800 CONICS IN A SMOOTH QUARTIC SURFACE

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ABSTRACT. We construct an example of a smooth spatial quartic surface that contains 800 irreducible conics.

1. INTRODUCTION

This short note was motivated by Barth, Bauer [1], Bauer [2], and my recent paper [4]. Generalizing [2], define $N_{2n}(d)$ as the maximal number of smooth rational curves of degree d that can lie in a smooth degree $2n$ $K3$-surface $X \subset \mathbb{P}^{n+1}$. (All algebraic varieties considered in this note are over $\mathbb{C}$.) The bounds $N_{2n}(1)$ have a long history and currently are well known, whereas for $d = 2$ the only known value is $N_6(2) = 285$ (see [4]). In the most classical case $2n = 4$ (spatial quartics), the best known examples have 352 or 432 conics (see [1, 2]), whereas the best known upper bound is 5016 (see [2], with a reference to S. A. Strømme).

For $d = 1$, the extremal configurations (for various values of n) tend to exhibit similar behaviour. Hence, contemplating the findings of [4], one may speculate that

- it is easier to count *all* conics, both irreducible and reducible, and
- nevertheless, in extremal configurations all conics are irreducible.

On the other hand, famous *Schur's quartic* (the one on which the maximum $N_4(1)$ is attained) has 720 conics (mostly reducible), suggesting that 432 should be far from the maximum $N_4(2)$. Therefore, in this note I suggest a very simple (although also implicit) construction of a smooth quartic with 800 irreducible conics.

Theorem 1.1 (see §3.3)**.** *There exists a smooth quartic surface $X_4 \subset \mathbb{P}^3$ containing 800 irreducible conics.*

I conjecture that $N_4(2) = 800$ and, moreover, 800 is the sharp upper bound on the total number of conics (irreducible or reducible) in a smooth spatial quartic.

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2. THE LEECH LATTICE (see [3])

2.1. **The Golay code.** The (*extended binary*) *Golay code* is the only binary code of length 24, dimension 12, and minimal Hamming distance 8. We regard codewords as subsets of $\Omega := \{1, \ldots, 24\}$ and denote this collection of subsets by $\mathcal{C}$; clearly, $|\mathcal{C}| = 2^{12}$. The code $\mathcal{C}$ is invariant under the complement $o \mapsto \Omega \smallsetminus o$. Apart from $\varnothing$

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and Ω itself, it consists of 759 *octads* (codewords of length 8), 759 complements thereof, and 2576 *dodecads* (codewords of length 12).

The setwise stabilizer of $\mathcal{C}$ in the full symmetric group $\mathbb{S}(\Omega)$ is the Mathieu group M_{24} of order 244823040; the actions of this group on Ω and $\mathcal{C}$ are described in detail in § 2 of [3, Chapter 10].

2.2. **The square 4 vectors.** The *Leech lattice* is the only root-free unimodular even positive definite lattice of rank 24. For the construction, consider the standard Euclidean lattice $E := \bigoplus_i \mathbb{Z}e_i$, $i \in \Omega$, and divide the form by 8, so that $e_i^2 = 1/8$. (Thus, we avoid the factor $8^{-1/2}$ appearing throughout in [3].) Then, the Leech lattice is the sublattice $\Lambda \subset E$ spanned over $\mathbb{Z}$ by the square 4 vectors of the form

$$(\mp 3, \pm 1^{23}) \qquad \text{(the upper signs are taken on a codeword } o \in \mathcal{C}\text{).} \tag{2.1}$$

(We use the notation of [3]: $a^m, b^n, \ldots$ means that there are m coordinates equal to a, n coordinates equal to b, *etc.*) Apart from (2.1), the square 4 vectors in Λ are

$$(\pm 2^8, 0^{16}) \qquad (\pm 2 \text{ are taken on an octad, the number of } + \text{ is even}), \text{ or} \tag{2.2}$$

$$(\pm 4^2, 0^{22}) \qquad \text{(no further restrictions).} \tag{2.3}$$

Altogether, there are 196560 square 4 vectors: $24 \cdot |\mathcal{C}| = 98304$ vectors as in (2.1), $2^7 \cdot 759 = 97152$ vectors as in (2.2), and $2^2 \cdot C(24, 2) = 1104$ vectors as in (2.3).

3. The construction

In this section, we prove Theorem 1.1.

3.1. **The lattice S.** Consider the lattice $V := \mathbb{Z}\hbar + \mathbb{Z}a + \mathbb{Z}u_1 + \mathbb{Z}u_2 + \mathbb{Z}u_3$ with the Gram matrix

$$\begin{bmatrix} 4 & 2 & 0 & 0 & 0 \\ 2 & 4 & 2 & 0 & 1 \\ 0 & 2 & 4 & 2 & -1 \\ 0 & 0 & 2 & 4 & 0 \\ 0 & 1 & -1 & 0 & 4 \end{bmatrix}.$$

It can be shown that, up to $O(\Lambda)$, there is a unique primitive isometric embedding $V \to \Lambda$; however, for our example, we merely choose a particular model. Fix an ordered quintuple $Q := (1, 2, 3, 4, 5) \subset \Omega$ and choose one of the four octads O such that $O \cap Q = \{1, 2, 4, 5\}$ (*cf. sextets* in § 2.5 of [3, Chapter 10]); upon reordering Ω, we can assume that $O = \{1, 2, 4, 5, 6, 7, 8, 9\}$ (the underlined positions in the top row of Table 1). Then, the generators of V can be chosen as shown in the upper part of Table 1. (For better readability, we represent zeros by dots; all components beyond $\bar{O} := Q \cup O$ are zeros.)

The choice of Q and O is unique up to M_{24}; furthermore, the subgroup $G \subset M_{24}$ stabilising Q pointwise and O as a set can be identified with the alternating group $\mathbb{A}(O \smallsetminus Q)$; in particular, it acts simply transitively on the set of ordered pairs

$$(p, q): \quad p, q \in O \smallsetminus Q = \{6, 7, 8, 9\}, \quad p \neq q. \tag{3.1}$$

Define a *conic* as a square 4 vector $l \in \Lambda$ such that

$$l \cdot \hbar = 2, \quad l \cdot a = 1, \quad l \cdot u_1 = l \cdot u_2 = l \cdot u_3 = 0.$$

This strange condition can be recast as follows: $l \cdot \hbar = 2$ and l (as well as $\hbar$) lies in the rank 20 lattice

$$S := \bar{V}^\perp \subset \Lambda, \quad \text{where } \bar{V} := \hbar_V^\perp.$$

Table 1. The lattice V and the conics

#	$\underline{1}$	$\underline{2}$	3	$\underline{4}$	$\underline{5}$	$\underline{6}$	$\underline{7}$	$\underline{8}$	$\underline{9}$	
$\hbar$	4	4	·	·	·	·	·	·	·	
a	·	4	4	·	·	·	·	·	·	
u_1	·	·	4	4	·	·	·	·	·	
u_2	·	·	·	4	4	·	·	·	·	
u_3	-2	2	·	-2	2	2	2	2	2	
1	1	3	-1	1	-1	1	1	-1^*	-1^*	$\pm 1^{15}$
2	3	1	1	-1	1	1	1	-1^*	-1^*	$\pm 1^{15}$
3	2	2	·	·	·	·	·	·	·	$\pm 2^6, 0^9$
4	2	2	·	·	·	·	·	2^*	-2^*	$\pm 2^4, 0^{11}$
	fixed $= Q$					movable in $O \smallsetminus Q$				

Table 2. The number of conics in S

$$
\begin{aligned}
\mathbf{1}\colon &\quad C(4,2)\cdot\underline{16} = 96 \quad (\text{sets } o \in \mathcal{C} \text{ such that } o \cap \bar{O} = \{2,3,5,p,q\}),\\
\mathbf{2}\colon &\quad C(4,2)\cdot\underline{16} = 96 \quad (\text{sets } o \in \mathcal{C} \text{ such that } o \cap \bar{O} = \{1,4,p,q\}),\\
\mathbf{3}\colon &\quad 2^5\cdot\underline{10} = 320 \quad (\text{octads } o \in \mathcal{C} \text{ such that } o \cap \bar{O} = \{1,2\}),\\
\mathbf{4}\colon &\quad 2^3\cdot P(4,2)\cdot\underline{3} = 288 \quad (\text{octads } o \in \mathcal{C} \text{ such that } o \cap \bar{O} = \{1,2,p,q\}).
\end{aligned}
$$

Using §2.2, we conclude that each conic fits one of the four patterns shown at the bottom of Table 1: there are two for (2.1) and two for (2.2). (If l is as in (2.3), we have $l \cdot a = 0 \bmod 2$.) The number of conics within each pattern is computed as shown in Table 2, where

- the ordered or unordered pair (p,q) as in (3.1) designates the two variable special positions marked with a * in Table 1,
- the underlined factor counts certain codewords $o \in \mathcal{C}$; the restrictions given by (2.1) or (2.2) are described in the parentheses, and
- the other factors account for the choice of (p,q) and/or signs in ± 2.

These counts sum up to 800.

3.2. **The Néron–Severi lattice.** Observe that $\hbar \in 2S^\vee$: indeed, $\hbar - 2a \in \bar{V}$ and we have $x \cdot \hbar = 2x \cdot a = 0 \bmod 2$ for any $x \in S$. Thus, we can apply to $S \ni \hbar$ the construction of [4], *i.e.*, consider the orthogonal complement $\hbar_S^\perp = V^\perp \subset \Lambda$, reverse the sign of the form, and pass to the index 2 extension

$$N := \left(-(\hbar_S^\perp) \oplus \mathbb{Z}h\right)_2^\sim, \quad h^2 = 4,$$

containing the vector $c := c(l) := l - \frac{1}{2}\hbar + \frac{1}{2}h$ for some (equivalently, any) conic $l \in S$. These new vectors $c \in N$ are also called *conics*; one obviously has

$$c^2 = -2 \quad \text{and} \quad c \cdot h = 2. \tag{3.2}$$

They are in a bijection with the conics in S; hence, there are 800 of them.

Starting from

$$\operatorname{discr} V \cong \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \oplus \left[\tfrac{1}{8}\right] \oplus \left[\tfrac{2}{5}\right]$$

(see Nikulin [6] for the concept of *discriminant form* $\operatorname{discr} V := V^\vee/V$ and related techniques), we easily compute

$$\mathcal{N} := \operatorname{discr} N \cong \left[\tfrac{5}{4}\right] \oplus \left[\tfrac{1}{8}\right] \oplus \left[\tfrac{2}{5}\right] \cong \left[-\tfrac{1}{4}\right] \oplus \left[-\tfrac{5}{8}\right] \oplus \left[\tfrac{2}{5}\right].$$

Therefore, $-\mathcal{N} \cong \operatorname{discr} T$, where $T := \mathbb{Z}b \oplus \mathbb{Z}c$, $b^2 = 4$, $c^2 = 40$. Then, it follows from [6] that there is a primitive isometric embedding of the hyperbolic lattice N to the intersection lattice H_2 of a $K3$-surface, so that $T \cong N^\perp$ play the rôle of the transcendental lattice. Finally, by the surjectivity of the period map [5], we conclude that there exists a $K3$-surface X with $NS(X) \cong N$.

3.3. **Proof of Theorem 1.1.** The Néron–Severi lattice $NS(X) \cong N$ constructed in the previous section is equipped with a distinguished polarisation $h \in N$, $h^2 = 4$. Since the original lattice $S \subset \Lambda$ is root free, N does *not* contain any of the following "bad" vectors:

- $e \in N$ such that $e^2 = -2$ and $e \cdot h = 0$ (*exceptional divisors*) or
- $e \in N$ such that $e^2 = 0$ and $e \cdot h = 2$ (2-*isotropic vectors*)

(see [4] for details). Hence, by Nikulin [7] and Saint-Donat [8], the linear system $|h|$ is fixed point free and maps X onto a smooth quartic surface $X_4 \subset \mathbb{P}^3$.

The lattice N contains 800 conics c as in (3.2). By the Riemann–Roch theorem, each class c is effective, *i.e.*, represented by a curve $C \subset X_4$ of projective degree 2. Since X is smooth and contains no lines (or other curves of odd degree, as we have $h \in 2N^\vee$ by the construction), each of these curves C is irreducible. This concludes the proof of Theorem 1.1. □

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