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$		5368	$926 \to 690$
1:	$[2]_\bullet^+$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	2^*	1	$\underline{1}$
2:	$[2]_\bullet^-$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	1^{**}	8	$\underline{2}$
3:	$[2]_\circ$	$[0]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	3^{**}	120	$20 \to \underline{18}$ ✓
4:	$[2]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_\circ$	1^{**}	96	$16 \to \underline{14}$ ✓
5:	$[2]_3$	$[0]_3$	$[0]_0$	$[0]_0$	$[0]_1$	3^{**}	320	$48 \to \underline{40}$ ✓
6:	$[2]_3$	$[0]_5$	$[0]_5$	$[0]_5$	$[0]_0$	2^*	432	$72 \to \underline{54}$ ✓
7:	$[2]_4$	$[0]_4$	$[0]_2$	$[0]_0$	$[0]_0$	6^*	225	$45 \to \underline{29}$ ✓
8:	$[2]_4$	$[0]_5$	$[0]_0$	$[0]_1$	$[0]_2$	6^*	288	$48 \to \underline{36}$ ✓
3:	$\llbracket \frac{5}{6}\rrbracket_5$	$\llbracket \frac{5}{6}\rrbracket_5$	$\llbracket \frac{4}{3}\rrbracket_4$	$\llbracket 0\rrbracket_0$	$\llbracket 1\rrbracket_1$		4408	$824 \to 704$
1:	$[\frac{2}{3}]_4$	$[\frac{1}{6}]_1$	$[\frac{2}{3}]_5$	$[0]_0$	$[\frac{1}{2}]_3$	2^*	200	$40 \to \underline{32}$ ✓
2:	$[\frac{5}{6}]_5$	$[0]_0$	$[\frac{2}{3}]_2$	$[0]_1$	$[\frac{1}{2}]_3$	2^*	144	$24 \to \underline{22}$ ✓
3:	$[\frac{5}{6}]_5$	$[0]_0$	$[\frac{2}{3}]_5$	$[0]_4$	$[\frac{1}{2}]_2$	2^*	120	$24 \to \underline{18}$ ✓
4:	$[1]_\circ$	$[1]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	1	25	$\underline{5}$
6:	$[1]_\circ$	$[0]_0$	$[1]_\circ$	$[0]_0$	$[0]_0$	2	40	$\underline{8}$
8:	$[1]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	$[1]_\circ$	2	30	$\underline{6}$
10:	$[\frac{1}{2}]_3$	$[\frac{1}{2}]_3$	$[0]_0$	$[0]_0$	$[1]_1$	1	100	$20 \to \underline{16}$ ✓
12:	$[\frac{1}{2}]_3$	$[\frac{5}{6}]_5$	$[\frac{2}{3}]_5$	$[0]_5$	$[0]_0$	2	120	$20 \to \underline{18}$ ✓
14:	$[\frac{1}{2}]_3$	$[0]_0$	$[1]_3$	$[0]_0$	$[\frac{1}{2}]_2$	2	160	$32 \to \underline{26}$ ✓
16:	$[\frac{2}{3}]_4$	$[\frac{2}{3}]_4$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	1	150	$30 \to \underline{24}$ ✓
18:	$[\frac{2}{3}]_4$	$[\frac{5}{6}]_5$	$[0]_0$	$[0]_1$	$[\frac{1}{2}]_2$	2	120	$20 \to \underline{18}$ ✓
20:	$[\frac{2}{3}]_4$	$[0]_0$	$[\frac{4}{3}]_4$	$[0]_2$	$[0]_0$	2	75	$15 \to \underline{11}$ ✓
22:	$[\frac{2}{3}]_4$	$[0]_0$	$[\frac{1}{3}]_1$	$[0]_5$	$[1]_1$	2	120	$20 \to \underline{18}$ ✓
24:	$[\frac{5}{6}]_5$	$[\frac{5}{6}]_5$	$[\frac{4}{3}]_4$	$[0]_0$	$[-1]_1$	1	1	$\underline{1}$
26:	$[\frac{5}{6}]_5$	$[\frac{5}{6}]_5$	$[\frac{1}{3}]_4$	$[0]_0$	$[0]_1$	1	48	$12 \to \underline{10}$ $*$
28:	$[\frac{5}{6}]_5$	$[\frac{5}{6}]_5$	$[\frac{\bar{2}}{3}]_4$	$[0]_0$	$[1]_1$	1	6	$\underline{2}$
30:	$[\frac{5}{6}]_5$	$[\frac{5}{6}]_5$	$[\frac{1}{3}]_1$	$[0]_3$	$[0]_0$	1	80	$\underline{12}$
4:	$\llbracket 2\rrbracket_\circ$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$	$\llbracket 2\rrbracket_\circ$		4888	$814 \to 688$
1:	$[1]_\circ^+$	$[0]_0$	$[0]_0$	$[0]_0$	$[1]_\circ$	2^*	32	$\underline{8}$
2:	$[1]_3$	$[0]_3$	$[0]_0$	$[0]_0$	$[1]_1$	3^{**}	240	$36 \to \underline{34}$ ✓

3:	$[1]_4$	$[0]_5$	$[0]_0$	$[0]_1$	$[1]_2$	6^*	288	$48 \to \underline{40}\,\checkmark$
4:	$[1]_5$	$[0]_4$	$[0]_0$	$[0]_5$	$[1]_3$	12^*	180	$30 \to \underline{24}\,\checkmark$
5:	$[2]_\bullet$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	1	12	$\underline{3}$
7:	$[2]_\circ$	$[0]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	3	30	$\underline{5}$
9:	$[2]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_\circ^+$	3	2	$\underline{1}$
5:	$[\![\frac{3}{2}]\!]_3$	$[\![\frac{3}{2}]\!]_3$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![1]\!]_1$		4408	$836 \to 712$
1:	$[\frac{3}{2}]_3$	$[0]_0$	$[0]_3$	$[0]_0$	$[\frac{1}{2}]_2$	4^*	80	$\underline{12}$
2:	$[1]_4$	$[\frac{1}{2}]_5$	$[0]_0$	$[0]_1$	$[\frac{1}{2}]_2$	8^*	216	$36 \to \underline{34}\,\checkmark$
3:	$[2]_\bullet$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	2	9	$3 \to \underline{2}\,\checkmark$
5:	$[1]_\circ$	$[1]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	1	81	$27 \to \underline{19}\,\checkmark$
7:	$[1]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	$[1]_\circ$	2	54	$18 \to \underline{12}\,*$
9:	$[\frac{3}{2}]_3$	$[\frac{3}{2}]_3$	$[0]_0$	$[0]_0$	$[-1]_1$	1	1	$\underline{1}$
11:	$[\frac{3}{2}]_3$	$[\frac{1}{2}]_5$	$[0]_5$	$[0]_5$	$[0]_0$	4	108	$18 \to \underline{16}\,\checkmark$
13:	$[1]_4$	$[1]_4$	$[0]_2$	$[0]_0$	$[0]_0$	4	135	$27 \to \underline{21}\,\checkmark$
6:	$[\![2]\!]_\circ$	$[\![2]\!]_\circ$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![0]\!]_0$		4888	$888 \to 690$
1:	$[2]_\circ$	$[0]_0$	$[0]_\circ$	$[0]_0$	$[0]_0$	4^*	30	$\underline{5}$
2:	$[2]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_\circ$	2^*	24	$\underline{4}$
3:	$[1]_\circ^+$	$[1]_\circ^+$	$[0]_0$	$[0]_0$	$[0]_0$	4^*	16	$\underline{4}$
4:	$[1]_3$	$[1]_3$	$[0]_0$	$[0]_0$	$[0]_1$	1^{**}	288	$72 \to \underline{42}\,\checkmark$
5:	$[1]_3$	$[1]_5$	$[0]_5$	$[0]_5$	$[0]_0$	4^*	216	$36 \to \underline{30}\,\checkmark$
6:	$[1]_4$	$[1]_4$	$[0]_2$	$[0]_0$	$[0]_0$	4^*	240	$48 \to \underline{34}\,\checkmark$
7:	$[1]_4$	$[1]_5$	$[0]_0$	$[0]_1$	$[0]_2$	8^*	192	$32 \to \underline{26}\,\checkmark$
8:	$[1]_5$	$[1]_5$	$[0]_4$	$[0]_0$	$[0]_1$	4^*	120	$24 \to \underline{16}\,\checkmark$
9:	$[1]_5$	$[1]_5$	$[0]_1$	$[0]_3$	$[0]_0$	4^*	120	$18 \to \underline{16}\,\checkmark$
10:	$[2]_\bullet$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	2	12	$\underline{3}$
7:	$[\![\frac{4}{3}]\!]_4$	$[\![\frac{4}{3}]\!]_4$	$[\![\frac{4}{3}]\!]_2$	$[\![0]\!]_0$	$[\![0]\!]_0$		4408	$936 \to 716$
1:	$[1]_3$	$[\frac{2}{3}]_5$	$[\frac{1}{3}]_5$	$[0]_5$	$[0]_0$	6^*	192	$32 \to \underline{30}\,\checkmark$
2:	$[\frac{4}{3}]_4$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_4$	$[0]_0$	6^*	90	$18 \to \underline{14}\,\checkmark$
3:	$[\frac{4}{3}]_4$	$[\frac{2}{3}]_5$	$[0]_0$	$[0]_1$	$[0]_2$	6^*	96	$16 \to \underline{14}\,*$
4:	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_4$	$[0]_0$	$[0]_0$	1^{**}	216	$72 \to \underline{38}\,\checkmark$
5:	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_5$	$[\frac{2}{3}]_1$	$[0]_0$	$[0]_3$	3^{**}	192	$48 \to \underline{30}\,\checkmark$
6:	$[\frac{2}{3}]_5$	$[\frac{2}{3}]_5$	$[\frac{2}{3}]_1$	$[0]_3$	$[0]_0$	1^{**}	160	$\underline{24}$
7:	$[2]_\bullet$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	3	6	$\underline{2}$
9:	$[1]_\circ$	$[1]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	3	64	$16 \to \underline{14}\,\checkmark$
11:	$[1]_3$	$[1]_3$	$[0]_0$	$[0]_0$	$[0]_1$	3	128	$32 \to \underline{20}\,*$
8:	$[\![\frac{5}{6}]\!]_5$	$[\![\frac{5}{6}]\!]_5$	$[\![\frac{5}{6}]\!]_1$	$[\![\frac{3}{2}]\!]_3$	$[\![0]\!]_0$		4408	$884 \to 702$

1:	$[1]_\circ$	$[1]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	3	25	$\underline{5}$	
3:	$[1]_\circ$	$[0]_0$	$[0]_0$	$[1]_\circ$	$[0]_0$	3	45	$\underline{9}$	
5:	$\left[\frac{1}{2}\right]_3$	$\left[\frac{5}{6}\right]_5$	$\left[\frac{1}{6}\right]_5$	$\left[\frac{1}{2}\right]_5$	$[0]_0$	6	150	$30 \to$	$\underline{24}$ ✓
7:	$\left[\frac{1}{2}\right]_3$	$[0]_0$	$[0]_0$	$\left[\frac{3}{2}\right]_3$	$[0]_3$	3	80	$16 \to$	$\underline{12}$ *
9:	$\left[\frac{2}{3}\right]_4$	$\left[\frac{2}{3}\right]_4$	$\left[\frac{2}{3}\right]_2$	$[0]_0$	$[0]_0$	1	125	$25 \to$	$\underline{19}$ ✓
11:	$\left[\frac{2}{3}\right]_4$	$\left[\frac{5}{6}\right]_5$	$[0]_0$	$\left[\frac{1}{2}\right]_1$	$[0]_2$	6	120	$24 \to$	$\underline{18}$ *
13:	$\left[\frac{5}{6}\right]_5$	$\left[\frac{5}{6}\right]_5$	$\left[\frac{5}{6}\right]_1$	$\left[\frac{\bar{1}}{2}\right]_3$	$[0]_0$	1	9	$3 \to$	$\underline{2}$ ✓

3.3.1. *Vector 1.*

```
1  := rec(fixed := -5, min := 24),
2  := rec(fixed := -1, min := 48, chain := 3),
3  := rec(fixed :=  2, min := 64, chain := 2),
4  := rec(),
```

3.3.2. *Vector 2.*

```
1  := rec(),
2  := rec(),
3  := rec(),
4  := rec(),
5  := rec(fixed := -5, min := 40),
6  := rec(fixed :=  2, min := 54, chain := [2, 2]),
7  := rec(fixed :=  2, min := 29, chain := [3, 2]),
8  := rec(fixed := -5, min := 36, chain := 2),
```

3.3.3. *Vector 3.*

```
1  := rec(fixed :=  5, min := 32, chain := 2),
2  := rec(fixed := -3, min := 22),
3  := rec(fixed :=  3, min := 18),
4  := rec(),
6  := rec(),
8  := rec(),
10 := rec(fixed :=  1, min := 16, chain := 2),
12 := rec(fixed :=  3, min := 18),
14 := rec(fixed :=  3, min := 26),
16 := rec(fixed :=  1, min := 24, chain := 2),
18 := rec(fixed :=  5, min := 18, chain := 2),
20 := rec(fixed :=  1, min := 11),
22 := rec(fixed :=  3, min := 18, chain := 2),
24 := rec(),
26 := rec(),
28 := rec(),
30 := rec(fixed :=  3, min := 12),
```

3.3.4. *Vector 4.*

```
1  := rec(),
2  := rec(fixed := -1, min := 34),
3  := rec(fixed :=  1, min := 40, chain := 2),
4  := rec(fixed :=  4, min := 24, chain := 3),
5  := rec(),
7  := rec(),
9  := rec(),
```

3.3.5. *Vector 5.*

```
1  := rec(fixed :=  5, min := 12),
2  := rec(fixed :=  5, min := 34),
3  := rec(),
5  := rec(fixed :=  1, min := 19),
7  := rec(),
9  := rec(),
11 := rec(fixed :=  2, min := 16),
13 := rec(fixed :=  1, min := 21)
```

3.3.6. *Vector 6.*

```
1  := rec(),
2  := rec(),
3  := rec(),
4  := rec(fixed :=  1, min := 42, chain := 2),
5  := rec(fixed := -1, min := 30, chain := 2),
6  := rec(fixed :=  1, min := 34),
7  := rec(fixed :=  1, min := 26, chain := 2),
8  := rec(fixed := -5, min := 16, chain := 2),
9  := rec(fixed := 3,  min := 16, chain := 3),
10 := rec(),
```

3.3.7. *Vector 7.*

```
1  := rec(fixed :=  1, min := 30),
2  := rec(fixed := -2, min := 14),
3  := rec(fixed := -5, min := 14),
4  := rec(fixed :=  1, min := 38),
5  := rec(fixed := -5, min := 30),
6  := rec(),
7  := rec(),
9  := rec(fixed :=  1, min := 14),
11 := rec(fixed := -5, min := 20, chain := 2),
```

3.3.8. *Vector 8.*

```
1  := rec(),
3  := rec(),
5  := rec(fixed :=  4, min := 24),
7  := rec(fixed := -5, min := 12),
9  := rec(fixed :=  1, min := 19, chain := 3),
11 := rec(fixed := -5, min := 18, chain := 2),
13 := rec(),
```

4. Further work required

The next three sections deal with the remaining five Niemeier lattices rationally generated by roots, constituting the proof of **Theorem 5.1.

Even after the exact bounds $b(\mathfrak{o})$ for all combinatorial orbits have been computed by brute force §1.4, the cumulative bound $b(\mathfrak{O}_h)$ still exceeds 720, sometimes more than twice. (These offending bounds are shown in bold in the tables.) Hence, we need to use the puzzle assembly §1.5 to obtain better reduced bounds $\bar{b}$ for some of the orbits or unions thereof.

Each subsection below starts with the computation of the sets $\mathcal{B}_m(\mathfrak{o})$ and, hence, bounds $b(\mathfrak{o})$, in the same notation as in §3 (see §1.4). Then, some of these sets need to be recomputed with a lower goal m, indicated as a `GAP` comment

`# min :=` m.

Upon that, we outline the major steps of the further computation, in the $\mathcal{B}^*_*(\ldots)$-notation of §1.1. (The order of the arguments is crucial as it may affect the convergence.) If clusters are used (see §1.7), they are described "geometrically", usually in terms of the supports $\operatorname{supp}\mathfrak{o}$.

Convention 4.1. We summarize the result indicating the final bound on $\bar{b}(\mathfrak{O}_h)$ and its ingredients. Thus, the last line

$$\bar{b}(\mathfrak{O}) \leqslant 142_{1,6} + 480_2 + 88_3 + 8_4 = 718$$

in §4.3.2 means that, *modulo sets of large rank that have been handled separately* (see §1.2 and §1.6), for a geometric subset $\mathfrak{C} \subset \mathfrak{O}_h$ one has

$$|\mathfrak{C} \cap (\bar{\mathfrak{o}}_1 \cup \bar{\mathfrak{o}}_6^+)| \leqslant 142, \quad |\mathfrak{C} \cap \bar{\mathfrak{o}}_2| \leqslant 480, \quad |\mathfrak{C} \cap \bar{\mathfrak{o}}_3| \leqslant 88, \quad |\mathfrak{C} \cap \bar{\mathfrak{o}}_4^+| \leqslant 8$$

and, therefore, $|\mathfrak{C}| \leqslant 718$. Note that here $\bar{\mathfrak{o}}_4^+ = \bar{\mathfrak{o}}_4 \cup \bar{\mathfrak{o}}_5$ and $\bar{\mathfrak{o}}_6^+ = \bar{\mathfrak{o}}_6 \cup \bar{\mathfrak{o}}_7$, even though only $\bar{\mathfrak{o}}_4$ and $\bar{\mathfrak{o}}_6$ are used in the computation.

4.1. **The code.** In the subsequent sections, upon a brief verbal explanation, we present also the uppermost layer of the code, primarily so that the author himself does not forget how the results can be reproduced. The general paradigm is that, for each set $\mathcal{B}^*_m(\ldots, \mathcal{S})$ to be computed, a dedicated data object is created which holds the necessary input and a number of parameters fine-tuning the algorithm. The puzzle assembly algorithm §1.5 itself is invoked *via*

`update(`⟨*data*⟩, m`);` where m is the goal in $\mathcal{B}^*_m$,

upon which ⟨*data*⟩`.RES` is the output. The exceptional "large" sets discarded as in §1.2, §1.6 are stored in the global variable

`\\saved;` collects admissible sets $\mathfrak{C}$ of size $|\mathfrak{C}| \geqslant 600$ encountered in the computation.

Thus, the working horse is the function

`@data(`n`, [ src |` $\mathfrak{c}$ `]);` here and below n refers to the relevant orbit $\bar{\mathfrak{o}}_n$ as in the tables.

The optional arguments may include a reference to a previously updated data object `src` and/or a cluster $\mathfrak{c} \subset \bar{\mathfrak{o}}_n$. Most other parameters, including the minimal and maximal goals as functions of m and a particular set $\mathfrak{C} \in$ `src.RES`, are set automatically; this is controlled by the function

`\\rc.spp;` called from within `@data`, which *may* be defined in (3.1).

Clusters are created by

`@(`*arg*`);` or `@@(`*arg*`);` referring, respectively, to `\\rc.dm;` or `\\rc.dmm;` also defined in (3.1).

Their arguments are case specific; typically,

`@(`n`);` returns the list of the "smallest" clusters for the orbit $\bar{\mathfrak{o}}_n$, whereas

`@(`$\mathfrak{c}$`);` returns the "next level" clusters containing the given cluster $\mathfrak{c}$.

The other function `@@(`*arg*`);` is more special: we use it when more then one system of clusters is considered for the same orbit (*cf.* §5.2 below).

As explained in §1.2, by default we compute $\mathcal{B}''$. For $\mathcal{B}'''$ or $\mathcal{B}^{\mathrm{iv}}$, we use the function

`@extend(src,` δ`);` after `update`, pushes the rank by $\delta = 1$ (default) or 2 units, see §1.2.

Finally, option $**$(2) in $**$Warning 4.5 (*cf.* Remark 1.20) is implemented *via*

`@select(`n`,`$\mathcal{S}$`-set,src);` creates several data objects updated by a single call to `update`; the input of one of the objects contains all sets $\mathfrak{C} \in$ `src.RES`, the others—only the "necessary" ones, based on the stabilizer stab $\mathfrak{C}$ and its action on $\mathcal{S}$`-set`.

4.2. **Pipes.** To save memory and avoid mistakes, we often use *pipes*: upon the completion of a chunk of the computation, a data object sends its partial output to the next one. A pipe is updated by a single call to `update`, and the ultimate goal m is recomputed automatically for each pipe member on the fly (hence, no mistakes).

Most pipes are created automatically, within `\\rc.spp;` in this function, the correct processing of the upper/lower bounds is usually achieved by `spp_cmax(src);` in a few more involved cases (*cf.* §5.1 below), the relationship of the pipe members is encoded directly.

A simplest pipe is constructed by

`@chain(`$n_1, n_2, \ldots$`);` a straightforward computation of $\mathcal{B}^*_m(\bar{\mathfrak{o}}_{n_1}, \bar{\mathfrak{o}}_{n_2}, \ldots)$, see (1.3).

4.3. **The lattice** $N(6\mathbf{D}_4)$**.** There are 3 classes; $b(\mathfrak{O}) \leqslant 732$.

1:	$[\![4]\!]^3_\bullet$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![0]\!]_0$		5848	1402 → **722**
1:	$[2]^2_\bullet$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	2^*	4	$\underline{1}$
2:	$[2]_\circ$	$[0]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	5^{**}	144	24 → $\underline{20}$ ✓
3:	$[2]_3$	$[0]_3$	$[0]_3$	$[0]_3$	$[0]_0$	$[0]_0$	10^{**}	512	128 → $\underline{62}$ ✓
2:	$[\![1]\!]_3$	$[\![1]\!]_3$	$[\![1]\!]_3$	$[\![1]\!]_3$	$[\![0]\!]_0$	$[\![0]\!]_0$		4408	1144 → **732**
1:	$[1]_3$	$[1]_3$	$[0]_0$	$[0]_0$	$[0]_3$	$[0]_3$	6^*	64	16 → $\underline{10}$ *
2:	$[1]_3$	$[\frac{1}{2}]_2$	$[\frac{1}{2}]_1$	$[0]_0$	$[0]_1$	$[0]_0$	24^*	128	32 → $\underline{20}$ *
3:	$[\frac{1}{2}]_2$	$[\frac{1}{2}]_2$	$[\frac{1}{2}]_2$	$[\frac{1}{2}]_2$	$[0]_0$	$[0]_0$	2^*	256	64 → $\underline{44}$ ✓
4:	$[2]^3_\bullet$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	4	1	$\underline{1}$
6:	$[1]_\circ$	$[1]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	6	36	12 → $\underline{8}$ *
3:	$[\![2]\!]_\circ$	$[\![2]\!]_\circ$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![0]\!]_0$		4888	1220 → 712
1:	$[2]_\circ$	$[0]_0$	$[0]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	8^*	24	$\underline{4}$
2:	$[1]_\circ$	$[1]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	1^{**}	64	24 → $\underline{20}$ ✓
3:	$[1]_3$	$[1]_3$	$[0]_3$	$[0]_3$	$[0]_0$	$[0]_0$	18^{**}	256	64 → $\underline{36}$ ✓
4:	$[2]^3_\bullet$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	6	2	1

4.3.1. *Vector* 1.

```
1  := rec(),
2  := rec(),
3  := rec(fixed :=  2, min := 62,  chain := 2),
```

Besides, by the puzzle assembly §1.5 we show that $\mathcal{B}_{90}(\bar{\mathfrak{o}}_2) = \varnothing$, reducing the bound by 12:

```
ds := @data(2);;       update(ds,  90);;
```

4.3.2. *Vector* 2.

```
1  := rec(),
2  := rec(fixed :=  2, min := 20,  chain := 2),
3  := rec(fixed :=  1, min := 44),
4  := rec(),
6  := rec(),
```

Besides, we use the puzzle assembly §1.5 to show that $\mathcal{B}_{144}(\bar{\mathfrak{o}}_6^+, \bar{\mathfrak{o}}_1) = \varnothing$:

```
ch := @chain(6, 1);;   update(ch, 144);;
```

Hence,

$$\bar{b}(\mathfrak{O}) \leqslant 142_{1,6} + 480_2 + 88_3 + 8_4 = 718.$$

4.3.3. *Vector* 3.

```
1  := rec(),
2  := rec(),
3  := rec(fixed := -3, min := 36,  chain := 3),
```

4.4. **The lattice** $N(6\mathbf{A}_4)$**.** There are 4 classes; $b(\mathfrak{O}) \leqslant 820$.

1:	$\llbracket 4\rrbracket_\bullet$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$		5400	1080 → **790**
1:	$[2]_\circ$	$[0]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	5^{**}	80	16 → $\underline{14}$ ✓
2:	$[2]_3$	$[0]_3$	$[0]_1$	$[0]_0$	$[0]_0$	$[0]_1$	20^*	250	50 → $\underline{36}$ ✓
2:	$\llbracket \frac{4}{5}\rrbracket_4$	$\llbracket \frac{4}{5}\rrbracket_4$	$\llbracket \frac{4}{5}\rrbracket_1$	$\llbracket \frac{4}{5}\rrbracket_1$	$\llbracket \frac{4}{5}\rrbracket_4$	$\llbracket 0\rrbracket_0$		4440	1000 → **820**
1:	$[\frac{3}{5}]_3$	$[\frac{2}{5}]_2$	$[0]_0$	$[\frac{1}{5}]_4$	$[\frac{4}{5}]_4$	$[0]_0$	20^*	96	24 → $\underline{18}$ ✓
2:	$[\frac{2}{5}]_2$	$[\frac{4}{5}]_4$	$[0]_0$	$[0]_0$	$[\frac{4}{5}]_4$	$[0]_2$	10^*	60	12 → $\underline{10}$ ✓
3:	$[1]_\circ$	$[1]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	10	16	$\underline{4}$
5:	$[\frac{3}{5}]_3$	$[\frac{3}{5}]_3$	$[\frac{4}{5}]_1$	$[0]_0$	$[0]_0$	$[0]_1$	10	80	16 → $\underline{14}$ ✓
3:	$\llbracket 2\rrbracket_\circ$	$\llbracket 2\rrbracket_\circ$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$		4920	992 → **800**
1:	$[2]_\circ$	$[0]_0$	$[0]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	8^*	20	$\underline{4}$
2:	$[1]^+_\circ$	$[1]^+_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	4^*	9	$\underline{3}$
3:	$[1]_3$	$[1]_3$	$[0]_1$	$[0]_0$	$[0]_0$	$[0]_1$	4^*	225	45 → $\underline{37}$ ✓
4:	$[1]_3$	$[1]_4$	$[0]_4$	$[0]_0$	$[0]_2$	$[0]_0$	16^*	150	30 → $\underline{24}$ ✓
5:	$[1]_4$	$[1]_4$	$[0]_3$	$[0]_0$	$[0]_0$	$[0]_3$	4^*	100	20 → $\underline{16}$ ✓
6:	$[1]_4$	$[1]_4$	$[0]_1$	$[0]_1$	$[0]_4$	$[0]_0$	8^*	125	25 → $\underline{19}$ ✓
7:	$[2]_\bullet$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	2	6	$\underline{2}$
4:	$\llbracket \frac{4}{5}\rrbracket_4$	$\llbracket \frac{4}{5}\rrbracket_4$	$\llbracket \frac{6}{5}\rrbracket_3$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$	$\llbracket \frac{6}{5}\rrbracket_3$		4440	944 → **820**
1:	$[\frac{3}{5}]_3$	$[\frac{1}{5}]_1$	$[\frac{6}{5}]_3$	$[0]_1$	$[0]_0$	$[0]_0$	4^*	80	16 → $\underline{14}$ ✓
2:	$[\frac{2}{5}]_2$	$[\frac{2}{5}]_2$	$[\frac{3}{5}]_4$	$[0]_0$	$[0]_0$	$[\frac{3}{5}]_4$	1^{**}	144	48 → $\underline{28}$ ✓
3:	$[\frac{4}{5}]_4$	$[0]_0$	$[\frac{6}{5}]_3$	$[0]_4$	$[0]_3$	$[0]_0$	4^*	50	10 → $\underline{8}$ ✓
4:	$[\frac{4}{5}]_4$	$[0]_0$	$[\frac{4}{5}]_2$	$[0]_2$	$[0]_0$	$[\frac{2}{5}]_1$	4^*	90	18 → $\underline{16}$ ✓
5:	$[\frac{4}{5}]_4$	$[0]_0$	$[\frac{3}{5}]_4$	$[0]_1$	$[0]_1$	$[\frac{3}{5}]_4$	2^*	100	20 → $\underline{18}$ ✓
6:	$[1]_\circ$	$[1]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	1	16	$\underline{4}$
8:	$[1]_\circ$	$[0]_0$	$[1]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	4	24	$\underline{6}$
10:	$[\frac{3}{5}]_3$	$[\frac{3}{5}]_3$	$[\frac{2}{5}]_1$	$[0]_0$	$[0]_0$	$[\frac{2}{5}]_1$	1	144	36 → $\underline{28}$ ✓
12:	$[\frac{3}{5}]_3$	$[\frac{4}{5}]_4$	$[\frac{3}{5}]_4$	$[0]_0$	$[0]_2$	$[0]_0$	4	80	16 → $\underline{14}$ ✓
14:	$[\frac{3}{5}]_3$	$[0]_0$	$[\frac{4}{5}]_2$	$[0]_0$	$[0]_4$	$[\frac{3}{5}]_4$	4	120	24 → $\underline{22}$ ✓
16:	$[\frac{2}{5}]_2$	$[\frac{4}{5}]_4$	$[\frac{4}{5}]_2$	$[0]_4$	$[0]_0$	$[0]_0$	4	90	18 → $\underline{16}$ ✓
18:	$[\frac{4}{5}]_4$	$[\frac{4}{5}]_4$	$[\frac{6}{5}]_3$	$[0]_0$	$[0]_0$	$[\frac{4}{5}]_3$	2	3	$\underline{1}$
20:	$[\frac{4}{5}]_4$	$[\frac{4}{5}]_4$	$[\frac{1}{5}]_3$	$[0]_0$	$[0]_0$	$[\frac{1}{5}]_3$	1	36	12 → $\underline{10}$ ✓
22:	$[\frac{4}{5}]_4$	$[\frac{4}{5}]_4$	$[\frac{2}{5}]_1$	$[0]_1$	$[0]_4$	$[0]_0$	2	75	15 → $\underline{13}$ ✓

4.4.1. *Vector 1.*

```
1  := rec(),
2  := rec(fixed :=  3, min := 36),    # min := 34
```

By the puzzle assembly §1.5, we find that

$$\mathcal{B}''_{54}(\bar{\mathfrak{o}}_1) = \varnothing :$$

```
ds := @data(1);;               update(ds,  54);;
```

Next, we consider the five clusters (see §1.7)

$$\mathfrak{c}_p := \{\mathfrak{o} \subset \bar{\mathfrak{o}}_2 \mid p \notin \operatorname{supp} \mathfrak{o}\}, \quad p \in \bar{\Omega},$$

and, by the puzzle assembly §1.5, we find that

$$\mathcal{B}''_{650}(\bar{\mathfrak{o}}_2) = \varnothing :$$

```
ds := @data(2, @(2)[1]);;     update(ds, 668);;   # a pipe is used
```

Hence,

$$\bar{b}(\mathfrak{O}) \leqslant 52_1 + 666_2 = 718.$$

4.4.2. *Vector 2.*

```
1  := rec(fixed :=  1, min := 16),
2  := rec(),
3  := rec(),
5  := rec(fixed :=  1, min := 13),
```

By the puzzle assembly §1.5 we have

$$\mathcal{B}''_{330}(\bar{\mathfrak{o}}_1) = \mathcal{B}''_{82}(\bar{\mathfrak{o}}_2) = \mathcal{B}'''{}_{64}(\bar{\mathfrak{o}}_3^+) = \mathcal{B}''_{250}(\bar{\mathfrak{o}}_5^+) = \varnothing :$$

```
ds := @data(1);;       update(ds, 330);;
ds := @data(2);;       update(ds,  82);;
ds := @data(3);;       update(ds,  64);;      @extend(ds);
ds := @data(5);;       update(ds, 250);;
```

Hence,

$$\bar{b}(\mathfrak{O}) \leqslant 328_1 + 80_2 + 62_3 + 248_5 = 718.$$

4.4.3. *Vector 3.*

```
1  := rec(),
2  := rec(),
3  := rec(fixed :=  3, min := 37, chain := 3),
4  := rec(fixed :=  1, min := 24),               # min := 20
5  := rec(fixed :=  3, min := 16, chain := 2),
6  := rec(fixed :=  3, min := 19, chain := 3),    # min := 16
7  := rec(),
```

We subdivide $\bar{\mathfrak{o}}_4$ into four pairwise disjoint clusters (see §1.7)

$$\mathfrak{c}_o := \bigl\{\mathfrak{o} \subset \bar{\mathfrak{o}}_4 \bigm| \operatorname{supp}\mathfrak{o} \smallsetminus \operatorname{supp}\hbar = o\bigr\}, \quad o \subset \bar{\Omega},$$

where the 2-element set o takes the four values $\{3,4\}, \{3,5\}, \{4,6\}, \{5,6\}$. Setting `push := 3` (see §1.6), by the puzzle assembly §1.5 we show that

$$\mathcal{B}''_{158}(\mathfrak{c}_{o'} < \mathfrak{c}_{o'} \cup \mathfrak{c}_{o''}) = \varnothing \quad \text{whenever } o' \cap o'' \neq \varnothing; \quad \text{hence,} \quad \mathcal{B}''_{314}(\bar{\mathfrak{o}}_4) = \varnothing :$$

```
ds := @data(4, @(4)[1]);;         update(ds, 314);;
ss := @select(4, @(ds.dm), ds);;  update(ss, 314);;
```

Next, we subdivide $\bar{\mathfrak{o}}_6$ into two disjoint clusters (see §1.7) $\mathfrak{c}_=, \mathfrak{c}_{\neq}$, according to whether the discriminant classes at the two points of $\operatorname{supp}\hbar = \{1,2\}$ are or are not equal. (The pair of classes can take values $(4,4)$, $(1,1)$ in $\mathfrak{c}_=$ or $(4,1)$, $(1,4)$ in $\mathfrak{c}_{\neq}$.) By the puzzle assembly §1.5 we show that

$$\mathcal{B}''_{130}(\mathfrak{c}_= < \bar{\mathfrak{o}}_6) = \varnothing :$$

```
ds := @data(6, @(6)[1]);;         update(ds, 130);;   # a pipe is used
```

Thus,

$$\bar{b}(\mathfrak{D}) \leqslant 32_1 + 12_2 + 148_3 + 312_4 + 64_5 + 128_6 + 8_7 = 704.$$

4.4.4. *Vector 4.*

```
1  := rec(fixed :=  1, min := 14, chain := 2),
2  := rec(),
3  := rec(),
4  := rec(fixed :=  3, min := 16),
5  := rec(fixed :=  3, min := 18),
6  := rec(),
8  := rec(),
10 := rec(fixed :=  3, min := 28),
12 := rec(fixed :=  3, min := 14),    # min := 12
14 := rec(fixed :=  3, min := 22),    # min := 16
16 := rec(fixed := -1, min := 16),    # min := 12
18 := rec(),
20 := rec(),
22 := rec(fixed :=  3, min := 13),
```

By the puzzle assembly §1.5 we establish that

$$\mathcal{B}''_{96}(\bar{\mathfrak{o}}^+_{12}) = \mathcal{B}''_{138}(\bar{\mathfrak{o}}^+_{14}) = \mathcal{B}''_{104}(\bar{\mathfrak{o}}^+_{16}) = \mathcal{B}_{96}(\bar{\mathfrak{o}}^+_8, \bar{\mathfrak{o}}^+_6, \bar{\mathfrak{o}}^+_{18}, \bar{\mathfrak{o}}^+_{20}, \bar{\mathfrak{o}}_3) = \varnothing :$$

```
ds := @data(12);;                  update(ds,  96);;  push := 3;
ds := @data(14);;                  update(ds, 138);;
ds := @data(16);;                  update(ds, 104);;
ch := @chain(8, 6, 18, 20, 3);;    update(ch,  96);
```

Hence,

$$\bar{b}(\mathfrak{D}) \leqslant 56_1 + 28_2 + 94_{3,6,8,18,20} + 64_4 + 36_5 + 56_{10} + 94_{12} + 136_{14} + 102_{16} + 52_{22} = 718.$$

4.5. **The lattice** $N(8\mathbf{A}_3)$**.** There are 4 classes; $b(\mathfrak{O}) \leqslant 1024$.

1:	$[\![4]\!]_\bullet$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![0]\!]_0$		5432	1484 →	**952**
1:	$[2]_\circ$	$[0]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	7^{**}	48	12 →	$\underline{10}$ ✓
2:	$[2]_2$	$[0]_2$	$[0]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_2$	$[0]_0$	7^{**}	216	72 →	$\underline{38}$ ✓
3:	$[2]_2$	$[0]_3$	$[0]_3$	$[0]_0$	$[0]_3$	$[0]_0$	$[0]_0$	$[0]_1$	14^{*}	256	64 →	$\underline{44}$ ✓
2:	$[\![2]\!]_\circ$	$[\![2]\!]_\circ$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![0]\!]_0$		4952	1280 →	**948**
1:	$[2]_\circ$	$[0]_0$	$[0]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	12^{*}	12	$\underline{3}$	
2:	$[1]_\circ^+$	$[1]_\circ^+$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	4^{*}	4	$\underline{2}$	
3:	$[1]_2$	$[1]_2$	$[0]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_2$	$[0]_0$	3^{**}	144	48 →	$\underline{28}$ ✓
4:	$[1]_2$	$[1]_3$	$[0]_3$	$[0]_0$	$[0]_3$	$[0]_0$	$[0]_0$	$[0]_1$	16^{*}	128	32 →	$\underline{24}$ ✓
5:	$[1]_3$	$[1]_3$	$[0]_2$	$[0]_3$	$[0]_1$	$[0]_0$	$[0]_0$	$[0]_0$	24^{*}	96	24 →	$\underline{18}$ ✓
6:	$[2]_\bullet$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	2	2	$\underline{1}$	
3:	$[\![1]\!]_2$	$[\![1]\!]_2$	$[\![1]\!]_2$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![1]\!]_2$	$[\![0]\!]_0$		4472	1272 →	**1024**
1:	$[1]_2$	$[1]_2$	$[0]_0$	$[0]_2$	$[0]_2$	$[0]_0$	$[0]_0$	$[0]_0$	12^{*}	36	12 →	$\underline{8}$ ✓
2:	$[1]_2$	$[\frac{1}{2}]_3$	$[\frac{1}{2}]_3$	$[0]_0$	$[0]_3$	$[0]_0$	$[0]_0$	$[0]_1$	48^{*}	64	16 →	$\underline{14}$ ✓
3:	$[\frac{1}{2}]_3$	$[\frac{1}{2}]_3$	$[\frac{1}{2}]_3$	$[0]_0$	$[0]_0$	$[0]_2$	$[\frac{1}{2}]_1$	$[0]_0$	8^{**}	96	32 →	$\underline{22}$ ✓
4:	$[2]_\bullet$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	4	1	$\underline{1}$	
6:	$[1]_\circ$	$[1]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	6	16	8 →	$\underline{6}$ ✓
4:	$[\![\frac{3}{4}]\!]_3$	$[\![\frac{3}{4}]\!]_3$	$[\![\frac{3}{4}]\!]_3$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![1]\!]_2$	$[\![\frac{3}{4}]\!]_1$	$[\![0]\!]_0$		4472	1204 →	**996**
1:	$[\frac{1}{2}]_2$	$[\frac{3}{4}]_3$	$[\frac{1}{4}]_1$	$[0]_1$	$[0]_0$	$[\frac{1}{2}]_1$	$[0]_0$	$[0]_0$	24^{*}	72	18 →	$\underline{16}$ ✓
2:	$[\frac{3}{4}]_3$	$[\frac{3}{4}]_3$	$[0]_0$	$[0]_2$	$[0]_0$	$[\frac{1}{2}]_3$	$[0]_0$	$[0]_1$	12^{*}	48	12 →	$\underline{10}$ ✓
3:	$[1]_\circ$	$[1]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	6	9	$\underline{3}$	
5:	$[1]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[1]_\circ$	$[0]_0$	$[0]_0$	4	12	$\underline{4}$	
7:	$[\frac{1}{2}]_2$	$[\frac{1}{2}]_2$	$[\frac{1}{2}]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[\frac{1}{2}]_2$	$[0]_0$	1	81	27 →	$\underline{19}$ ✓
9:	$[\frac{1}{2}]_2$	$[\frac{1}{2}]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[1]_2$	$[0]_0$	$[0]_2$	6	54	18 →	$\underline{12}$ ✓
11:	$[\frac{1}{2}]_2$	$[\frac{3}{4}]_3$	$[\frac{3}{4}]_3$	$[0]_0$	$[0]_3$	$[0]_0$	$[0]_0$	$[0]_1$	12	48	12 →	$\underline{10}$ ✓
13:	$[\frac{3}{4}]_3$	$[\frac{3}{4}]_3$	$[\frac{3}{4}]_3$	$[0]_0$	$[0]_0$	$[-1]_2$	$[\frac{3}{4}]_1$	$[0]_0$	1	1	$\underline{1}$	

4.5.1. *Vector* 1.

```
1  := rec(),
2  := rec(fixed := -2, min := 38),                   # min :=  6
3  := rec(fixed :=  2, min := 44,  chain := 2),      # min := 28
```

We use a non-standard, "non-cluster" approach, splitting $\mathfrak{O}$ into seven pairwise disjoint triples of the form

$$(\mathfrak{o}_3^+, \mathfrak{o}_1, \mathfrak{o}_2), \qquad \mathfrak{o}_i \in \bar{\mathfrak{o}}_i, \quad \operatorname{supp}\mathfrak{o}_3 \cap \operatorname{supp}\mathfrak{o}_2 = \operatorname{supp}\mathfrak{o}_1 \cap \operatorname{supp}\mathfrak{o}_2 = \operatorname{supp}\hbar.$$

(Here, $\mathfrak{o}_2$ and $\mathfrak{o}_3^+$ determine each other uniquely, whereas for $\mathfrak{o}_1$ choices can be made.) By the puzzle assembly §1.5 we show that, for each such triple,

$$\mathcal{B}'''{}_{104}(\mathfrak{o}_3^+, \mathfrak{o}_1, \mathfrak{o}_2) = \varnothing; \quad \text{hence,} \quad \bar{b}(\mathfrak{O}) \leqslant 7 \cdot 102 = 714:$$

```
ds := @data(3, @()[1]);;           update(ds,  56);;
ss := @select(1, @(ds.dm), ds);;  update(ss, 104);;   # a pipe is used
                                   @extend(ss);
```

There is a unique geometric set $\mathfrak{L} \subset \mathfrak{O}$ of size $|\mathfrak{L}| = 728$. In the notation of **§3.3, assuming that $h|_1 = \mathbf{1}_{\mathcal{I}}$ and letting $r := \{1,2\}$, $s := \{1,3\}$, the lattice $\operatorname{ort}_h \mathfrak{C}$ (see **Convention 5.2) is generated by any three roots generating D_1 and two extra vectors u, v:

$$u|_1 = v|_1 = v|_k = [r], \qquad u|_k = [s] \quad \text{for } k = 2, \ldots, 8.$$

4.5.2. *Vector* 2.

```
1  := rec(),
2  := rec(),
3  := rec(fixed :=  1, min := 28),    # min := 18
4  := rec(fixed :=  1, min := 24),    # min := 19
5  := rec(fixed :=  4, min := 18),    # min := 14
6  := rec(),
```

By the puzzle assembly §1.5 we have

$$\mathcal{B}''_{70}(\bar{\mathfrak{o}}_3) = \mathcal{B}''_{36}(\bar{\mathfrak{o}}_1, \bar{\mathfrak{o}}_2, \bar{\mathfrak{o}}_6^+) = \varnothing:$$

```
ds := @data(3);;                   update(ds,  70);;   # push := 4;
ch := @chain(1, 2, 6);;            update(ch,  36);;
```

There are 24 self-dual four-element sets $\mathfrak{c}_i \subset \bar{\mathfrak{o}}_5$ such that $\operatorname{rk}\mathfrak{c}_i = 15$. They constitute two $O_\hbar(N)$-orbits, both of length 12, and only one of them has disjoint pairs $(\mathfrak{c}_i, \mathfrak{c}_j)$ such that $\operatorname{rk}(\mathfrak{c}_i \cup \mathfrak{c}_j) = 18$. The elements $\mathfrak{c}_i$ of this latter orbit are taken for clusters, and the 24 pairs $\mathfrak{p}_k := \mathfrak{c}_i \cup \mathfrak{c}_j$ as above, for larger clusters, see §1.7. Setting `push := 3`, see §1.6, we use the puzzle assembly §1.5 to show that

$$\mathcal{B}''_{318}(\mathfrak{c}_i < \mathfrak{p}_k < \bar{\mathfrak{o}}_5) = \varnothing:$$

```
ds := @data(5, @(5)[1]);;          update(ds, 318);;
ss := @select(5, @(ds.dm), ds);;  update(ss, 318);;   # a pipe is used
```

Likewise, there are twelve self-dual four-element sets $\mathfrak{c}'_i \subset \bar{\mathfrak{o}}_4$ such that $\operatorname{rk}\mathfrak{c}_i = 17$, and they form six disjoint pairs $\mathfrak{p}'_k := \mathfrak{c}'_i \cup \mathfrak{c}'_j$ such that $\operatorname{rk}\mathfrak{p}'_k = 20$. By the puzzle assembly §1.5 with `push := 4` (see §1.6), we find

$$\mathcal{B}''_{152}(\mathfrak{c}'_i < \mathfrak{p}'_k) = \varnothing; \quad \text{hence,} \quad \mathcal{B}''_{302}(\bar{\mathfrak{o}}4) = \varnothing:$$

```
ds := @data(4, @(4)[1]);;          update(ds, 302);;
ss := @select(4, @(ds.dm), ds);;  update(ss, 302);;
```

Hence,

$$\bar{b}(\mathfrak{O}) \leqslant 34_{1,2,6} + 68_3 + 300_4 + 316_5 = 718.$$

Remark 4.2. Strictly speaking, the flags $\mathfrak{c}_i \subset \mathfrak{p}_k$ are not homogeneous (see **§4.1), but the group $O_\hbar(N)$ acts transitively on the set of such flags; hence, we can still use the arguments of §1.7.

4.5.3. *Vector 3.*

```
1  := rec(),
2  := rec(fixed :=  2, min := 14),    # min := 6
3  := rec(),
4  := rec(),
6  := rec(),
```

Consider the six pairwise disjoint clusters (see §1.7)

$$\mathfrak{c}_o := \bigl\{\mathfrak{o} \subset \bar{\mathfrak{o}}_2 \bigm| \operatorname{supp}\mathfrak{o} \cap o = \varnothing\bigr\}, \quad o \subset \bar{\Omega}, \quad |o| = 2,$$

each consisting of eight combinatorial orbits. They split into three pairs $(\mathfrak{c}_o, \mathfrak{c}_{\bar{o}})$, $\bar{o} := \bar{\Omega} \smallsetminus o$. By the puzzle assembly §1.5 we find that

$$\mathcal{B}''_{158}(\mathfrak{c}_o < \mathfrak{c}_o \cup \mathfrak{c}_{\bar{o}}) = \varnothing; \quad \text{hence,} \quad \mathcal{B}''_{470}(\bar{\mathfrak{o}}_2) = \varnothing :$$

```
ds := @data(2, @(2)[1]);;     update(ds, 470);;   # a pipe is used
```

Next, by the puzzle assembly §1.5 we show that

$$\mathcal{B}''_{126}(\bar{\mathfrak{o}}_1, \bar{\mathfrak{o}}_6^{+}) = \varnothing :$$

```
ch := @chain(1, 6);;          update(ch, 126);;
```

This implies the bound

$$\bar{b}(\mathfrak{O}) \leqslant 124_{1,6} + 468_2 + 176_3 + 8_4 = 776. \tag{4.3}$$

To reduce this down to 718, we use the puzzle assembly §1.5 to show that

$$\mathcal{B}'''_{120,172}(\bar{\mathfrak{o}}_3; \bar{\mathfrak{o}}_1) = \varnothing :$$

```
ds := @data(3);;              update(ds, 120);;
ds := @data(1, ds);;          update(ds, 172);;   @extend(ds);
```

Hence, 176_3 in (4.3) can be reduced down to 118_3, as $|\mathfrak{L} \cap \bar{\mathfrak{o}}_3| \geqslant 120$ would imply that

$$|\mathfrak{L}| \leqslant 170_{1,3} + 468_2 + 8_4 + 72_6 = 718.$$

4.5.4. *Vector 4.*

```
  1  := rec(fixed :=  1, min := 16),    # min := 10
  2  := rec(),
  3  := rec(),
  5  := rec(),
  7  := rec(fixed :=  1, min := 19),    # min := 14
  9  := rec(),
  11 := rec(),
  13 := rec(),
```

We apply the following uniform approach to the orbits $\bar{\mathfrak{o}}_n$, $n = 1, 9, 11$. Subdivide $\bar{\mathfrak{o}}_n$ into three pairwise disjoint clusters (see §1.7)

$$\mathfrak{c}_p := \bigl\{\mathfrak{o} \subset \bar{\mathfrak{o}}_n \bigm| \operatorname{supp}\mathfrak{o} \ni p \text{ (if } n = 1, 9) \text{ or } \operatorname{supp}\mathfrak{o} \not\ni p \text{ (if } n = 11)\bigr\}, \quad p \in \bar{\Omega} = \{4, 5, 8\}.$$

Then, for $(i, j, k) = (4, 5, 8)$ or $(4, 8, 5)$ and a certain even integer m, we iterate the puzzle assembly §1.5 to find that

$$\mathcal{B}'''_{3m+2}(\mathfrak{c}_i^+ < \mathfrak{c}_i^+ \cup \mathfrak{c}_j^+ < \mathfrak{c}_i^+ \cup \mathfrak{c}_j^+ \cup \mathfrak{c}_k^+) = \varnothing,$$

implying that $\bar{b}(\bar{\mathfrak{o}}_n^+) \leqslant 3m$. In this manner, we show that

$$\text{(4.4)} \qquad \text{(a) } \bar{b}(\bar{\mathfrak{o}}_1) \leqslant 264, \quad \text{(b) } \bar{b}(\bar{\mathfrak{o}}_9^+) \leqslant 96, \quad \text{(c) } \bar{b}(\bar{\mathfrak{o}}_{11}^+) \leqslant 180:$$

```
  ds := @data(1, @(1)[1]);;         update(ds, 266);;
  ss := @select(1, @(ds.dm), ds);;  update(ss, 266);;   # a pipe is used
  ds := @data(9, @(9)[1]);;         update(ds,  98);;
  ss := @select(9, @(ds.dm), ds);;  update(ss,  98);;   # a pipe is used
                                    @extend(ss);;
  ds := @data(11, @(11)[1]);;       update(ds, 182);;
  ss := @select(11, @(ds.dm), ds);; update(ss, 182);;   # a pipe is used
```

Next, by the puzzle assembly §1.5 we establish that

$$\mathcal{B}''_{90,186}(\bar{\mathfrak{o}}_2; \bar{\mathfrak{o}}_9^+) = \varnothing:$$

```
  ds := @data(2);;          update(ds,  90);;
  ds := @data(9, ds);;      update(ds, 186);;
```

Finally, iterating the puzzle assembly §1.5, we find that

$$\mathcal{B}''_{90,270}(\bar{\mathfrak{o}}_3^+, \bar{\mathfrak{o}}_5^+, \bar{\mathfrak{o}}_7^+; \bar{\mathfrak{o}}_{11}^+) = \varnothing:$$

```
  ch := @chain(3, 5, 7);;    update(ch,  90);;
  ds := @data(11, ch);;      update(ds, 270);;
```

Combining this with (4.4c), *cf.* (1.7), we arrive at

$$\bar{b}(\mathfrak{O}) \leqslant 264_1 + 184_{2,9} + 268_{3,5,7,11} + 2_{13} = 718.$$

5. The lattice $N(12\mathbf{A}_2)$

To simplify the tables, we change the notation as follows: we pick a standard Dynkin basis a_1, a_2 for each $\mathbf{A}_2$-summand and use the following pictographs:

$$(\circ\!-\ ) \mapsto a_1, \qquad (\ -\!\circ) \mapsto a_2, \qquad (\ \circ\ ) \mapsto a_1 + a_2,$$
$$(\longrightarrow) \mapsto \tfrac{1}{3}(2a_1 + a_2), \quad (\longleftarrow) \mapsto \tfrac{1}{3}(a_1 + 2a_2),$$

Certainly, in the l-rows these vectors are merely *representatives* of the combinatorial orbits.

There are 2 classes; $b(\mathfrak{D}) \leqslant 1260$.

1:	$\circ\!-$	$\circ\!-$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$		4984	1664 $\to$	**1184**
1:	$\circ\!-$	$\cdot$	$\circ$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	20*	6	$\underline{2}$	
2:	$\circ$	$\circ$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	4*	1	$\underline{1}$	
3:	$\longrightarrow$	$\longrightarrow$	$\longrightarrow$	$\longrightarrow$	$\cdot$	$\longrightarrow$	$\cdot$	$\cdot$	$\longrightarrow$	$\cdot$	$\cdot$	$\cdot$	60*	81	27 $\to$	$\underline{19}$ ✓
2:	$\longrightarrow$	$\longrightarrow$	$\longrightarrow$	$\longrightarrow$	$\cdot$	$\longrightarrow$	$\cdot$	$\cdot$	$\longrightarrow$	$\cdot$	$\cdot$	$\cdot$		4504	1532 $\to$	**1260**
1:	$\longrightarrow$	$\longrightarrow$	$\longrightarrow$	$\cdot$	$\longrightarrow$	$\cdot$	$\cdot$	$\longrightarrow$	$\cdot$	$\cdot$	$\cdot$	$\longrightarrow$	40*	27	9 $\to$	$\underline{7}$ ✓
2:	$\longrightarrow$	$\longrightarrow$	$\longleftarrow$	$\longleftarrow$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\longrightarrow$	$\cdot$	$\longleftarrow$	90*	36	12 $\to$	$\underline{10}$ ✓
3:	$\longleftarrow$	$\longleftarrow$	$\longleftarrow$	$\longleftarrow$	$\cdot$	$\longleftarrow$	$\cdot$	$\cdot$	$\longleftarrow$	$\cdot$	$\cdot$	$\cdot$	1**	64	32 $\to$	$\underline{20}$ ✓
4:	$\circ$	$\circ$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	15	4	$\underline{2}$	

5.1. Vector 1.

```
1  := rec(),
2  := rec(),
3  := rec(fixed := 3,   min := 19),        # min := 10
```

By the puzzle assembly §1.5 we establish that

$$\mathcal{B}''_{34}(\bar{\mathfrak{o}}_1, \bar{\mathfrak{o}}_2) = \varnothing :$$

```
ch := @chain(1, 2);;                 update(ch, 34);;
```

Next, we consider the 30 clusters (see §1.7)

$$\mathfrak{c}_\mathfrak{s} := \bigl\{\mathfrak{o} \in \bar{\mathfrak{o}}_3 \bigm| \operatorname{supp}\mathfrak{o} \cap \operatorname{supp}\mathfrak{s} = \operatorname{supp}\hbar\bigr\}, \quad \mathfrak{s} \in \bar{\mathfrak{o}}_3;$$

we have $|\mathfrak{c}_\mathfrak{s}| = 6$ and each orbit $\mathfrak{o} \subset \bar{\mathfrak{o}}_3$ is contained in three clusters. Hence, each cluster is contained in six *hooks* $\mathfrak{h}$ and three *tridents* $\mathfrak{t}_\mathfrak{o}$:

$$\mathfrak{h} := \mathfrak{c}_{\mathfrak{s}'} \cup \mathfrak{c}_{\mathfrak{s}''}, \quad |\mathfrak{c}_{\mathfrak{s}'} \cap \mathfrak{c}_{\mathfrak{s}''}| = 2; \qquad \mathfrak{t}_\mathfrak{o} := \bigcup_{\mathfrak{c}_\mathfrak{s} \ni \mathfrak{o}} \mathfrak{c}_\mathfrak{s}, \quad \mathfrak{o} \in \bar{\mathfrak{o}}_3;$$

we have $|\mathfrak{h}| = 10$ and $|\mathfrak{t}_\mathfrak{o}| = 14$. Using the flags $(\mathfrak{c}_\mathfrak{s} \smallsetminus \mathfrak{o}^+) < (\mathfrak{h} \smallsetminus \mathfrak{o}^+) < (\mathfrak{t}_\mathfrak{o} \smallsetminus \mathfrak{o}^+)$ and puzzle assembly §1.5, we establish that

$$\mathcal{B}^{\mathrm{iv}}_{162}(\mathfrak{t}_\mathfrak{o}) = \varnothing, \quad \text{hence} \quad \mathcal{B}^{\mathrm{iv}}_{686}(\bar{\mathfrak{o}}_3) = \varnothing :$$

```
ds := @data(3, @(3)[1]);;          update(ds, 162);;
ss := @select(3, @(ds.dm), ds);;   update(ss, 162);;    # a pipe is used
                                   @extend(ss, 2);;
```

It follows that

$$\bar{b}(\mathfrak{D}) \leqslant 32_{1,2} + 684_3 = 716.$$

To avoid the computation of the full set $\mathcal{B}_0(\mathfrak{o})$, $\mathfrak{o} \in \bar{\mathfrak{o}}_3$, we set `push := 3` (or even `push := 4` at the very last step), see §1.6, and essentially reduce the algorithm to brute force §1.4.

We found but three large geometric sets $\mathfrak{C}$, of sizes 608, 608, 600, which are graph isomorphic to some of those listed in §6.1 below.

Remark 5.1. The chain $\mathfrak{c}_\mathfrak{s} \subset \mathfrak{h} \subset \mathfrak{t}_\mathfrak{o}$ is not homogeneous in the sense of **§4.1, and we need to modify the approach of §1.7. We can assume that $\mathfrak{h}$ is the largest hook in $\mathfrak{t}_\mathfrak{o}$ and $\mathfrak{c}_\mathfrak{s}$ is the largest cluster in $\mathfrak{h}$ (hence, also in $\mathfrak{t}_\mathfrak{o}$). Then, for the "central" pair $\mathfrak{o}^+$ in $\mathfrak{h}$ and $\mathfrak{t}_\mathfrak{o}$, we have

$$|\mathfrak{C}\cap\mathfrak{h}| \geqslant \frac{2}{3}|\mathfrak{C}\cap\mathfrak{t}_\mathfrak{o}| + \frac{1}{3}|\mathfrak{C}\cap\mathfrak{o}^+| \geqslant 108,$$
$$|\mathfrak{C}\cap\mathfrak{c}_\mathfrak{s}| \geqslant \frac{1}{3}|\mathfrak{C}\cap\mathfrak{t}_\mathfrak{o}| + \frac{2}{3}|\mathfrak{C}\cap\mathfrak{o}^+| \geqslant 54,$$
$$|\mathfrak{C}\cap\mathfrak{c}_\mathfrak{s}| \geqslant \frac{1}{2}|\mathfrak{C}\cap\mathfrak{h}| + \frac{1}{2}|\mathfrak{C}\cap\mathfrak{o}^+| \geqslant 54.$$

Thus, we use the conservative estimates on the right aiming at $\mathcal{B}_{54}(\mathfrak{c}_\mathfrak{s})$ and $\mathcal{B}_{108}(\mathfrak{C};\mathfrak{h})$, but we invoke the more refined estimates in the middle when setting the lower and upper bounds, as in (1.14), in the computation of $\mathcal{B}_{108}(\mathfrak{C};\mathfrak{h})$ and $\mathcal{B}_{162}(\mathfrak{C};\mathfrak{t}_\mathfrak{o})$. For example, a set $\mathfrak{C}\subset\mathfrak{c}_\mathfrak{s}$ would have to be rejected (for a particular hook $\mathfrak{h}$ or trident $\mathfrak{t}_\mathfrak{o}$, which both determine the position of $\mathfrak{o}^+\subset\mathfrak{c}_\mathfrak{s}$) unless

$$3|\mathfrak{C}| - 2|\mathfrak{C}\cap\mathfrak{o}^+| \geqslant 162.$$

5.2. **Vector 2.**

```
1  := rec(),
2  := rec(),
3  := rec(),
4  := rec(),
```

For each combinatorial orbit $\mathfrak{o}\in\bar{\mathfrak{o}}_1$, there is a unique *complimentary combinatorial orbit* $\mathfrak{o}'\in\bar{\mathfrak{o}}_1$, characterized by the property that $\operatorname{supp}\mathfrak{o}' = \Omega \smallsetminus \operatorname{supp}\mathfrak{o}$, and we consider the 10 pairwise disjoint clusters (see §1.7)

$$\mathfrak{p}_i := \{\mathfrak{o},\mathfrak{o}^*,\mathfrak{o}',\mathfrak{o}'^*\},\quad i=1,\ldots,10.$$

The unions $\mathfrak{p}_{ij} := \mathfrak{p}_i\cup\mathfrak{p}_j$, $i\neq j$, also constitute a single $O_\hbar(N)$-orbit, and by the puzzle assembly §1.5 we show that

$$\mathcal{B}^{\mathrm{iv}}_{172;210}(\mathfrak{p}_{ij} < \bar{\mathfrak{o}}_1; \bar{\mathfrak{o}}_4^+, \bar{\mathfrak{o}}_3) = \mathcal{B}''_{40;210}(\bar{\mathfrak{o}}_3, \bar{\mathfrak{o}}_4^+; \mathfrak{p}_i < \bar{\mathfrak{o}}_1) = \varnothing, \quad \textit{cf.}\ (1.7):$$

```
ds := @data(1, @@(2)[1]);;        update(ds, 172);;   # a pipe is used
ds := @data(4, ds);;              update(ds, 190);;
ds := @data(3, ds);;              update(ds, 210);;   @extend(ds, 2);

ch := @chain(3, 4);;              update(ch,  40);;
ss := @select(1, @@(1), ch);;     update(ss, 210);;   # a pipe is used
```

Next, we consider the 15 pairwise disjoint clusters (see §1.7)

$$\mathfrak{c}_o := \bigl\{\mathfrak{o}\in\bar{\mathfrak{o}}_2 \bigm| \operatorname{supp}\mathfrak{o}\subset\operatorname{supp}\hbar\cup o\bigr\},\quad o\subset\bar\Omega,\ |o|=2,$$

and the 20 triple unions

$$\mathfrak{t}_s := \bigcup_{o\subset s}\mathfrak{c}_o,\quad s=\operatorname{supp}\mathfrak{o}\smallsetminus\operatorname{supp}\hbar\ \text{ for some } \mathfrak{o}\in\bar{\mathfrak{o}}_1.$$

By the puzzle assembly §1.5, we have

$$\mathcal{B}''_{512}(\mathfrak{c}_o < \mathfrak{t}_s < \mathfrak{t}_s\cup\mathfrak{t}_{\bar s} < \bar{\mathfrak{o}}_2) = \varnothing$$

for each pair $o\subset s$ as above and $\bar s := \bar\Omega\smallsetminus s$:

```
ds := @data(2, @(2)[1]);;         update(ds, 512);;
ss := @select(2, @(ds.dm), ds);;  update(ss, 512);;   # a pipe is used
```

It follows that $\bar b(\bar{\mathfrak{o}}_2) \leqslant 510$ and

$$\bar b(\mathfrak{O}) \leqslant 208_{1,3,4} + 510_2 = 718.$$

We found but three large geometric sets $\mathfrak{C}$, of sizes 660, 640, 640, which are graph isomorphic to some of those listed in §6.1 below.

6. The lattice $N(24\mathbf{A}_1)$

When depicting vectors in $(24\mathbf{A}_1)^\vee$, we use the following notation for the coefficients:

$$(\bullet)\mapsto 1,\quad (\circ)\mapsto -1,\quad (-)\mapsto \tfrac{1}{2},\quad (=)\mapsto -\tfrac{1}{2},\quad (+)\mapsto \tfrac{3}{2}.$$

For the Golay code $\mathcal{C}$ we use the generating matrix as in sage:

(6.1)

—···········—·—·———···——
·—··········—————··—··—·
··—·········——·—··—·—·——
···—········——···———·——·
····—·······——··——·——··—
·····—·······——··——·——·—
······—·······——··——·———
·······—····—·——·————···
········—····—·——·————··
·········—····—·——·————·
··········—·—·———···——·—
···········—·—·———···———

There are 2 classes; $b(\mathfrak{O})\leqslant 1736$.

1:	●●······················		5016	2508 → **1584**	
1:	●·●·····················	44^*	2	<u>1</u>	
2:	—————··—··—·—···········	77^{**}	64	32 →	<u>20</u> ✓
2:	—————··—··—·—···········		4536	2296 → **1736**	
1:	————·—········—··—·····—	280^*	16	8 →	<u>6</u> ✓
2:	●●······················	28	1	<u>1</u>	

6.1. Vector 1.

```
1  := rec(),
2  := rec(),
```

By the puzzle assembly §1.5 we establish that

$$\mathcal{B}''_{28;720}(\bar{\mathfrak{o}}_1;\bar{\mathfrak{o}}_2)=\varnothing:$$

```
ds := @data(1);;             update(ds,  28);;
ds := @data(2, ds);;         update(ds, 720);;
```

Next, we consider the 77 clusters (see §1.7)

$$\mathfrak{c}_{\mathfrak{s}} := \bigl\{\mathfrak{o}\in\bar{\mathfrak{o}}_2 \bigm| \operatorname{supp}\mathfrak{o}\cap\operatorname{supp}\mathfrak{s}=\operatorname{supp}\hbar\bigr\},\quad \mathfrak{s}\in\bar{\mathfrak{o}}_2,$$

and, by the puzzle assembly §1.5, we show that

$$\mathcal{B}''_{694}(\mathfrak{c}_{\mathfrak{s}}<\bar{\mathfrak{o}}_2)=\varnothing:$$

```
ds := @data(2, @(2)[1]);;     update(ds, 694);;    # a pipe is used
```

In view of (1.7), it follows that

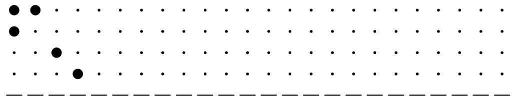
$$\bar{b}(\mathfrak{O})\leqslant 26_1+692_2=718.$$

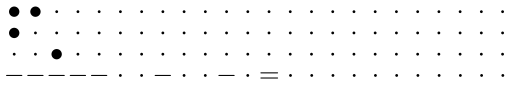
There are four exceptional sets. The complete list of the sets $\mathfrak{C}$ of size $|\mathfrak{C}|\geqslant 600$ that have been observed is as follows, where we show the lattice $\operatorname{ort}_h\mathfrak{C}$ of **Convention 5.2, the transcendental lattice(s) T of the corresponding quartic(s), and the (itemized according to T) numbers (r,c) of, respectively, real and pairs of complex conjugate deformation families of quartics:

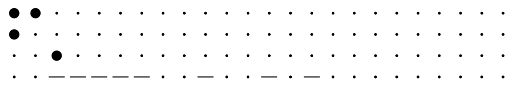
$|\mathfrak{C}|=800$:

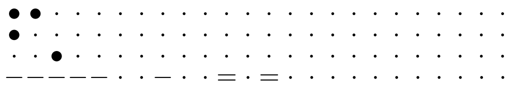
●●······················
●·······················
··●·····················
—————··—··—·—···········
————————————————————————

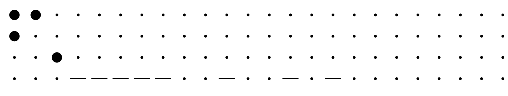
$[4,0,40]$ $(1,0)$

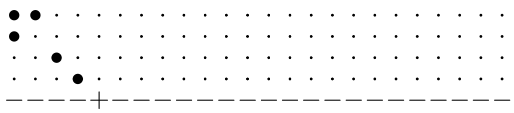
$|\mathfrak{C}| = 800$:  $[4, 0, 40]$ $(1, 0)$

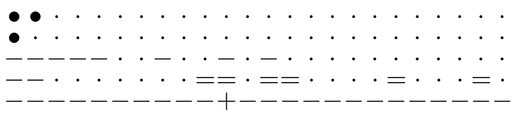
$|\mathfrak{C}| = 736$:  $[12, 0, 16]$ $(1, 0)$

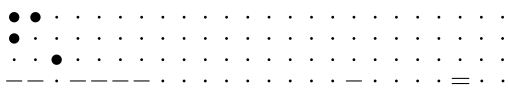
$|\mathfrak{C}| = 728$:  $[14, 0, 14]$ $(1, 0)$

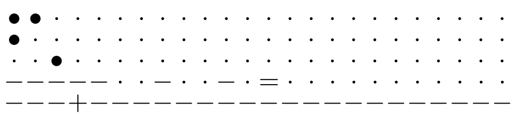
$|\mathfrak{C}| = 704$:  $[8, 4, 28]$ $(0, 1)$

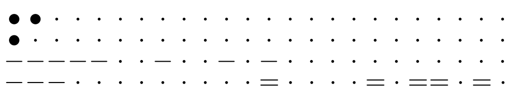
$|\mathfrak{C}| = 704$:  $[8, 4, 28]$ $(0, 1)$

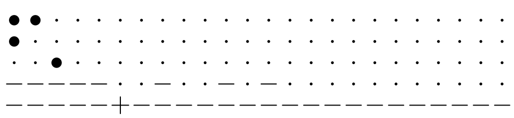
$|\mathfrak{C}| = 680$:  $[12, 4, 20]$ $(0, 1)$

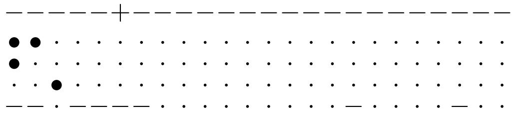
$|\mathfrak{C}| = 680$:  $[12, 4, 20]$ $(0, 1)$

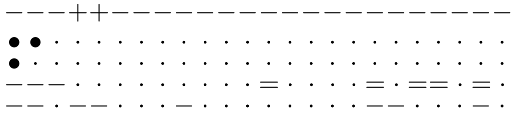
$|\mathfrak{C}| = 680$:  $[4, 2, 56]$ $(1, 0)$; $[16, 6, 16]$ $(1, 0)$

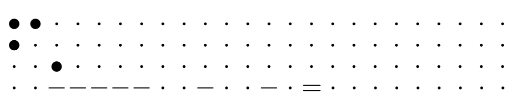
$|\mathfrak{C}| = 664$:  $[4, 0, 60]$ $(1, 0)$

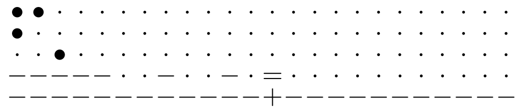
$|\mathfrak{C}| = 664$:  $[4, 0, 60]$ $(1, 0)$

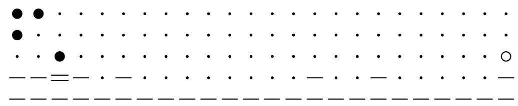
$|\mathfrak{C}| = 660$:  $[4, 0, 60]$ $(1, 0)$

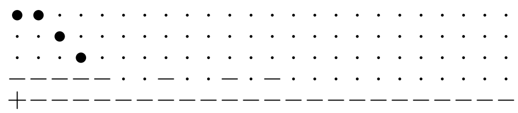
$|\mathfrak{C}| = 656$:  $[2, 0, 122]$ $(1, 0)$; $[10, 4, 26]$ $(0, 1)$

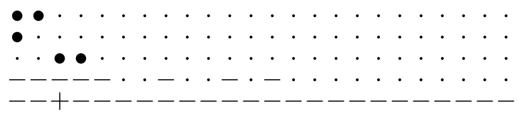
$|\mathfrak{C}| = 656$:  $[2, 0, 122]$ $(1, 0)$; $[10, 4, 26]$ $(0, 1)$

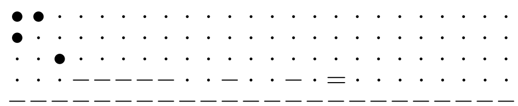
$|\mathfrak{C}| = 656$:  $[2, 0, 122]$ $(1, 0)$; $[10, 4, 26]$ $(0, 1)$

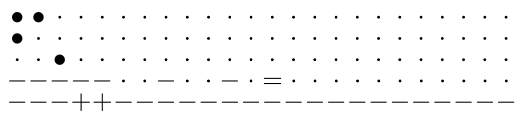
$|\mathfrak{C}| = 640$:  $[4, 0, 72]$ $(1, 0)$

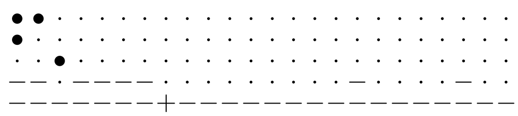
$|\mathfrak{C}| = 640$:  $[4, 0, 72]$ $(1, 0)$

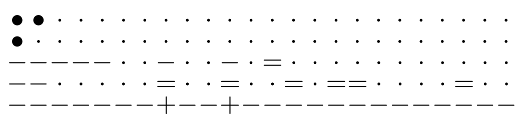
$|\mathfrak{C}| = 640$:  $[4, 0, 72]$ $(1, 0)$

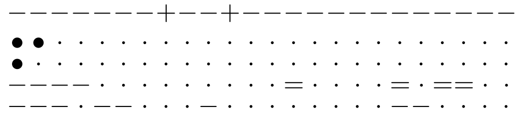
$|\mathfrak{C}| = 640$:  $[4, 0, 72]$ $(1, 0)$

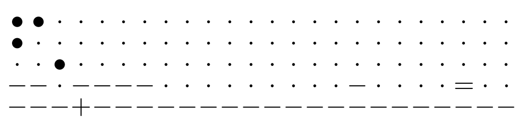
$|\mathfrak{C}| = 628$:  $[4, 0, 66]$ $(1, 0)$ $[12, 0, 22]$ $(1, 0)$

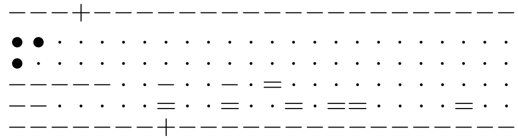
$|\mathfrak{C}| = 624$:  $[12, 4, 24]$ $(0, 2)$

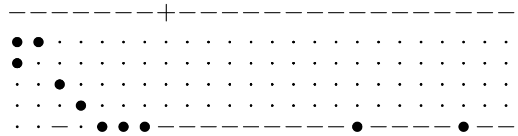
$|\mathfrak{C}| = 624$:  $[10, 2, 28]$ $(0, 1)$

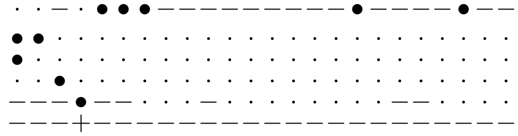
$|\mathfrak{C}| = 624$:  $[10, 2, 28]$ $(0, 1)$

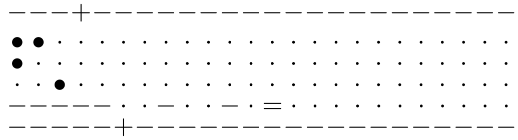
$|\mathfrak{C}| = 622$:  $[10, 2, 28]$ $(0, 1)$

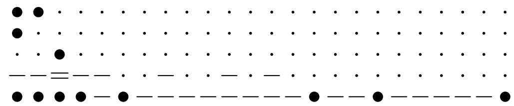
$|\mathfrak{C}| = 622$:  $[10, 2, 28]$ $(0, 1)$

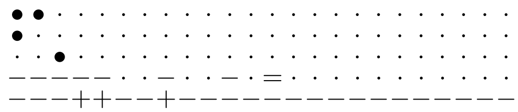
$|\mathfrak{C}| = 622$:  $[10, 2, 28]$ $(0, 1)$

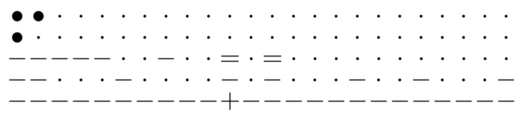
$|\mathfrak{C}| = 620$:  $[14, 0, 20]$ $(1, 0)$

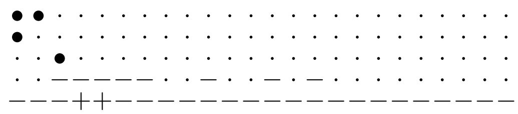
$|\mathfrak{C}| = 620$:  $[14, 0, 20]$ $(1, 0)$

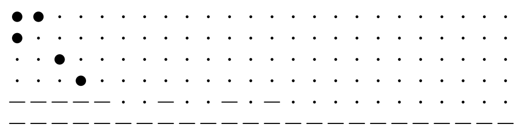
$|\mathfrak{C}| = 620$:  $[12, 4, 24]$ $(0, 2)$

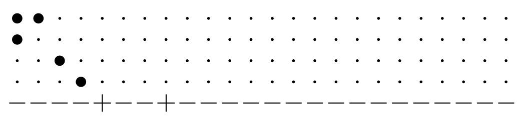
$|\mathfrak{C}| = 616$:

$[4, 0, 72]$ $(2, 0)$

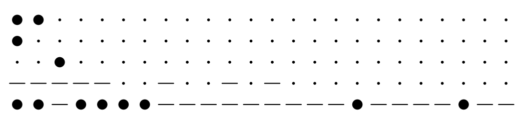
$|\mathfrak{C}| = 616$:

$[4, 0, 72]$ $(2, 0)$

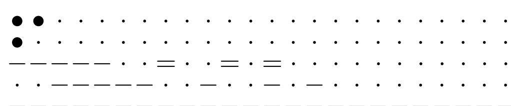
$|\mathfrak{C}| = 616$:

$[4, 0, 72]$ $(2, 0)$

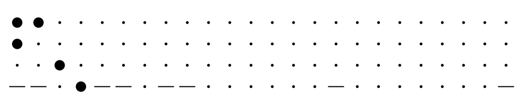
$|\mathfrak{C}| = 616$:

$[2, 0, 138]$ $(1, 0)$
$[12, 6, 26]$ $(1, 0)$

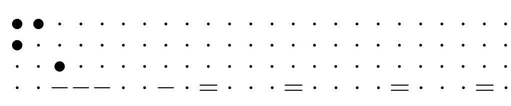
$|\mathfrak{C}| = 608$:

$\mathrm{rk} = 19$ $(1, 0)$

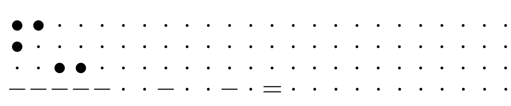
$|\mathfrak{C}| = 608$:

$[8, 0, 36]$ $(1, 0)$

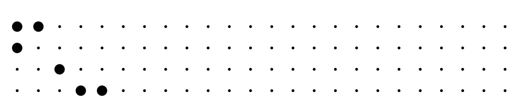
$|\mathfrak{C}| = 608$:

$[14, 4, 22]$ $(0, 1)$

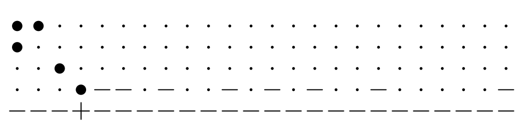
$|\mathfrak{C}| = 608$:

$[14, 4, 22]$ $(0, 1)$

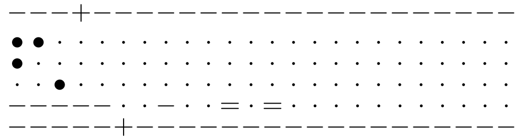
$|\mathfrak{C}| = 608$:

$[14, 4, 22]$ $(0, 1)$

$|\mathfrak{C}| = 608$:

$[14, 4, 22]$ $(0, 1)$

$|\mathfrak{C}| = 608$:

$[4, 0, 76]$ $(1, 0)$
$[16, 4, 20]$ $(0, 1)$

$|\mathfrak{C}| = 608$:

$[4, 0, 76]$ $(1, 0)$
$[16, 4, 20]$ $(0, 1)$

$|\mathfrak{C}| = 608$:

$[4, 0, 76]$ $(1, 0)$
$[16, 4, 20]$ $(0, 1)$

$|\mathfrak{C}| = 600$:

$[4, 0, 72]$ $(2, 0)$

6.2. Vector 2.

```
1  := rec(),
2  := rec(),
```

We consider the 30 clusters

$$\mathfrak{c}_o := \{\mathfrak{o} \in \bar{\mathfrak{o}}_1 \mid \operatorname{supp} \mathfrak{o} \subset o\}, \quad o \in \mathcal{C}_8 \text{ (an octad)}, \ o \cap \operatorname{supp} \hbar = \varnothing,$$

and, for o as above, we let $\bar{o} := \bar{\Omega} \smallsetminus o$. By the puzzle assembly §1.5 we show that

$$\mathcal{B}''_{702}(\mathfrak{c}_o < \mathfrak{c}_o \cup \mathfrak{c}_{\bar{o}} < \bar{\mathfrak{o}}_1) = \varnothing :$$

```
ds := @data(1, @(1)[1]);;         update(ds, 702);;   # a pipe is used
```

Likewise, we show that

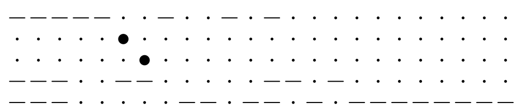
$$\mathcal{B}''_{18;720}(\bar{\mathfrak{o}}_2^+; \mathfrak{c}_o < \mathfrak{c}_o \cup \mathfrak{c}_{\bar{o}} < \bar{\mathfrak{o}}_1) = \varnothing :$$

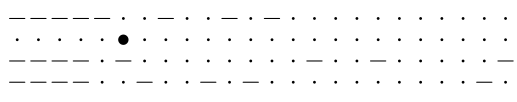
```
ds := @data(2);;                  update(ds, 18);;
ss := @select(1, @(1), ds);;      update(ss, 720);;   # a pipe is used
```

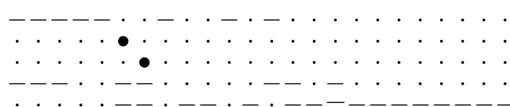
Hence, by (1.7),

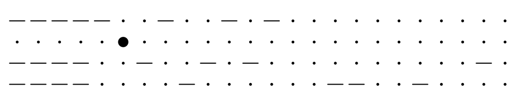
$$\bar{b}(\mathfrak{O}) \leqslant 700_1 + 16_2 = 716.$$

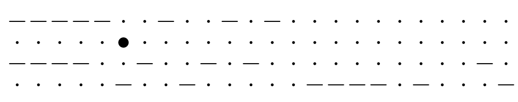
The complete list of the sets $\mathfrak{C}$ of size $|\mathfrak{C}| \geqslant 600$ that have been observed is as follows (in the same notation as in §6.1):

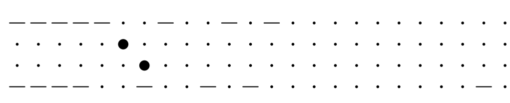
$|\mathfrak{C}| = 704$:  $[8, 4, 28]$ $(0, 1)$

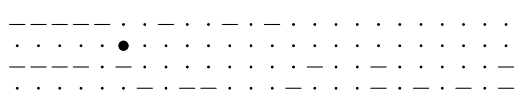
$|\mathfrak{C}| = 704$:  $[8, 4, 28]$ $(0, 1)$

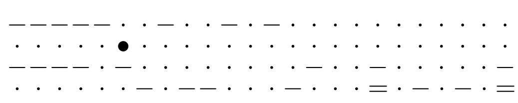
$|\mathfrak{C}| = 680$:  $[12, 4, 20]$ $(0, 1)$

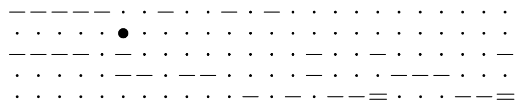
$|\mathfrak{C}| = 680$:  $[12, 4, 20]$ $(0, 1)$

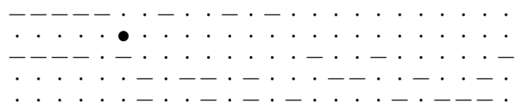
$|\mathfrak{C}| = 664$:  $[4, 0, 60]$ $(1, 0)$

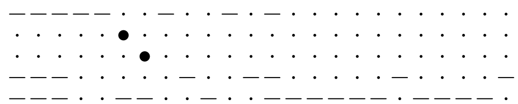
$|\mathfrak{C}| = 656$:  $[2, 0, 122]$ $(1, 0)$ $[10, 4, 26]$ $(0, 1)$

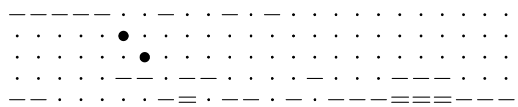
$|\mathfrak{C}| = 656$:  $[2, 0, 122]$ $(1, 0)$ $[10, 4, 26]$ $(0, 1)$

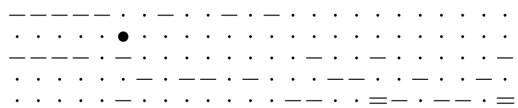
$|\mathfrak{C}| = 656$:  $[2, 0, 122]$ $(1, 0)$ $[10, 4, 26]$ $(0, 1)$

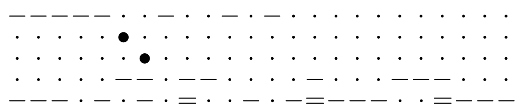
$|\mathfrak{C}| = 656$:  $[2, 0, 122]$ $(1, 0)$ $[10, 4, 26]$ $(0, 1)$

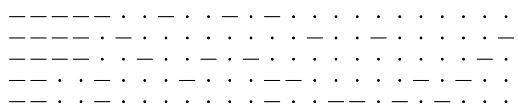
$|\mathfrak{C}| = 656$: 

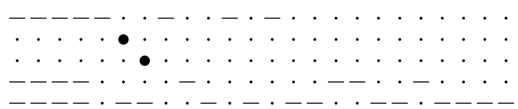
$|\mathfrak{C}| = 624$:  $[12, 4, 24]$ $(0, 2)$

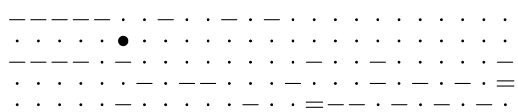
$|\mathfrak{C}| = 624$:  $[12, 4, 24]$ $(0, 2)$

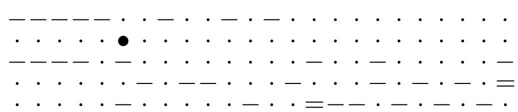
$|\mathfrak{C}| = 624$:  $[12, 4, 24]$ $(0, 2)$

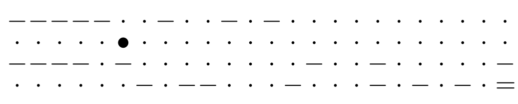
$|\mathfrak{C}| = 624$:  $[10, 2, 28]$ $(0, 1)$

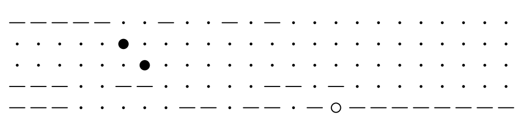
$|\mathfrak{C}| = 624$: 

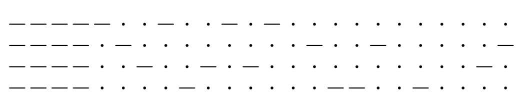
$|\mathfrak{C}| = 620$:  $[12, 4, 24]$ $(0, 2)$

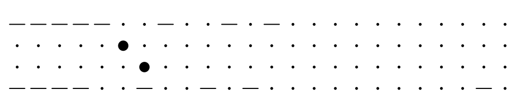
$|\mathfrak{C}| = 620$:  $[12, 4, 24]$ $(0, 2)$

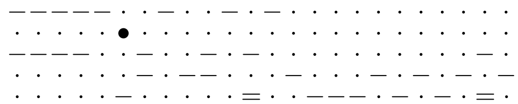
$|\mathfrak{C}| = 620$:  $[12, 4, 24]$ $(0, 2)$

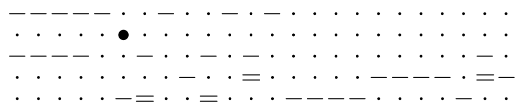
$|\mathfrak{C}| = 620$:  $[12, 4, 24]$ $(0, 2)$

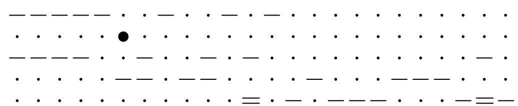
$|\mathfrak{C}| = 616$:  $[4, 0, 72]$ $(2, 0)$

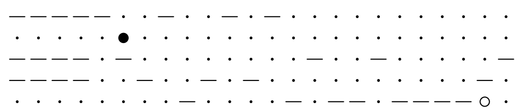
$|\mathfrak{C}| = 616$:  $[4, 0, 72]$ $(2, 0)$

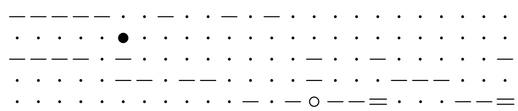
$|\mathfrak{C}| = 616$:  $[2, 0, 138]$ $(1, 0)$ $[12, 6, 26]$ $(1, 0)$

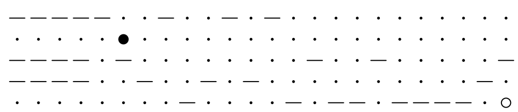
$|\mathfrak{C}| = 616$:  $[2, 0, 138]$ $(1, 0)$ $[12, 6, 26]$ $(1, 0)$

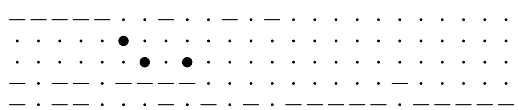
$|\mathfrak{C}| = 616$:  $[2, 0, 138]$ $(1, 0)$ $[12, 6, 26]$ $(1, 0)$

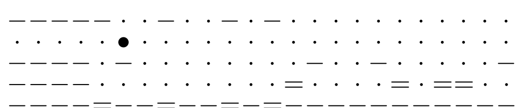
$|\mathfrak{C}| = 616$:  $[2, 0, 138]$ $(1, 0)$ $[12, 6, 26]$ $(1, 0)$

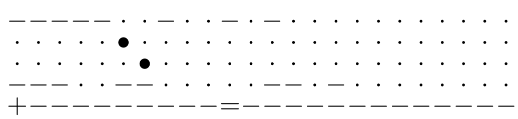
$|\mathfrak{C}| = 608$:  $[14, 4, 22]$ $(0, 1)$

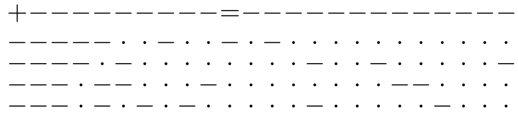
$|\mathfrak{C}| = 608$:  $[14, 4, 22]$ $(0, 1)$

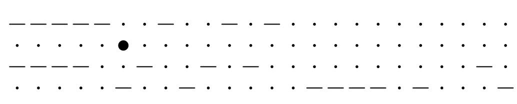
$|\mathfrak{C}| = 608$:  $[8, 0, 36]$ $(1, 0)$

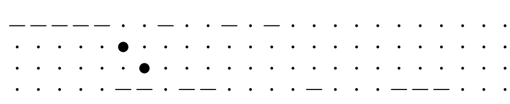
$|\mathfrak{C}| = 608$:  $[4, 0, 76]$ $(1, 0)$ $[16, 4, 20]$ $(0, 1)$

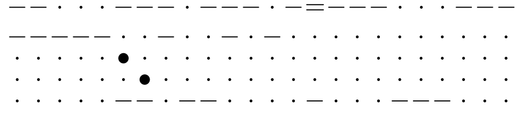
$|\mathfrak{C}| = 608$:  $[4, 0, 76]$ $(1, 0)$ $[16, 4, 20]$ $(0, 1)$

$|\mathfrak{C}| = 600$:  $[4, 0, 72]$ $(2, 0)$

$|\mathfrak{C}| = 600$:  $[4, 0, 72]$ $(2, 0)$

$|\mathfrak{C}| = 600$:  $[12, 0, 24]$ $(1, 0)$

$|\mathfrak{C}| = 600$:  $[12, 0, 24]$ $(1, 0)$

$|\mathfrak{C}| = 600$:  $[12, 0, 24]$ $(1, 0)$

7. The Leech lattice

The Leech lattice Λ has a unique, up to $O(\Lambda)$, 4-polarization $h \in \Lambda$. The proof of **Theorem 6.1 is based on **Lemma 6.2. Namely, we start with $\mathfrak{O}_h$ and build chains

$$\mathfrak{O}_h =: \mathfrak{C}_0 \supset \mathfrak{C}_1 \supset \ldots$$

of self-dual (*a priori* not saturated) subsets recursively: once a set $\mathfrak{C}_k$ has been constructed, we

(1) consider the quotient projection $(\operatorname{mod} 2)$: $\operatorname{span}_{\mathbb{Z}} \mathfrak{C}_k \to V_k := (\operatorname{span}_{\mathbb{Z}} \mathfrak{C}_k) \otimes \mathbb{F}_2$;
(2) compute the $(\operatorname{stab} \mathfrak{C}_k)$-orbits on the annihilator $h^\perp \subset V_k^\vee$ (as all sets are self-dual);
(3) for a representative v of each orbit, let $\mathfrak{C}_{k+1} := \mathfrak{C}_k \cap (\operatorname{mod} 2)^{-1}(v^\perp)$;
(4) discard the subsets $\mathfrak{C}_{k+1}$ of cardinality

$$|\mathfrak{C}_{k+1}| < 720 + \max\{0, 2(\operatorname{rk} \mathfrak{C}_{k+1} - 20)\}$$

(as at least two conics are lost when the rank is reduced by one unit).

Several tricks described in §7.1–§7.5 below are used to speed up the algorithm.

Remark 7.1. In fact, we work with the index 2 sublattice $\Lambda' \subset \Lambda$ generated by $\mathfrak{O}$; it has a larger orthogonal group: $[O_h(\Lambda') : O_h(\Lambda)] = 2$.

7.1. **Follow the proof of **Lemma 6.2 literally.** We do not accept a new set $\mathfrak{C}_{k+1}$ unless

$$\text{the group} \quad \operatorname{span} \mathfrak{C}_{k+1} / \operatorname{span}_{\mathbb{Z}} \mathfrak{C}_{k+1} \quad \text{is cyclic or free abelian.}$$

More precisely, we keep track of the last

$$\textit{saturated} \text{ set } \mathfrak{S} := \mathfrak{C}_r,\ r \leqslant k, \quad \text{its lattice } \operatorname{span}_{\mathbb{Z}} \mathfrak{S}, \quad \text{and group } H := \operatorname{stab} \mathfrak{S}, \tag{7.2}$$

and it is the quotient

$$Q_{k+1} := \operatorname{span}_{\mathbb{Z}} \mathfrak{S} / \operatorname{span}_{\mathbb{Z}} \mathfrak{C}_{k+1} \tag{7.3}$$

that we check. If $Q_k \neq 0$ is a finite cyclic group, generated by (the image of) a vector $\bar\kappa \in \operatorname{span}_{\mathbb{Z}} \mathfrak{S}$, we keep the vector $\kappa := |Q_k|\bar\kappa \in \operatorname{span}_{\mathbb{Z}} \mathfrak{C}_k$ and, at the next step, reduce $V_k^\vee$ to the affine space

$$\bar V_k^\vee := \{v \in V_k^\vee \mid \langle v, \kappa\rangle \neq 0\}.$$

7.2. **Compute and sort rank by rank, set by set, index by index.** The saturated sets $\mathfrak{S}$ as in (7.2) play a special rôle: each such set is considered separately and is exhausted index by index (referring to the size $|Q_{k+1}|$ in (7.3)) until the rank drops. Thus, for each new set $\mathfrak{C}_{k+1} \subset \mathfrak{S}$,

- if $\operatorname{rk} \mathfrak{C}_{k+1} < \operatorname{rk} \mathfrak{S}$, then $\mathfrak{C}_{k+1}$ is carried over to the next rank, see below;
- otherwise, $\mathfrak{C}_{k+1}$ is stored for a subsequent step within $\mathfrak{S}$.

In the latter case, we retain a single representative of each H-orbit. To compare two sets $\mathfrak{C}'$, $\mathfrak{C}''$, we compare their

$$\text{ranks } \operatorname{rk} \mathfrak{C}^*, \quad \text{sizes } |\mathfrak{C}^*|, \quad \text{indices } |Q^*|, \quad \text{graph isomorphism classes} \tag{7.4}$$

(for the latter, the sets are stored according to the canonical labelling given by the `digraph` package, so that it remains to compare their adjacency matrices); if everything coincides, we check if $\mathfrak{C}'$, $\mathfrak{C}''$ belong to the same orbit *via*

```
RepresentativeAction(H, Set(C'), Set(C''), OnSets) <> fail;
```

When storing a set $\mathfrak{C}_{k+1}$ of lower rank $\operatorname{rk} \mathfrak{C}_{k+1} < \operatorname{rk} \mathfrak{S}$, we stop at (7.4): indeed, we are primarily interested in subsets $\mathfrak{C} \subset \mathfrak{O}$ as abstract graphs rather than particular embeddings into Λ. However, if a new set $\mathfrak{C}'$ has the stabilizer $\operatorname{stab} \mathfrak{C}'$ (to be used as H in a subsequent step) larger that that of a set $\mathfrak{C}''$ already present on the list, then $\mathfrak{C}'$ replaces $\mathfrak{C}''$.

Remark 7.5. *A posteriori*, we show (see §7.6 below or **§6.2 for more details) that each of the large sets of rank at most 20 found admits a unique, up to $O_h(\Lambda)$, embedding to $\mathfrak{O}$.

7.3. **Choose a standard basis for $V_k^\vee$.** When computing the orbits (and even the spaces $\bar{V}_k^\vee$), a great deal of time is spent to the mere listing of $V_k^\vee$, converting it from a vector space to a set on which the group $G_k := \operatorname{stab} \mathfrak{C}_k$ acts. On the other hand, $V_k^\vee$ is an $\mathbb{F}_2$-vector space of dimension $19 \leqslant \dim V_k^\vee \leqslant 23$. Therefore, we conjugate the G_k-action to the standard basis and keep the five "standard" vector spaces pre-listed as sets.

7.4. **Compute representatives rather than orbits.** For small groups (we use $|G| \leqslant 2^{13}$: this may be arbitrary, but it works well), in item (2) above, instead of computing the whole orbits and selecting representatives *via*

```
V := List(OrbitsDomain(G, V), o -> o[1]);
```

we compute a list of representatives directly:

```
G := List(G);
V := Filtered(V, v -> ForAll(G, v <= v*g));
```

7.5. **Count both hits *and misses*.** In items (3), (4) above, for each orbit representative $v \in V_k^\vee$, a "natural" way to proceed would be to compute $\mathfrak{C}_{k+1}$ first, item (3), and then discard it unless $|\mathfrak{C}_{k+1}|$ is large, item (4):

```
new := Filtered(old, u -> u*o = Z(2)*0);
if Length(new) < min_size then return fail; fi;
return new;
```

where we let `old :=` $\mathfrak{C}_k$ and `new :=` $\mathfrak{C}_{k+1}$ or `fail`. However, the number of orbits is often huge (it is not uncommon that $|G_k| = 4$ or even 2), and the following less straightforward code reduces the time dramatically:

```
new := [ ];
c   := Length(old) - min_count;       # = 720
for u in old do
  if u*v = Z(2)*0 then
    Add(new, u);
  else
    c := c - 1;
    if c < 0 then return fail; fi;    # too many misses: abort!!
  fi;
od;
return new;
```

In other words, we count the misses as well, not only hits, and we abort prematurely as soon as too many misses have been encountered. This works well because the vast majority of sets $\mathfrak{C}_k$ to be dealt with are relatively small, $|\mathfrak{C}_k| < 800$ (the numbers growing quite fast as $|\mathfrak{C}_k|$ approaches the goal 720), and we can abort after but a few misses. In particular, we avoid the creation of large lists never to be used, followed by garbage collection.

7.6. **The results.** The output of the algorithm is but nine sets $\mathfrak{C}_k \subset \mathfrak{O}$ of rank $\operatorname{rk}\mathfrak{C} \leqslant 20$ and size $|\mathfrak{C}| \geqslant 720$. All sets are saturated and of rank 20, and their sizes are

$$|\mathfrak{C}| = 896, \underline{800}, 768, 760, 740, \underline{736}, 736, \underline{728}, \underline{720},$$

where underlined are the geometric ones. The sets are described below by means of the Gram matrix of the lattice $\operatorname{ort}_h \mathfrak{C}$ of **Convention 5.2, where the first basis vector is always h. For the four geometric sets we show also the generators of $\operatorname{ort}_h \mathfrak{C}$ in the "standard" model $\Lambda \subset \mathbf{H}_{24}(\frac{1}{8})$, see **§6.2, using the following notation for the coefficients:

$$(\bullet) \mapsto 4, \quad (\circ) \mapsto -4, \quad (+) \mapsto 2, \quad (-) \mapsto -2.$$

For transparency, we assume that $\{1, \ldots, 8\}$ and $\{5, \ldots, 12\}$ are octads of the Golay code, and we cut the display at position 14, extending each vector by zeroes. The four geometric sets $\mathfrak{C}$ are

$$(7.6) \qquad |\mathfrak{C}| = 800\colon \begin{bmatrix} 4 & 0 & 0 & 0 & 2 \\ 0 & 4 & 0 & 1 & 2 \\ 0 & 0 & 4 & 1 & 2 \\ 0 & 1 & 1 & 4 & 0 \\ 2 & 2 & 2 & 0 & 4 \end{bmatrix} \quad \begin{array}{cccccccccccccc} \cdot & \cdot & \cdot & \cdot & \bullet & \cdot & \cdot & \circ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \bullet & \cdot & \cdot & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & + & + & - & + & - & + & + & + & \cdot & \cdot \\ + & + & + & + & + & + & + & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & + & + & - & - & + & + & + & + & \cdot & \cdot \end{array}$$

$$(7.7) \qquad |\mathfrak{C}| = 736\colon \begin{bmatrix} 4 & 2 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 2 \\ 0 & 0 & 4 & 1 & 2 \\ 0 & 0 & 1 & 4 & 2 \\ 0 & 2 & 2 & 2 & 4 \end{bmatrix} \quad \begin{array}{cccccccccccccc} \circ & \cdot & \cdot & \cdot & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \bullet & \cdot & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot \\ + & + & + & + & + & + & + & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & + & + & + & + & + & + & + & + & \cdot & \cdot \end{array}$$

$$(7.8) \qquad |\mathfrak{C}| = 728\colon \begin{bmatrix} 4 & 2 & 2 & 2 & 2 \\ 2 & 4 & 0 & 1 & 0 \\ 2 & 0 & 4 & 0 & 1 \\ 2 & 1 & 0 & 4 & 2 \\ 2 & 0 & 1 & 2 & 4 \end{bmatrix} \quad \begin{array}{cccccccccccccc} \cdot & \cdot & \cdot & \cdot & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ + & + & + & + & + & + & + & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ + & - & - & - & + & + & - & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & + & + & + & - & - & + & + & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & + & + & - & - & + & + & + & + & \cdot & \cdot \end{array}$$

$$(7.9) \qquad |\mathfrak{C}| = 720\colon \begin{bmatrix} 4 & 0 & 0 & 2 & 2 \\ 0 & 4 & 0 & 0 & 2 \\ 0 & 0 & 4 & 0 & 2 \\ 2 & 0 & 0 & 4 & 1 \\ 2 & 2 & 2 & 1 & 4 \end{bmatrix} \quad \begin{array}{cccccccccccccc} \bullet & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot \\ + & + & + & + & + & + & + & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

The other five sets $\mathfrak{C}$ are as follows (listed in the decreasing order):

$$(7.10) \quad (a) \begin{bmatrix} 4 & 0 & 0 & 2 & 2 \\ 0 & 4 & 0 & 0 & 2 \\ 0 & 0 & 4 & 0 & 2 \\ 2 & 0 & 0 & 4 & 0 \\ 2 & 2 & 2 & 0 & 4 \end{bmatrix}, (b) \begin{bmatrix} 4 & 0 & 0 & 2 & 0 \\ 0 & 4 & 0 & 2 & 2 \\ 0 & 0 & 4 & 0 & 2 \\ 2 & 2 & 0 & 4 & 0 \\ 0 & 2 & 2 & 0 & 4 \end{bmatrix}, (c) \begin{bmatrix} 4 & 2 & 2 & 2 & 2 \\ 2 & 4 & 1 & 0 & 0 \\ 2 & 1 & 4 & 0 & 0 \\ 2 & 0 & 0 & 4 & 1 \\ 2 & 0 & 0 & 1 & 4 \end{bmatrix}, (d) \begin{bmatrix} 4 & 0 & 0 & 0 & 2 \\ 0 & 4 & 0 & 0 & 2 \\ 0 & 0 & 4 & 0 & 1 \\ 0 & 0 & 0 & 4 & 2 \\ 2 & 2 & 1 & 2 & 4 \end{bmatrix}.$$

The non-geometric sets of size 768 and 736 share the same lattice (7.10b), which has two distinct embeddings to $\Lambda \ni h$; for the other seven sets in (7.6)–(7.10), an embedding $(\operatorname{ort}_h \mathfrak{C} \ni h) \hookrightarrow (\Lambda \ni h)$ is unique up to $O_h(\Lambda)$: since only square 4 vectors are involved, this fact can easily be established by brute force.

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