Math 535 – Topology of Algebraic Varieties, Fall 2010
Final exam (due on January 5, 2011)

Name: ______________________________
I.D.: ______________________________

Most of this problem will probably be done in class, so do whatever is left.

Problem 1. Let X be a $K3$-surface and $\varphi\colon X \to \mathbb{C}\mathrm{p}^n$ a generically one-to-one map (i.e., φ is one-to-one except that some curves are contracted to points). Let $h \in H_2(X)$ be the class of the pull-back of a generic hyperplane section.

(1) Characterize the curves in X that are contracted by φ.
(2) Prove that the image of φ has at worst simple singularities.
(3) Give an estimate on the total Milnor number of these singularities.

The other two problems are related to the notion of linear equivalence of divisors, that we didn't discuss formally. Two divisors D_1 and D_2 on a variety X are *linearly equivalent*, $D_1 \sim D_2$, if $D_1 - D_2$ is the divisor of zeroes and poles of a meromorphic function on X. The set of all *effective* divisors linearly equivalent to D is called a *linear system* and denoted $|D|$; it is a projective space. Important is the fact that linear systems is what the Riemann–Roch theorem is about: $\dim|D| = \ell(D) - 1$ in our notation. (Accidentally, K_X is, in fact, a class of linear equivalence of divisors; that is why it is called the canonical *class*!)

Problem 2. Let $C \subset \mathbb{C}\mathrm{p}^2$ be a nonsingular cubic curve. Recall that declaring one of the inflection points $\mathbf{0}$ makes it a group: one has $a + b + c = 0$ if and only if a, b, c are collinear, and then $-a$ is defined via $\mathbf{0} + a + (-a) = 0$.

(1) Show that the relation $a+b+c = 0$, and hence the group law, can be defined intrinsically as follows: $a + b + c = 0$ if and only if $(a) + (b) + (c) \sim 3(\mathbf{0})$ as divisors. (*Hint*: Compare dimensions.)
(2) Show that $3d$ points $a_1, \ldots , a_{3d} \in C$ lie on a curve of degree d if and only if $a_1 + \ldots + a_{3d} = 0$ with respect to the group law.

Item (2) above and the fact that, as an abstract group (i.e., without taking into account the complex structure), C is just $S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$, makes some geometric problems involving cubics very simple. Here are a few examples.

Problem 3. Let $C \subset \mathbb{C}\mathrm{p}^2$ be as above.

(1) Show that there is a triple of distinct points $p_1, p_2, p_3 \in C$ such that, for all i, the line $(p_i p_{i+1})$ is tangent to C at p_i. (We number the points cyclically: $p_{i+3k} = p_i$.) How many such triples are there?
(2) Take points $p_1, p_2, p_3 \in C$ as above for the coordinate vertices. What is the equation of C in this coordinates? After all rescalings, are their still any parameters left? Can you relate these parameters to the j-invariant?
(3) Does there exist a nonsingular conic that intersects C at a single point (with appropriate multiplicity)? How many?
(4) Along these lines, think of a problem of your own. State a problem and explain its solution.