Math 240-3, Spring 1999

Solutions to Midterm 2

Problem 1. Find the solution to $yy'' = (y')^2 + y^2y'$ satisfying the initial conditions $y(0) = y'(0) = 1$.

Solution: Let $y' = p(y)$, so that $y'' = p\,dp/dy$. The equation becomes $ypp' = p^2 + y^2p$. It has an obvious solution $p = 0$, which does not satisfy the initial conditions. Dividing by yp one gets a linear first order equation $p' - \frac{1}{y}p = y$, whose general solution (found, say, using the formula) is $p(y) = y(y + C)$. The initial condition $p(1) = 1$ gives $C = 0$, i.e., $p(y) = y^2$. Now, solving $dy/dx = y^2$ (by separating the variables) gives $y = -1/(x + C)$, and from the initial condition $y(0) = 1$ one gets $C = -1$. Finally, $\boxed{y = 1/(1 - x)}$.

Problem 2. Find a general solution to $y''' + 4y'' + 4y' + 3y = 0$.

Solution: The characteristic equation is $t^3 + 4t^2 + 4t + 3 = 0$. Possible integral roots are ±1 and ±3, and $t = -3$ is a root; and dividing by $(t+3)$ gives the equation $t^2+t+1 = 0$ and two more roots $-1/2\pm i\sqrt{3}/2$. The solution is now straightforward: $\boxed{y = C_1e^{-3x} + C_2e^{-x/2}\sin(x\sqrt{3}/2) + C_3e^{-x/2}\cos(x\sqrt{3}/2)}$.

Problem 3. Given that $y = e^x$ is a solution of the associated homogeneous equation, find a general solution to $(1 - x)y'' + xy' - y = 2(x - 1)^2e^{-x}$.

Solution: The equation is linear and one can reduce its order by substituting $y = ve^x$. After plugging in (and dividing by $(1 - x)e^x$) one gets a first order linear equation in v':

$$v'' + \Big(1 + \frac{1}{1 - x}\Big)v' = 2(1 - x)e^{-2x},$$

whose solution can be found using the standard formula: $v' = 2(x - 1)e^{-2x} + C(x - 1)e^{-x}$. Integration gives $v = -Cxe^{-x} - xe^{-2x} + \frac{1}{2}e^{-2x}$, and the general solution to the original equation is $y = C_1e^x + ve^x$; after simplification, $\boxed{y = C_1e^x + C_2x + (\frac{1}{2} - x)e^{-x}}$.

Problem 4. Find a general solution to $y''' + y'' - y' - y = e^x + x^2$.

Solution: The characteristic polynomial is $t^3 + t^2 - t - 1 = (t^2 - 1)(t + 1)$, and its roots are $t_1 = 1$ of multiplicity 1 and $t_2 = -1$ of multiplicity 2. Hence, the general solution to the associated homogeneous equation is $y_c = C_1e^x + C_2e^{-x} + C_3xe^{-x}$. For the e^x part of the free term, since $t = 1$ is a simple root of the characteristic equation, the solution should be looked for in the form $y_1 = Axe^x$; plugging in gives $4AE^x = e^x$, i.e., $A + 1/4$. For the x part the solution has the form $Y - 2 = Ax^2 + Bx + C$ (a generic polynomial of degree 2), and plugging in gives $-Ax^2 - (2A + B)x + (2A - B - C) = x^2$, i.e., $A = -1$, $2A + B = 0$, and $2A - B - C = 0$, or $A = -1$, $B = 2$, and $C = -4$. Finally, the solution is $y = y_c + y_1 + y_2 = \boxed{C_1e^x + C_2e^{-x} + C_3xe^{-x} + \frac{1}{4}xe^x - x^2 + 2x - 4}$.

Problem 5. Find a general solution to $y''' + y' = \dfrac{1}{\cos x}$.

Solution: The general solution to the associated homogeneous equation is straightforward: $y_c = C_1 + C_2\sin x + C_3\cos x$. Variation of parameters gives the system

$$\begin{aligned} C_1' + C_2'\sin x + C_3'\cos x &= 0,\\ C_2'\cos x - C_3'\sin x &= 0,\\ -C_2'\sin x - C_3'\cos x &= 1/\cos x. \end{aligned}$$

From the last two equations $C_2' = -\tan x$ and $C_3' = 1$, and from the first one, $C_1' = -1/\cos x$. After integrating and substituting,

$$\boxed{y = C_1 + C_2\sin x + C_3\cos x + \frac{1}{2}\ln\left|\frac{1 - \sin x}{1 + \sin x}\right| + (\sin x)\ln|\cos x| + x\cos x}.$$

(To integrate C_1' one can use, e.g., $C_1' = \dfrac{\cos x}{1 - \sin^2 x}$.)