Topology of Algebraic Varieties

Math 535

by Alex Degtyarev

Motivation. Topology of algebraic varieties, both real and complex, singular and nonsingular, is a very active research area that goes back to such mathematicians as Klein, Hilbert, and Zariski. The subject illustrates the interplay of a wide variety of disciplines, from number theory to algebra and algebraic geometry to analysis to topology, as opposed to the modern tendency of narrow mathematical specialization. Alas, in spite of the popularity of the subject, there seems to be no comprehensive textbook and most ideas are either spread as a folklore or scattered throughout a number of surveys and technical papers.

Tentative course contents. I will emphasize breadth rather than depth, *i.e.*, I will try to introduce the basic concepts and ideas and discuss their applications, while merely sketching out lengthy technical proofs and replacing them with references. Topics to be included are as follows (subject to change):

(1) Introduction: algebraic varieties, rigidity *vs.* flexibility, algebro-geometric *vs.* topological properties. Deformation invariance of topology.
(2) Singular *vs.* nonsingular and real *vs.* complex varieties.
(3) Simple case: Riemann surfaces. Genus, Riemann–Hurwitz formula, the topological nature of discrete invariants.
(4) A brief introduction to singularities of curves and surfaces. Blow-ups and resolution of singularities.
(5) Singular plane curves. Components of the moduli space. The fundamental group and other topological invariants.
(6) Introduction to the Hodge theory (if needed).
(7) $K3$-surfaces: the global Torelli theorem, period spaces, curves and linear systems, topological classification (including singular, symmetric, and real surfaces).
(8) Polarized $K3$-surfaces: plane sextics and quartics in space.
(9) Hyperelliptic and trigonal curves in Hirzebruch surfaces; elliptic surfaces.

Prerequisites. The subject depends on a great deal of other topics, which cannot be assumed known anyway. For this reason, all necessary results from other areas of mathematics (algebraic topology, complex analysis, number theory, algebra, *etc.*) will be discussed on the fly as needed. A brief familiarity with topology is a plus, and the participation in (the first half of) Math 430/Math 505 by Ali Sinan Sertöz may be helpful. An active participation in the Bilkent/METU algebraic geometry seminar and in the METU topology seminar is highly recommended. The course should be accessible to all graduate and advanced undergraduate students.

Assessment. As in any advanced course, the principal reference material is your lecture notes. The grading will be based upon the in-class participation, homeworks, and a few take-home exams. Collaboration is encouraged; plagiarism will be frowned upon!

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Recommended literature.

- F. Hirzebruch, Topological methods in algebraic geometry. Springer-Verlag (any edition).

This is an original monograph, culminating in the (Hirzebruch–)Riemann–Roch theorem, that can serve as a great introduction to algebraic geometry. It contains everything a topologist needs to know and is much easier to read than the standard 'algebraic' textbooks (personal opinion).

- J. Milnor, Singular points of complex hypersurfaces. Annals of Mathematics Studies, No. 61. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo. 1968

Another original monograph that is a great introduction to the subject.

- V. I. Arnol′d, S. M. Guseĭn-Zade, A. N. Varchenko, Singularities of differentiable maps. Vol. I. The classification of critical points, caustics and wave fronts. Monographs in Mathematics, 82. Birkhäuser Boston, Inc., Boston, MA, 1985. Vol. II. Monodromy and asymptotics of integrals. Monographs in Mathematics, 83. Birkhäuser Boston, Inc., Boston, MA, 1988.

A classical and very well written survey/textbook/reference book on singularities, with a great deal of further useful references.