Math 310 Topology, Spring 2010

Solutions to Final exam

Problem 1. Let $p\colon X \to B$ be a covering.

(1) Prove that X is a surface if and only if so is B.
(2) Prove that, if B is an orientable surface, then so is X.

Solution: (1) A covering, in particular, is a surjective local homeomorphism. Hence, the local Euclidean property of X is equivalent to that of B.

I didn't mean the proof of the Hausdorff property of X or B: such things are usually taken for granted in homotopy theory. The fact that B Hausdorff implies X Hausdorff is quite obvious (but one should consider the two cases $p(x) = p(y)$ or $p(x) \neq p(y)$ separately). The converse seems more involved, and I am not even sure that it is true. Sorry about that!

(2) A local homeomorphism induces an orientation of X from one of B: just orient each sufficiently small neighborhood $U \in X$ so that $p|_U$ be orientation preserving. An alternative explanation is similar to the solution of Problem 2.

Problem 2. Let B be a nonorientable surface. Prove that there exists a unique, up to isomorphism, *orientation covering* of B, *i.e.*, a covering $\tilde{B} \to B$ with the following properties:

- $\tilde{B}$ is an orientable surface;
- any other covering $X \to B$ with X orientable factors through $\tilde{B}$.

What is the degree of this covering?

Remark. Certainly, the statements of Problems 1 and 2 hold for manifolds of any dimension.

Solution: The remark above is a hint: don't try to construct this covering using the known classification of surfaces!

The following two statements are intuitively obvious, and their formal proofs are similar to those of the path and homotopy lifting lemmas (referring to the Lebesgue lemma):

(1) any path γ in B transfers orientation from $\gamma(0)$ to $\gamma(1)$;
(2) the resulting transfer depends on the path homotopy class of γ only.

Hence, each loop in B is either orientation preserving or orientation reversing, and this property depends on the homotopy class of the loop only. Thus, one can define a map $w_1\colon \pi_1(B) \to \mathbb{Z}_2$, sending a loop to $0 \in \mathbb{Z}_2$ if and only if it is orientation preserving. This map is a homomorphism (reversing the orientation twice preserves it!), and its kernel is an index two subgroup that defines a certain covering $\tilde{B} \to B$. With this construction, all properties and the uniqueness are immediate, and the covering is seen to be of degree 2.

Alternatively, one can define $\tilde{B}$ as the set of pairs $(b, \mathfrak{o})$, where $b \in B$ and $\mathfrak{o}$ is an orientation of B at b. This set can be given a natural topology, so that the forgetful map $\tilde{B} \to B$, $(b, \mathfrak{o}) \mapsto b$, be a double covering. Then, one can prove that $\tilde{B}$ is connected if and only if B is not orientable. I leave this to you as an exercise.

Remark. Using Problem 3(1) and the classification of surfaces, one can easily conclude that the orientation double coverings are $(n-1)T^2 \to n\mathbb{R}\mathrm{p}^2$, $n \geqslant 1$.

Remark. The homomorphism w_1 constructed in the proof is usually regarded as an element of $H^1(B; \mathbb{Z}_2)$ and, as such, is a simplest example of a *characteristic class*; it is called the first *Stiefel–Whitney class* of B.

Problem 3. Let B be a 'nice' space (say, a finite simplicial complex), and let $f\colon X \to B$ be a covering of finite degree d.

(1) Prove that $\chi(X) = d \cdot \chi(B)$. (*Hint*: Just count cells.)
(2) List all topological spaces that can be covered by a torus.
(3) Same question for an orientable closed surface of genus 12.
(4) Same question for the real projective plane.

Remark. Item (1) in this problem is a special case of the so called *Riemann–Hurwitz formula*.

Solution: (1) Triangulate X by the pull-backs of the simplices in B. (The pull-back of each simplex consists of d copies of this simplex. For proof, one can notice that each simplex Δ is contractible, hence simply connected, hence any covering of Δ is trivial, *i.e.*, the covering space consists of d disjoint copies of Δ.)

Then, denoting by $\#_i$ the number of simplices of dimension d, one has $\#_i(X) = d \cdot \#_i(B)$ for each i; hence, $\chi(X) = d \cdot \chi(B)$. Note that there are no simple formulas for the individual Betti numbers!

For Items (2)–(4), observe that, if $p\colon X \to B$ is a covering and X is compact, then B is compact (as a continuous image of a compact space) and $\deg p$ is finite, as each fiber is closed, hence compact and discrete, hence finite. Thus, we can use Item (1). Furthermore, due to Problem 1, if X is a closed surface, then so is B (*i.e.*, B has no boundary).

(2) If X is a torus, then $\chi(B) = \chi(X)/d = 0$, hence B is either a torus or a Klein bottle. Both coverings do exist, for example, the degree 1 covering (identity) $T^2 \to T^2$ and the orientation double covering $T^2 \to K$. (Failure to prove the existence was a common mistake!)

(3) One has $\chi(X) = -22$, hence $\chi(B)$ is a negative divisor of -22, *i.e.*, $\chi(B)$ can take values -1, -2, -11, or -22, resulting in the following short list: $3\mathbb{R}\mathrm{p}^2$, $2T^2$, $4\mathbb{R}\mathrm{p}^2$, $13\mathbb{R}\mathrm{p}^2$, $12T^2$, and $24\mathbb{R}\mathrm{p}^2$. The possibilities $24\mathbb{R}\mathrm{p}^2$ and $4\mathbb{R}\mathrm{p}^2$ are **ruled out**: in the former case, the covering would have to be of degree one, *i.e.*, a homeomorphism, in the latter case, the covering would have to factor through the orientation double covering $3T^2 \to 4\mathbb{R}\mathrm{p}^2$, see remark after Problem 2, but $3T^2$ cannot be covered by $12T^2$. All other coverings do exist: $12T^2 \to 13\mathbb{R}\mathrm{p}^2$ is the orientation double covering; $12T^2 \to 3\mathbb{R}\mathrm{p}^2$ is the composition of the orientation double covering $2T^2 \to 3\mathbb{R}\mathrm{p}^2$ and a covering $12T^2 \to 2T^2$, and for $2T^2$ one can use the following observation: any surface nT^2, $n \geqslant 1$, has a covering of any positive degree d (which would automatically have the correct genus due to Item (1)). For the construction, just 'unwrap' one of the handles.

Problem 4.

(1) Let G be a graph, *i.e.*, a finite *CW*-complex of dimension one. Prove that $\pi_1(G)$ is a free group F_r on $r = 1 - \chi(G)$ generators.
(2) Let F_r be a free group on r generators, and let $H \subset F_r$ be a subgroup of finite index d. Prove the following *Schreier index formula*: H is a free group on $d(r-1)+1$ generators. (*Hint*: A previous problem may help.)
(3) Generalize Item (2): a **finite index** subgroup of a finitely generated group is finitely generated (and the number of generators can be made at most that given by the Schreier index formula).

SOLUTION: (1) Due to the homotopy invariance of both χ and π_1, it suffices to consider the case when G is a wedge of circles; for r circles, one has $\chi(G) = 1 - r$.

(2) The statement is an immediate consequence of Item (1) and Problem 3(1) (and the fact that any subgroup $H \subset \pi_1(G)$ of index d defines a covering of G of degree d).

(3) This is an immediate consequence of Item (2): given an epimorphism $\phi\colon F_r \twoheadrightarrow G$ and a subgroup $H \subset G$, one obtains an epimorphism $\phi^{-1}(H) \twoheadrightarrow H$, and $[F_r : \phi^{-1}(H)] = [G : H]$.