Solutions to Midterm II

Problem 1. Let P_4 be the space of polynomials of degree $\leqslant 4$. Prove that

$$(p,q)=\int_{-1}^{1} t^2 p(t)q(t)\,dt$$

is an inner product and, given this inner product, find a basis for $W^\perp$, where $W \subset P^4$ is the subspace spanned by 1, $t-1$, and $(t-1)^2$.

Solution: The expression in question is symmetric and bilinear, and we only need to check that it's positive definite. One has $(p,p)=\int_{-1}^{1} t^2p^2\,dt=\int_{-1}^{1}(tp)^2dt \geqslant 0$. If the integral is 0, then, as tp is continuous, one must have $tp=0$ identically on $[-1,1]$ (cf. the proof for the standard integral inner product). As p is a polynomial, one gets $p=0$.

Now, notice that $W=\mathrm{Span}\{1,t-1,(t-1)^2\}=P_2=\mathrm{Span}\{1,t,t^2\}$. Thus, we can replace the given vectors with 1, t, t^2 :). Let $p=a_0+a_1t+a_2t^2+a_3t^3+a_4t^4$. Then one has

$$(p,1)=\frac{2}{3}a_0+\frac{2}{5}a_2+\frac{2}{7}a_4=0,\quad (p,t^2)=\frac{2}{5}a_0+\frac{2}{7}a_2+\frac{2}{9}a_4=0,\quad \text{and}\quad (p,t)=\frac{2}{5}a_1+\frac{2}{7}a_3=0.$$

From the last equation one gets $a_1=-\frac{5}{7}a_3$. The first two form a system in a_0, a_2, a_4, which is not the easiest one, but still solvable. The solution is $a_0=\frac{5}{21}a_4$ and $a_2=-\frac{10}{9}a_4$. Thus, one can take for a basis for $W^\perp$ the polynomials $\boxed{-5t+7t^3 \text{ and } 15-70t^2+63t^4}$.

Problem 2. The inner product on $\mathbb{R}^4$ is given by $(a,b)=a_1b_1+a_2b_2+2a_3b_3+2a_4b_4$. Use the Gram-Schmidt process to find an orthonormal basis in $W=\mathrm{Span}\{u_1,u_2,u_3\}$, where

$$u_1=\begin{bmatrix}1\\4\\-2\\0\end{bmatrix},\qquad u_2=\begin{bmatrix}5\\4\\-1\\3\end{bmatrix},\qquad u_3=\begin{bmatrix}10\\5\\-5\\11\end{bmatrix}.$$

Solution: This one is really straightforward: just use the formulas. The Gram-Schmidt process gives

$$v_1=u_1=\begin{bmatrix}1\\4\\-2\\0\end{bmatrix},\qquad v_2=u_2-v_1=\begin{bmatrix}4\\0\\1\\3\end{bmatrix},\qquad v_3=u_3-2v_1-\frac{8}{3}v_2=\begin{bmatrix}-8/3\\-3\\-11/3\\3\end{bmatrix}.$$

For an orthonormal system these vectors should be normalized, i.e., divided, respectively, by 5, 6, and $\sqrt{61}$. Finally, one gets

$$\boxed{\frac{1}{5}\begin{bmatrix}1\\4\\-2\\0\end{bmatrix},\quad \frac{1}{6}\begin{bmatrix}4\\0\\1\\3\end{bmatrix},\quad \frac{1}{3\sqrt{61}}\begin{bmatrix}-8\\-9\\-11\\9\end{bmatrix}}.$$

Remark. When calculating the length, one should use **the same** inner product!

Problem 3. If A is nonsingular, prove that $A^{\mathrm{T}}A$ is positive definite.

Solution: We need to show that $\mathbf{x}^{\mathrm{T}}(A^{\mathrm{T}}A)\mathbf{x}>0$ for any $\mathbf{x}\neq 0$. One has $\mathbf{x}^{\mathrm{T}}(A^{\mathrm{T}}A)\mathbf{x}=(\mathbf{x}^{\mathrm{T}}A^{\mathrm{T}})A\mathbf{x}=(A\mathbf{x})^{\mathrm{T}}(A\mathbf{x})=(A\mathbf{x},A\mathbf{x})>0$ whenever $A\mathbf{x}\neq 0$ (where $(\,\cdot\,,\,\cdot\,)$ stands for the standard inner product). Since A is nonsingular, $A\mathbf{x}=0$ if and only if $\mathbf{x}=0$.

Problem 4. Find the polynomial $p(t)$ of degree $\leqslant 3$ such that $p(1) = p'(1) = 0$ and the value of

$$\int_0^1 \left|p(t) + 3t - 5\right|^2 dt$$

is minimal possible.

SOLUTION: Consider the space P_3 with the inner product $(p, q) = \int_0^1 pq\,dt$ and let $W \subset P_3$ be the subspace defined by $p(1) = p'(1) = 0$. Then the problem reduces to finding $\operatorname{proj}_W q$, where $q = (5 - 3t) = 2 - 3(t - 1)$.

One can take for a basis in W the polynomials $v_1 = (t - 1)^2$ and $v_2 = (t - 2)^3$ (cf. Midterm I). We need to orthogonalize it; applying the formulas gives us $u_1 = v_1$ and $u_2 = v_2 + \frac{5}{6}v_1$. (Leave it in this form!) Now, the projection is found by the formula:

$$p(t) = \frac{(q, u_1)}{(u_1, u_1)}u_1 + \frac{(q, u_2)}{(u_2, u_2)}u_2 = \frac{85}{12}u_1 + \frac{203}{10}u_2 = \boxed{24(t-1)^2 + \frac{203}{10}(t-1)^3}.$$

Remark. When evaluating integrals, keep the polynomials expanded in powers of $(t-1)$ and use substitution. For the record: $(q, u_1) = 17/12$, $(u_1, u_1) = 1/5$, $(q, u_2) = 29/360$, and $(u_2, u_2) = 1/252$.

Remark. In this particular case it might be easier to use the other method: to keep the original basis and to solve explicitly the (2×2)-system for the coefficients of the projection.

Problem 5. Let

$$C = \begin{bmatrix} 3 & -2 \\ -2 & a \end{bmatrix}.$$

Find the values of a for which the function $(x, y) = x^{\mathrm{T}}Cy$ is an inner product on $\mathbb{R}^2$.

SOLUTION: The function (x, y) above is obviously bilinear and symmetric; hence, the only condition to be verified is whether it is positive definite. One has $(x, x) = 3x_1^2 - 4x_1x_2 + ax_2^2 = 3(x_1 - \frac{2}{3}x_2)^2 + (a - \frac{4}{3})x_2^2$. This expression is a **sum** of two squares if and only if the second coefficient, $a - \frac{4}{3}$, is positive. Thus, the answer is $\boxed{a > \frac{4}{3}}$.