Solutions to Final Exam

Problem 1. Find the coefficients of the first and second fundamental forms, principal, Gaussian, and mean curvatures, the lines of curvature, and the asymptotic curves of Enneper’s surface
\[ \mathbf{x}(u, v) = \left( u - \frac{u^3}{3} + uw^2, \ v - \frac{v^3}{3} + u^2v, \ u^2 - v^2 \right). \]

Solution: This is just a direct calculation, so I skip it. Here is the answer:
\[ E = G = (1 + u^2 + v^2)^2, \quad F = 0; \quad e = -g = 2, \quad f = 0; \quad k_1 = -k_2 = \frac{2}{E}; \quad H = 0, \quad K = -\frac{4}{E^2}; \]
the lines of curvature are \( u = \text{const} \) and \( v = \text{const} \); the asymptotic curves are \( u \pm v = \text{const} \).

Problem 2. (1) A nonsingular linear map \( A: V \to W \) of 2-dimensional inner product spaces is called a similitude if there is a constant \( \lambda \) such that \( (Ax, Ay) = \lambda (x, y) \) for each pair \( x, y \in V \). (In other words, \( A \) is a constant times an isometry.) Show that if \( A \) is not a similitude, there is a unique (up to reordering and multiplication by \(-1\)) pair of orthonormal vectors \( \mathbf{e}_1, \mathbf{e}_2 \in V \) such that \( A\mathbf{e}_1, A\mathbf{e}_2 \) are orthogonal.

(2) Let \( \varphi: S_1 \to S_2 \) be a diffeomorphism of surfaces so that \( d\varphi \) is never a similitude. Use part (1) to show that a neighborhood of each point of \( S_1 \) admits an orthogonal parametrization \( \mathbf{r}_1 \) such that \( \varphi \circ \mathbf{r}_1 \) is an orthogonal parametrization of \( S_2 \).

Solution: (1) is an easy exercise in linear algebra. Given an orthonormal basis \( \mathbf{x}_1, \mathbf{x}_2 \) of \( V \), the vectors \( \mathbf{e}_1, \mathbf{e}_2 \) are found in the form \( \mathbf{e}_1 = \mathbf{x}_1 \cos \alpha + \mathbf{x}_2 \sin \alpha, \mathbf{e}_2 = -\mathbf{x}_1 \sin \alpha + \mathbf{x}_2 \cos \alpha \) from the equation \( (A\mathbf{e}_1, A\mathbf{e}_2) = 0 \), which is homogeneous in \( \cos \alpha, \sin \alpha \). In particular, it follows that \( \alpha \) can be chosen a continuous function of the coefficients of the matrix of \( A \).

(2) In view of therosteritem1 one can find two orthogonal differentiable unit vector fields \( \mathbf{e}_1, \mathbf{e}_2 \) on \( S_1 \) such that \( d\varphi(\mathbf{e}_1) \) and \( d\varphi(\mathbf{e}_2) \) are also orthogonal. The statement follows from the existence theorem applied to the line fields defined by \( \mathbf{e}_1, \mathbf{e}_2 \).

Problem 3. Let \( \mathbf{x} = \mathbf{x}(u, v) \) be a regular parametrized surface. A parallel surface to \( \mathbf{x} \) is a parametrized surface
\[ \mathbf{y}(u, v) = \mathbf{x}(u, v) + a\mathbf{N}(u, v), \]
where \( a \) is a constant. Prove that:

(1) \( \mathbf{y}_u \times \mathbf{y}_v = (1 - 2Ha + Ka^2)(\mathbf{x}_u \times \mathbf{x}_v) \), where \( K \) and \( H \) are the Gaussian and mean curvatures of \( \mathbf{x} \) respectively;

(2) at regular points the Gaussian and mean curvatures of \( \mathbf{y} \) are, respectively
\[ \frac{K}{1 - 2Ha + Ka^2} \quad \text{and} \quad \frac{H - Ka}{1 - 2Ha + Ka^2}; \]

(3) if \( \mathbf{x} \) has constant mean curvature \( H \equiv c = \text{const} \) and \( a = 1/2c \), then the parallel surface has constant Gaussian curvature \( K \equiv 4c^2 \).

Solution: (1) Differentiate \( \mathbf{y} \) to get \( \mathbf{y}_u = \mathbf{x}_u + a(a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v) \) and \( \mathbf{y}_v = \mathbf{x}_v + a(a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v) \), multiply, and use \( a_{11}a_{22} - a_{12}a_{21} = K \) and \( a_{11} + a_{22} = -2H \).

(2) Let \( K' \) and \( H' \) be the new curvatures, \( F', \ldots \), the new coefficients, etc. Denote, for shortness, \( A = (1 - 2Ha + Ka^2) \). From (1) it follows that \( N' = N \). Hence,
\[ \begin{cases} \mathbf{N}_u' = N_u = a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v = a_{11}(\mathbf{y}_u - a\mathbf{N}_u) + a_{21}(\mathbf{y}_v - a\mathbf{N}_v), \\ \mathbf{N}_v' = N_v = a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v = a_{12}(\mathbf{y}_u - a\mathbf{N}_u) + a_{22}(\mathbf{y}_v - a\mathbf{N}_v). \end{cases} \]
Now just resolve this system in \( N_u, N_v \) to get the new coefficients \( a'_{ij} \) (The determinant of this system is nonzero at regular points. Guess why?)
\[ \begin{cases} N_u' = \frac{a_{11} + Ka}{A}\mathbf{y}_u + \frac{a_{21}}{A}\mathbf{y}_v, \\ N_v' = \frac{a_{12}}{A}\mathbf{y}_u + \frac{a_{22} + Ka}{A}\mathbf{y}_v \end{cases} \]
and use \( H' = -\frac{1}{2}(a'_{11} + a'_{22}) \) and \( K' = a'_{11}a'_{22} - a'_{21}a'_{12} \).

(3) follows immediately from (2).
**Problem 4.** Let $\alpha: I \to S \subset \mathbb{R}^3$ be a curve on a regular surface $S$. Consider the ruled surface generated by the family $[\alpha(t), N(\alpha(t))]$. Prove that $\alpha$ is a line of curvature in $S$ if and only if this ruled surface is developable.

**Solution:** Let $N(t) = N(\alpha(t))$. Then $N' = dN(\alpha')$ by definition. If $\alpha$ is a line of curvature, then $\alpha' = \lambda N'$, which implies $(N, N', \alpha') = 0$, i.e., the surface is developable. If the surface is developable, then, by definition, $(N, N', \alpha') = 0$ and, since $\alpha' \perp N$ and $N' \perp N$, this implies $\alpha' \parallel N'$, i.e., $\alpha$ is a line of curvature.