Math 240-01, Spring 2000

Solutions to Midterm 1

Problem 1. Find the general solution to $(3x^2y + 12x^2 + 6xy + y^3)\,dx + 3(x^2 + y^2)\,dy = 0$.

SOLUTION: Let $M(x, y) = 3x^2y + 12x^2 + 6xy + y^3$ and $N(x, y) = 3(x^2 + y^2)$. Then $D = (\partial M/\partial y) - (\partial N/\partial x) = 3x^2 + 3y^2$, and $D/N = 1$ does not depend on y. Hence, $\exp(\int 1\,dx) = e^x$ is an integrating factor. Multiply the equation by e^x and find its first integral Φ from

$$\frac{\partial \Phi}{\partial x} = (3x^2y + 12x^2 + 6xy + y^3)e^x, \quad \frac{\partial \Phi}{\partial y} = 3(x^2 + y^2)e^x.$$

It is more convenient to use the second equation first; integration in respect to y gives us $\Phi = (3x^2y + y^3)e^x + C(x)$. Now substitute to the first equation to get $C'(x) = 12x^2e^x$. Integrating by parts two times, we obtain $C(x) = (12x^2 - 24x + 24)e^x$. Finally, the general solution is $\boxed{(3x^2y + 12x^2 - 24x + y^3 + 24)e^x = C}$.

Problem 2. Solve the Cauchy problem $-2xy\,dx + (x^2 + y^2)\,dy = 0$, $y(0) = 2$.

SOLUTION: The equation has homogeneous coefficients; hence, it can be solved by the substitution $y = vx$, $dy = x\,dv + v\,dx$. Substituting and cancelling x^2 we obtain $-2v\,dx + (v^2 + 1)(x\,dv + v\,dx) = 0$, or, after simplification, $v(v^2 - 1)\,dx = -(v^2 + 1)x\,dv$. Separate the variables and integrate:

$$\frac{dx}{x} = -\frac{(v^2 + 1)dv}{v(v^2 - 1)} \quad \left[= \frac{dv}{v} - \frac{2v\,dv}{v^2 - 1}\right] \quad \text{and} \quad \ln|x| = \ln\left|\frac{v}{v^2 - 1}\right| + C.$$

It remains to substitute back $v = y/x$ and simplify: $x = C_1yx/(y^2 - x^2)$, or $y^2 - x^2 = C_1y$. Using the initial conditions $x = 0$, $y = 2$, we find $C_1 = 2$ and, finally, the solution is $\boxed{y^2 - x^2 = 2y}$.

Remark. Alternatively, one can treat the equation as Bernoulli's equation in $x(y)$ (cf. Problem 3).

Problem 3. Find the orthogonal trajectories of the family of circles $x^2 + (y - a)^2 = a^2$, $a \in \mathbb{R}$.

SOLUTION: First, we need to represent the given family as the family of integral curves of a differential equation. Differentiate the given equation $x^2 + (y - a)^2 = a^2$ to get $2x + 2(y - a)y' = 0$ and eliminate a: from the last equation one gets $a = y + x/y'$, and substitution to the first one yields $x^2 + (x/y')^2 = (y + x/y')^2$, or, after simplification, $x^2 = y^2 + 2xy/y'$. Now substitute $y = Y$, $y' = -1/Y'$ to get an equation for the orthogonal trajectories: $x^2 = Y^2 - 2xYY'$. This can be solved as Bernoulli's equation (or else similar to Problem 2): $Y' - (1/2x)Y = -(x/2)Y^{-1}$. Let $Y = \sqrt{z}$. Then the equation transforms into $z1 - (1/x)z = -x$, and the solution is straightforward: $Y^2 = z = -x^2 + Cx$. After simplification one can see that the resulting curves are also circles: $\boxed{(x - C_1)^2 + Y^2 = C_1^2}$ (where $C_1 = C/2$).

Remark. As a matter of fact, the problem can easily be solved using elementary geometry.

Problem 4. A bullet of mass m is shot straight up at the velocity v_0. The air resistance is $|kv^2|$, where v is the current velocity of the bullet. Find:

(1) The moment when the bullet reaches its maximal altitude (10 pts).
(2) The maximal altitude of the bullet (10 pts).

SOLUTION: Note that the problem is **different** from the last year's one, as the resistance is proportional to v^2! Newton's second law gives the equation $mv' = -mg - kv^2$; it is separable, and its general solution is $\sqrt{m/kg}\,\arctan(v\sqrt{k/mg}) = C - t$. The constant $C = \sqrt{m/kg}\,\arctan(v_0\sqrt{k/mg})$ is found from the initial condition $v(0) = v_0$. Clearly, at the highest position one has $v = 0$; hence, the time is $t_0 = C = \boxed{\sqrt{m/kg}\,\arctan(v_0\sqrt{k/mg})}$.

To find the maximal altitude, resolve the obtained solution in v and integrate:

$$\frac{dx}{dt} = v = \sqrt{\frac{mg}{k}}\,\tan\sqrt{\frac{kg}{m}}(C-t) \quad \text{and, hence,} \quad x = \frac{m}{k}\ln\left|\cos\sqrt{\frac{kg}{m}}(C-t)\right| + C_1.$$

Now

$$C_1 = -\frac{m}{k}\ln\left|\cos C\sqrt{\frac{kg}{m}}\right| = -\frac{m}{2k}\ln\frac{mg}{mg+kv_0^2}$$

is found from the condition $x(0)=0$, and the position at time $t_0 = C$ (found above) is

$$x_{\max} = \frac{m}{k}\ln\cos(0) + C_1 = \boxed{\frac{m}{2k}\ln\frac{mg+kv_0^2}{mg}}.$$

Remark. A good way to verify your answer is to, first, check the dimensions (note that the dimension of k is sec/m) and, second, check that, when $k \to 0$, the limits of t_0 and $x_{\max}$ are the well known values v_0/g and $v_0^2/2g$, respectively. (The latter can easily be done using $\lim_{t\to 0}(\arctan t/t) = 1$ and $\lim_{t\to 0}(\ln(1+t)/t) = 1$.)

Problem 5. Solve the equation $(x^3 + xy^2 - y)\,dx + (y^3 + x^2y + x)\,dy = 0$.

SOLUTION: Rewrite the equation as $(x^2+y^2)(x\,dx + y\,dy) + (x\,dy - y\,dx) = 0$ and divide by (x^2+y^2); then we get $\frac{1}{2}d(x^2+y^2) - d\Big(\arctan\frac{y}{x}\Big) = 0$, and integration gives us the solution: $\boxed{x^2+y^2 = 2\arctan\frac{y}{x} + C}$.

Remark. To visualize the solutions you can notice that in polar coordinates the integral curves of the equation are given by $\rho = \sqrt{2\varphi + C}$. Thus, they look like 'parallel' expanding spirals starting at the origin.