

## Solutions to the Final Exam

**Problem 1.** A 1-kg weight is attached to the lower end of a spring suspended from the ceiling. In its equilibrium position the weight stretches the spring 10 cm. Starting at time  $t_0 = 0$  the weight is pulled down by a force of 30 N for  $\pi$  sec and then released. Find the displacement of the weight as function of time and the amplitude of the resulting free oscillation. (Assume  $g = 10$ .)

SOLUTION: Introduce the constants:  $m = 1$  kg is the mass of the weight,  $y_0 = 0.1$  m, the original displacement in the equilibrium position,  $k$ , the spring coefficient,  $F_0 = 30$  N, the value of the force applied to the weight, and  $a = \pi$  sec, the duration of the force. From Hook's law we have  $mg = ky_0$ , i.e.,  $k = mg/y_0$ . The force applied as function of time can be given by  $F(t) = F_0(1 - \alpha(t - a))$ , where  $\alpha$  is the step function. Newton's second law gives the equation  $my'' = -k(y + y_0) + mg + F(t)$ ,  $y(0) = 0$ ,  $y'(0) = 0$ , or  $y'' + (k/m)y = (F_0/m)(1 - \alpha(t - a))$ . (Here  $y$  is the displacement; the axis is directed straight down and the origin is chosen at the equilibrium position.) Let  $\hat{y} = L\{y\}$ . Apply the Laplace transform:

$$s^2 \hat{y} + \frac{k}{m} \hat{y} = \frac{F_0}{ms} (1 - e^{-as}), \quad \text{or} \quad \hat{y} = \frac{F_0/m}{s(s^2 + (k/m))} (1 - e^{-as}).$$

Hence (using properties of  $L^{-1}$ ),  $y = G(t) - G(t - a)\alpha(t - a)$ , where  $G(t)$  is the inverse transform of  $g(s) = F_0/[ms(s^2 + (k/m))]$ . To find  $G(t)$ , use the tables:

$$g(s) = \frac{F_0}{k} \left[ \frac{1}{s} - \frac{s}{s^2 + (k/m)} \right], \quad \text{whence} \quad G(s) = \frac{F_0}{k} \left( 1 - \cos \sqrt{\frac{k}{m}} t \right).$$

Finally, substituting the constants, we get

$$\boxed{y(t) = 0.3(1 - \alpha(t - \pi)) - 0.3(\cos 10t - \cos 10(t - \pi)\alpha(t - \pi))}.$$

When the force is removed (i.e.,  $t > \pi$ ), one has  $\alpha(t - \pi) = 1$  and, since  $\cos 10(t - \pi) = \cos 10t$ , we get  $y(t) = 0$  for  $t > \pi$ , i.e., the weight comes back to the equilibrium position and there is no oscillation! (Thus, the amplitude is  $\boxed{0}$ .) Surprising, isn't it?

**Problem 2.** Represent the general solution to  $(x^2 - 3)y'' + 6xy' + 6y = 0$  by a power series about the origin. Indicate the interval of convergence of the series.

SOLUTION: Let  $y = \sum_{n=0}^{\infty} c_n x^n$  be the series in question. Plug in to get

$$\sum_{n=0}^{\infty} n(n-1)c_n x^n - 3 \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} + 6 \sum_{n=0}^{\infty} n c_n x^n + 6 \sum_{n=0}^{\infty} c_n x^n = 0,$$

or, after shifting the index in the second sum,  $\sum_{n=0}^{\infty} [n(n-1)c_n - 3(n+2)(n+1)c_{n+2} + 6nc_n + 6c_n]x^n = 0$ . Equating the coefficients to 0, we get the recurrence relation  $(n^2 + 5n + 6)c_n = 3(n+2)(n+1)c_{n+2}$ , or  $c_{n+2} = (n+3)c_n/3(n+1)$ . Shift the index back:  $c_n = (n+1)c_{n-2}/3(n-1)$ ,  $n \geq 2$ . This obviously yields

$$c_{2k} = \frac{(2k+1)(2k-1)\dots(3)}{3^k(2k-1)(2k-3)\dots(1)} c_0 = \frac{2k+1}{3^k} c_0 \quad \text{and} \quad c_{2k+1} = \frac{(2k+2)(2k)\dots(4)}{3^k(2k)(2k-2)\dots(2)} c_1 = \frac{k+1}{3^k} c_1.$$

Thus, the solution is

$$\boxed{y = c_0 \sum_{k=0}^{\infty} \frac{2k+1}{3^k} x^{2k} + c_1 \sum_{k=0}^{\infty} \frac{k+1}{3^k} x^{2k+1}}.$$

The equation has two singular points,  $x = \pm\sqrt{3}$ . Hence, the interval of convergence of the series is  $(-\sqrt{3}, \sqrt{3})$ .

**Problem 3.** Find the inverse Laplace transforms of

$$(a) \quad f(s) = \frac{2s^2 - 2}{s^2(s^2 - 2s + 2)}, \quad (b) \quad f(s) = \frac{1}{s^2 + 6s + 13}.$$

SOLUTION: (a) Since  $s^2 - 2s + 2 = (s - 1)^2 + 1$  has no real roots, the elementary fraction decomposition has the form

$$\frac{2s^2 - 2}{s^2(s^2 - 2s + 2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{(s^2 - 2s + 2)}.$$

Get rid of the denominators,  $2s^2 - 2 = As(s^2 - 2s + 2) + B(s^2 - 2s + 2) + (Cs + D)s^2$ , and equate the coefficients:  $A + C = 0$ ,  $-2A + B + D = 2$ ,  $2A - 2B = 0$ ,  $2B = -2$ . Hence,  $A = B = -1$ ,  $C = D = 1$ , and

$$L^{-1}\left\{\frac{2s^2 - 2}{s^2(s^2 - 2s + 2)}\right\} = -L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{1}{s^2}\right\} + L^{-1}\left\{\frac{(s - 1) + 2}{(s - 1)^2 + 1}\right\} = \boxed{-1 - t + e^t \cos t + 2e^t \sin t}.$$

(b) This one is straightforward:

$$L^{-1}\left\{\frac{1}{s^2 + 6s + 13}\right\} = L^{-1}\left\{\frac{1}{(s + 3)^2 + 4}\right\} = \boxed{\frac{1}{2}e^{-3t} \sin 2t}.$$

**Problem 4.** Find the first five terms of the power series expansion about  $x_0 = 1$  of the solution to

$$y'' = (y')^3 + x^2, \quad y(1) = 1, \quad y'(1) = 1.$$

SOLUTION: We need the values of  $y$  and its first four derivatives at 1. From the initial conditions we have  $y(1) = 1$  and  $y'(1) = 1$ , and, differentiating the equation and plugging in, we obtain:

$$\begin{aligned} y'' &= (y')^3 + x^2, & y''(1) &= 2, \\ y''' &= 3(y')^2 y'' + 2x, & y'''(1) &= 8, \\ y^{IV} &= 6y'(y'')^2 + 3(y')^2 y''' + 2, & y^{IV}(1) &= 50. \end{aligned}$$

$$\text{Thus, } y = \sum_{n=0}^{\infty} y^{(n)}(1)(x - 1)^n/n! = \boxed{1 + (x - 1) + (x - 1)^2 + \frac{4}{3}(x - 1)^3 + \frac{25}{12}(x - 1)^4 + \dots}.$$

**Problem 5.** Show that

$$L^{-1}\left\{\frac{1}{\sqrt{s(s-1)}}\right\} = \frac{2e^t}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-x^2} dx.$$

SOLUTION: We have  $L^{-1}\{1/\sqrt{s}\} = 1/\sqrt{\pi t}$  and  $L^{-1}\{1/(s-1)\} = e^t$ . Use the convolution theorem:

$$L^{-1}\left\{\frac{1}{\sqrt{s(s-1)}}\right\} = \frac{1}{\sqrt{\pi}} \int_0^t e^{t-u} \frac{du}{\sqrt{u}} = \frac{2e^t}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-x^2} dx \quad (\text{substituting } u = x^2).$$