Math 535 – Topology of Algebraic Varieties, Fall 2010
Midterm II (due on December 22, 2010)

Name: ______________________________
I.D.: ______________________________

First problem is a mere dull computation. Everyone should do it!

Problem 1. Let X be the Euclidean space $\mathbb{R}^3$ with the standard Euclidean metric and orthonormal basis. Express 0-, 1-, 2, and 3-forms on X in terms of vector and scalar fields. Express the known operators grad, curl, and div in terms of d and the Hodge $*$-operator. Also, express the Laplace operator for forms in terms of (the components of) the corresponding vector/scalar fields.

The aim of the following two problems is to understand the intersection index form in the 2-(co-)homology of a Kähler surface. These results generalize to higher dimensions, but even the statements are way too more involved.

The following observation may be useful. As we know, in general a metric does not need to be standard, even locally: curvature is an obstruction. However, **at** any given point a metric **is** standard. Hence, in all fiberwise computations (*i.e.*, those not involving d), you can always use your favorite Hermitian metric. In addition, in Problem 3 you can assume all forms harmonic. (Why?)

Problem 2. Let X be a Kähler surface and ω the fundamental form of its Kähler metric. Show that:

(1) ω is closed and coclosed, hence harmonic;
(2) $\omega \wedge \bar{\omega}$ is a volume form, *i.e.*, locally it is $f\, dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2$ for some **positive** real valued function f.

Assuming X compact, show that the product of two harmonic forms is harmonic.

Problem 3. In the settings of problem 2, assume X compact and consider the sesquilinear form $\alpha \circ \beta := \int_X \alpha \wedge \bar{\beta}$, where α and β are two forms. It defines a sesquilinear form on $H^2(X; \mathbb{C})$. (Why?) Show that:

(1) the Hodge subspaces $H^{p,q}$ are pairwise orthogonal with respect to $\circ$;
(2) if $\alpha \neq 0$ belongs to $H^{2,0}$ or $H^{0,2}$, then $\alpha^2 > 0$;
(3) the Kähler form ω is of type $(1,1)$ and $\omega^2 > 0$;
(4) if $\alpha \neq 0$ belongs to $H^{1,1}$ and $\alpha \circ \omega = 0$, then $\alpha^2 < 0$.

Use these facts to compute the Hodge numbers in terms of $b_i(X)$ and $\sigma(X)$. (*Hint*: In item (4), the following observation may prove helpful: if α is harmonic and $\alpha \circ \omega = 0$, then $\alpha \wedge \bar{\omega} = 0$ as a 4-form. Why?)

The last problem is also purely technical: it's supposed to give you a feeling of what blow-ups and resolutions of singularities are.

Problem 4. Compute (in any reasonable sense) the embedded resolutions of all simple curve singularities and the resolutions of simple surface singularities (those that were not done in class). For surfaces, draw the exceptional curves and their corresponding dual graphs. Congratulations: now you know all Dynkin diagrams (without short roots)!