Math 240-3, Spring 1999

Solutions to Final Exam

Problem 1. When the *dolmuş* driver steps in, the springs under his seat are squeezed by 10 cm. How much (in per cent) does the seat rest reduce the vibration of the engine whose frequency is 1200 revolutions per minute? (Assume the seat massless. Just a hint: keep in mind that the angular frequency ω (the coefficient in sin) and the usual frequency f (in Hz) are related via $\omega = 2\pi f$.)

SOLUTION: Let m be the driver's mass, $l = 0.1\,\mathrm{m}$, the squeezing of the springs, and k, their coefficient of rigidity. Then $kl = mg$, i.e., $k = mg/l$. Let, further, $f = 20\,\mathrm{Hz}$ be the frequency of the engine vibration, $\omega = 2\pi f$, its circular frequency, and A, its amplitude. Then the equation is $my'' + ky = mg + kA\sin(\omega t)$, where $y(t)$ is the vertical position of the seat. (Note that the equation represents Newton's second law for the seat; since the vibration is transmitted to the seat via the springs, the force is $kA\sin(\omega t)$.) The general solution is

$$y(t) = C_1\sin(\alpha t) + C_2\cos(\alpha t) + \frac{mg}{k} + \frac{kA}{m(\alpha^2 - \omega^2)}\sin(\omega t),$$

where $\alpha = \sqrt{k/m} = \sqrt{g/l}$ is the intrinsic circular frequency of the seat-springs system. We are interested in the last term, which represents the vibration. Its amplitude is

$$\left|\frac{kA}{m(\alpha^2 - \omega^2)}\right| = \left|\frac{g}{g - \omega^2 l}\right| A \approx 0.006A.$$

Thus, the vibration is reduced by $100 - 0.6 = \boxed{99.4\%}$ (i.e., almost completely).

Problem 2. Find the inverse Laplace transforms of

$$\text{(a)} \quad f(s) = \frac{2s^4 + s^3 + s^2 + 1}{s(s^2+1)^2}, \qquad \text{(b)} \quad f(s) = \frac{1 - 2s}{s^2 + 4s + 5}.$$

SOLUTION: (a) Decompose $f(s)$ into partial fractions:

$$\frac{2s^4 + s^3 + s^2 + 1}{s(s^2+1)^2} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1} + \frac{Ds + E}{(s^2+1)^2},$$
$$2s^4 + s^3 + s^2 + 1 = A(s^2+1)^2 + (Bs + C)s(s^2+1) + (Ds + E)s.$$

Plugging in $s = 0$ gives $A = 1$, and the equation becomes $s^4 + s^3 - s^2 = (Bs + C)s(s^2 + 1) + (Ds + E)s$. Now, equating the coefficients of s^4, s^3, s^2, and s, one gets, respectively, $B = 1$, $C = 1$, $B + D = -1$, and $C + E = 0$, i.e., $D = -2$ and $E = -1$. It remains tu use the tables:

$$\mathcal{L}^{-1}[f(s)] = \mathcal{L}^{-1}\Big[\frac{1}{s}\Big] + \mathcal{L}^{-1}\Big[\frac{s}{s^2+1}\Big] + \mathcal{L}^{-1}\Big[\frac{1}{s^2+1}\Big] - 2\mathcal{L}^{-1}\Big[\frac{s}{(s^2+1)^2}\Big] - \mathcal{L}^{-1}\Big[\frac{1}{(s^2+1)^2}\Big] =$$
$$1 + \cos t + \sin t - t\sin t - \frac{1}{2}(\sin t - t\cos t) = \boxed{1 + \frac{1}{2}\sin t + \cos t - t\sin t + \frac{t}{2}\cos t}.$$

(b) One has

$$\mathcal{L}^{-1}[f(s)] = \mathcal{L}^{-1}\Big[\frac{1 - 2s}{(s+2)^2 + 1}\Big] = \mathcal{L}^{-1}\Big[\frac{5 - 2(s+2)}{(s+2)^2 + 1}\Big] = \boxed{e^{-2t}(5\sin t - 2\cos t)}.$$

Problem 3. Solve the initial value problem

$$y'' + y = \begin{cases} 1, & 0 \leqslant t < \pi, \\ 0, & t \geqslant \pi, \end{cases} \qquad y(0) = y'(0) = 0.$$

Solution: Let $\hat{y}(s) = \mathcal{L}[y]$. Then $\mathcal{L}[y''] = s^2\hat{y}$. The transform f of the right hand side can be found by straightforward integration:

$$f(s) = \int_0^\pi e^{-st}\,dt = -\frac{e^{-st}}{s}\Big|_{t=0}^{\pi} = \frac{1}{s}(1-e^{-\pi s}).$$

Thus, the equation transforms to $\hat{y}(s^2+1) = f(s)$ and $\hat{y}(s) = (1-e^{-\pi s})/s(s^2+1)$. To find the inverse transform, use partial fractions and tables. Let $g(t) = 1/s(s^2+1)$. Then

$$\mathcal{L}^{-1}[g(s)] = \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] = 1-\cos t, \quad \text{and}$$
$$y = \mathcal{L}^{-1}[(1-e^{-\pi s})g(s)] = \boxed{1-\cos t - \alpha(t-\pi)(1+\cos t)},$$

where α is the Heaviside step function. (Certainly, I use the fact that $\cos(t-\pi) = -\cos t$.)

Problem 4. Represent the general solution to $(x^2+4)y''+6xy'+4y=0$ by a power series about the origin. Indicate an interval of convergence of the series.

Solution: Let $y = \sum_{n=0}^\infty a_n x^n$ be the series. Plugging in gives

$$\sum_{n=0}^\infty n(n-1)a_n x^n + 4\sum_{n=0}^\infty n(n-1)a_n x^{n-2} + 6\sum_{n=0}^\infty n a_n x^n + 4\sum_{n=0}^\infty a_n x^n = 0.$$

In the second sum the first two terms are zero; hence, it can be reindexed to $\sum_{n=0}^\infty (n+1)(n+2)a_{n+2}x^n$. Now, equating the coefficients of equal powers of x gives a recurrence relation $4(n+1)(n+2)a_{n+2} = -n(n-1)a_n - 6na_n - 4a_n$, or $a_{n+2} = -a_n(n+4)/4(n+2)$. By induction one obtains

$$a_{2k} = \frac{2k+2}{2\cdot(-4)^k}\,a_0,\ a_{2k+1} = \frac{2k+3}{3\cdot(-4)^k}\,a_1, \text{ and } \boxed{y = a_0\sum_{k=0}^\infty (-1)^k\frac{2k+2}{2\cdot 4^k}\,x^{2k} + a_1\sum_{k=0}^\infty(-1)^k\frac{2k+3}{3\cdot 4^k}\,x^{2k+1}},$$

where a_0, a_1 are arbitrary constants. Since the singular points of the equation are $x = \pm 2i$, the series converges at least in the disk of radius 2 or, for real arguments, in the interval $(-2,2)$. On the other hand, it is obvious that the series diverges at $x=2$.

Problem 5. Find the first five terms of the power series expansion of the solution to

$$y'' = \sin y', \qquad y(0) = 0, \quad y'(0) = 1.$$

Solution: We need the values of y and its derivatives up to y^{IV} at 0. The first two are given: $y(0)=0$ and $y'(0) = 0$. The others are found from the equation (differentiating and using the chain rule):

$$y''(0) = \sin y'|_{x=0} = \sin 1,$$
$$y'''(0) = \frac{d}{dx}(\sin y')|_{x=0} = (y''\cos y')|_{x=0} = \sin 1\cos 1,$$
$$y^{\text{IV}}(0) = \frac{d}{dx}(y''\cos y')|_{x=0} = \big(y'''\cos y' - (y'')^2\sin y'\big)|_{x=0} = \sin 1\cos^2 1 - \sin^3 1 = \sin 1\cos 2.$$

Thus,

$$y = \sum_{n=0}^4 \frac{y^{(n)}(0)}{n!}\,x^n + \ldots = \boxed{x + \frac{\sin 1}{2}\,x^2 + \frac{\sin 1\cos 1}{6}\,x^3 + \frac{\sin 1\cos 2}{24}\,x^4 + \ldots}.$$

An alternative (more difficult) way would be to try to solve the equation (substituting $y' = v(x)$) to arrive at $y' = 2\arctan(Ce^x)$, where $C = \tan(1/2)$. (Note the constant which comes from the initial condition $y'(0) = 1$!) Then again use $y(0)=0$, differentiate the above expression to obtain the other derivatives at 0, and use the Maclaurin formila ...