Math 310 Topology, Spring 1999

Solutions to Midterm 2

Problem 1. Let $X$ be Hausdorff and $A \subset X$ dense and locally compact. Prove that $A$ is open. Deduce from this that any locally compact subspace of a Hausdorff space is relatively closed, i.e., open in its closure.

Solution: Pick a point $a \in A$ and assume that each neighborhood $U$ of $a$ in $X$ contains limit points of $A$ which are not in $A$. Clearly, limit points of $A$ which are in $U$ are also limit points of $U \cap A$; hence, $U \cap A$ is not closed in $X$. On the other hand, $a$ has a compact neighborhood $U \cap A$; since $X$ is Hausdorff, such a neighborhood must be closed in $X$.

In general, if $A$ is locally compact, it is dense and, hence, open in $\text{Cl} \ A$.

Problem 2. Let $X$ be a topological space and $A, B \subset X$ compact subspaces.

1. Is $A \cup B$ compact?
2. Is $A \cap B$ compact?
3. The same question under the assumption that $X$ is Hausdorff.

Solution: (1) Yes. Given an open covering of $A \cup B$, it also covers $A$ and $B$. Hence, there is a finite subcovering of $A$ and a finite subcovering of $B$; their union is finite and covers $A \cup B$.

(2) No. Let $Y$ be an infinite set and $X = Y \cup \{a, b\}$ (where $a, b$ are two extra points), so that open are any subset of $Y$, $Y \cup \{a\}$, $Y \cup \{b\}$, and $X$ itself. Let $A = Y \cup \{a\}$ and $B = Y \cup \{b\}$. Then $A$ and $B$ are compact (say, any open covering of $A$ must contain $A$ itself), while $A \cap P = Y$ is an infinite discrete space, hence, noncompact.

(3) Yes. Due to (1) only the intersection part needs proof. Since $X$ is Hausdorff and $B$ is compact, it is closed. Thus, $A \cap B$ is closed in $A$ and, hence, compact.

Problem 3. Let $X$ be a Hausdorff space, $\{K_\alpha\}_{\alpha \in A}$ a family of compact subspaces, and $U \subset X$ an open set containing $\bigcap_{\alpha \in A} K_\alpha$. Prove that $U$ contains $\bigcap_{\alpha \in A} K_\alpha$ for some finite subset $A \subset \Lambda$.

Solution: We can assume that $X$ itself is compact (otherwise replace it with one of $K_\alpha$); hence, so is $X \setminus U$. Since $X$ is Hausdorff, all $K_\alpha$ are closed and $\{X \setminus K_\alpha\}$ is an open covering of $X \setminus U$. Take a finite subcovering; the corresponding $K_\alpha$’s are what we need.

Problem 4. Prove that any collection of disjoint balls (of nonzero radii) in $\mathbb{R}^n$ is countable.

Solution: Any ball contains a rational point; since the balls are disjoint, all the points are distinct and, thus, they enumerate the balls. The set of rational points is countable.

Problem 5. Let $X_1 \subset X_2 \subset \ldots \subset X_n \subset \ldots$ be an increasing sequence of subspaces so that each $X_i$ is is closed in $X_{i+1}$. Let $X = \bigcup_{i=1}^{\infty} X_i$ with the so-called weak or direct limit topology: a set $F \subset X$ is closed if and only if $F \cap X_i$ is closed in $X_i$ for each $i$.

1. Show that each $X_i$ is a closed subspace of $X$.

2. Show that if each $X_i$ is normal, so is $X$.

Solution: (1) Each $X_n$ is closed (this is obvious); why is it a subspace? If $A$ is closed in $X$, then $A \cap X_n$ is closed in $X_n$ by the definition of the topology in $X$. If $A$ is closed in $X_n$, then $A \cap X_m$ is closed in $X_m$ for $m \leq n$ (as $X_m \subset X_n$ is a subspace) and $A$ is closed in $X_m$ for $m \geq n$ as $X_n \subset X_m$ is a closed subspace. Thus, $A$ is also closed in $X$.

(2) It suffices to prove that, given a closed set $A \subset X$ and a function $f: A \to I$, it admits a continuous extension $g$ to $X$. Construct $g$ by induction, using Tietze’s extension theorem. Let $g_1: X_1 \to I$ be an extension of $f|_{A \cap X_1}$ to $X_1$. Assuming that $g_{n-1}: X_{n-1} \to I$ such that $g_{n-1} = f$ on $A \cap X_{n-1}$ is already constructed, denote by $f_n: X_{n-1} \cup (A \cap X_n) \to I$ the result of pasting $g_{n-1}$ and $f|_{A \cap X_n}$ (it is continuous due to the pasting lemma) and let $g_n: X_n \to I$ be an extension of $f_n$ to $X_n$. Finally, define $g: X \to I$ via $g(x) = g_{n}(x)$ if $x \in X_n$. This function is continuous due to the fact that $X$ has weak topology: if $K \subset I$ is closed, then $g^{-1}(K) \cap X_n = g_n^{-1}(K)$ is closed in $X_n$ for all $n$ and, hence, $g^{-1}(K)$ is closed in $X$. 
