Math 240-3, Spring 1999

Solutions to Midterm 1

Problem 1. Find the general solution to $(e^x - \sin y)\,dx + \cos y\,dy = 0$.

Solution: Let $\sin y = z$. Then $\cos y\,dy = dz$ and one obtains a linear equation $z' - z = -e^x$, whose general solution is $z = e^x(-x + C)$. Thus, $\boxed{\sin y = e^x(C - x)}$, or $\boxed{y = \pi k + (-1)^k \arcsin e^x(C - x),\ k \in \mathbb{Z}}$.

Remark. Another way would be to find an integrating factor.

Problem 2. Solve the Cauchy problem $(x^2 + 3xy + y^2)\,dx - x^2\,dy = 0$, $y(1) = -\frac{1}{2}$.

Solution: The equation is homogeneous. Let $y = xz$. Then $x^2(1 + 3z + z^2)\,dx - x^2(x\,dz + z\,dx) = 0$, or $(z+1)^2\,dx = x\,dz$. Separate the variables and integrate: $\ln|x| = -(z+1)^{-1} + C$, or $\ln|x| = C - ((y/x)+1)^{-1}$. To find C, let $x = 1$ and $y = -\frac{1}{2}$. Then $C = 2$. Finally, $\boxed{\ln x = 2 - x/(x + y)}$. (The $|\cdot|$ is dropped as the initial value $x = 1$ is positive.) The result can easily be resolved in y:

$$\boxed{y = \frac{x}{2 - \ln x} - x}, \qquad x \in (0, e^2).$$

Problem 3. A cylindrical tank of radius R and height H has a round opening of radius r in the bottom. The tank is full of water. Find the time necessary to empty it. (*Hint*: water escapes from the tank through the opening at the rate $\sqrt{2gh}$, where h is its current level.)

Solution: Let $h(t)$ be the level of the water and $V(t)$ its volume at time t. Clearly, $V = \pi R^2 h$. Furthermore, $dV/dt = -\pi r^2\sqrt{2gh}$. (Here πr^2 is the area of the opening; the "$-$" sign indicates that the volume is decreasing.) Thus, we get an equation $\pi R^2 h' = -\pi r^2\sqrt{2gh}$. Separate the variables and integrate:

$$\sqrt{\frac{2h}{g}} = C - \frac{r^2}{R^2}t. \qquad \text{Hence,}\quad C = \sqrt{\frac{2H}{g}} \quad \text{and} \quad \boxed{t_0 = \frac{R^2}{r^2}\sqrt{\frac{2H}{g}}}.$$

(Here C is found from the initial condition $h(0) = H$, and the time t_0 in question, from $h(t_0) = 0$.)

Problem 4. A bullet of mass m is shot straight up at the velocity v_0. The air resistance is $|kv|$, where v is the current velocity of the bullet. Find the position (altitude) of the bullet at time t.

Solution: The equation is obtained immediately from Newton's second law: $mv' = -mg - kv$. (Here $v = v(t)$ is the velocity of the bullet at time t and $v' = a$, the acceleration.) Separate the variables, integrate, and resolve in v:

$$\frac{m}{k}\ln\left|\frac{k}{m}v + g\right| = C - t, \quad \text{or} \quad v = C_1\exp\left(-\frac{k}{m}t\right) - \frac{mg}{k}, \quad \text{where} \quad C_1 = \frac{mg}{k} + v_0.$$

(The constant is found from the condition $v(0) = v_0$.) Since $v = h'$, to find h it remains to integrate the expression for v: (the constant of integration is found from $h(0) = 0$)

$$\boxed{h(t) = \frac{m}{k}\left(\frac{mg}{k} + v_0\right)\left(1 - e^{-\frac{k}{m}t}\right) - \frac{mg}{k}t}.$$

Problem 5. Solve the equation $y(x^2y^2 - 1)\,dx + x(x^2y^2 + 1)\,dy = 0$.

Solution: Rewrite the equation as $x^2y^2(y\,dx + x\,dy) = y\,dx - x\,dy$. Divide by y^2 to get $x^2\,d(xy) = d(x/y)$. Let $xy = u$ and $x/y = v$. Then $x^2 = uv$ and the equation transforms into $uv\,du = dv$. Separate the variables and integrate: $u^2/2 = \ln|v| + C$ or, in the old variables, $\boxed{x^2y^2 = \ln(x^2/y^2) + C_1}$. Alternatively, after simplification, $\boxed{x^2 = C_2 y^2 e^{x^2y^2}}$.