Math 220-04 Linear Algebra, Fall 1999

Solutions to Final Exam

Problem 1. If possible, diagonalize the matrix and find an orthogonal basis in which it has diagonal form:

$$A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}.$$

Solution: The characteristic polynomial is

$$\det \begin{bmatrix} -1-\lambda & 2 & 2 \\ 2 & -1-\lambda & 2 \\ 2 & 2 & -1-\lambda \end{bmatrix} = \lambda^3 + 3\lambda^2 - 9\lambda - 27 = (\lambda-3)(\lambda+3)^2.$$

(All roots are integers, so they can be found by trial and error, among the divisors of 27.) The corresponding homogeneous systems for eigenvectors and their solutions are:

$$\lambda = 3\colon \quad \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix} X = 0; \quad u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; \qquad \lambda = -3\colon \quad \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} X = 0; \quad u_{2,3} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Thus, the matrix can be diagonalized, and it has diagonal form in the basis $\{u_1, u_2, u_3\}$. Since A is symmetric, u_1 is orthogonal to u_2, u_3, and we only need to orthogonalize u_2, u_3. (By the way, the fact that A is symmetric also tells us that it **is** diagonalizable, i.e., we **must** find three independent eigenvectors!) Applying Gram-Schmidt to $\{u_2, u_3\}$, we replace u_3 with $[-1\ 2\ 1]^{\mathrm{T}}$. Finally, the matrix has diagonal form

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \quad \text{in the orthogonal basis} \quad \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

Problem 2. Find A^{-1} for

$$A = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix}.$$

Solution: The problem is straightforward, you can use any way you like. The correct answer is

$$A^{-1} = \frac{1}{2} \begin{bmatrix} -2 & -2 & -1 & -1 \\ -2 & -4 & -2 & -2 \\ -1 & -2 & -2 & -1 \\ -1 & -2 & -1 & -2 \end{bmatrix}.$$

Problem 3. Let A be a square matrix with integral entries. Prove that A^{-1} exists and has integral entries if and only if $\det A = \pm 1$.

Solution: Clearly, the determinant of an integral matrix is an integer (as it is obtained from the entries of the matrix by addition and multiplication only). Thus, for the *only if* part it suffices to notice that $\det A \det A^{-1} = \det(AA^{-1}) = \det I = 1$. Since the product of two integers is 1, they must both be ± 1. The *if* part follows from the formula $A^{-1} = (1/\det A)\,\mathrm{adj}\,A$ and the fact that, if A is integral, so is $\mathrm{adj}\,A$ (as its entries are determinants of integral matrices, see above).

Problem 4. Let $L\colon P_3 \to P_3$ be given by $L[p(t)] = p(t) + p(1)(t-3) - 2p'(1)(t-1)$. Find the Eigenvalues of L, their multiplicities, and the Eigenvectors of L. Is L diagonalizable?

Solution: Take for a basis the set $\{(t-1)^3, (t-1)^2, (t-1), 1\}$. Then L is represented by the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

and the characteristic polynomial and eigenvalues are found immediately: $f_A(\lambda) = (\lambda-1)^2(\lambda+1)^2$ has two roots, $\lambda = 1$ and -1, each of multiplicity 2. As a matter of fact, the eigenvectors are also found immediately, as the matrices obtained are already in (almost) row echelon form. A basis for the eigenspace corresponding to $\lambda = 1$ is $\{(t-1)^3, (t-1)^2\}$, and a basis for the eigenspace corresponding to $\lambda = -1$ is $\{(t-1)\}$. Thus, one eigenvector is missing and the operator is not diagonalizable.

Problem 5. Let

$$A_1 = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}, \quad A_4 = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}$$

be vectors in the space $M_{2,2}$ of (2×2)-matrices with the inner product $(A, B) = \operatorname{trace}(AB^{\mathrm{T}})$ and $W = \operatorname{Span}\{A_1, A_2, A_3, A_4\}$. Find a basis for $W^{\perp}$. What are $\dim W$ and $\dim W^{\perp}$?

SOLUTION: First of all, note that $(A, B) = \operatorname{trace}(AB^{\mathrm{T}})$ is just the standard inner product $(u, v) = u^{\mathrm{T}} v$ with respect to a natural basis for $M_{2,2}$. (A straightforward calculation discussed in class.) Thus, pick a natural basis, say, $\{E_{11}, E_{12}, E_{21}, E_{22}\}$, represent the given matrices as vectors, and write down the system for $W^{\perp}$:

$$\begin{bmatrix} 2 & 1 & 0 & 4 \\ 0 & 1 & 2 & 2 \\ 1 & 1 & 1 & 3 \\ -2 & 1 & 1 & 0 \end{bmatrix} X = 0.$$

(Note that the given vectors form the **columns** of the matrix, as they are transposed in the expression for the standard inner product.) The system has 1-dimensional solution space spanned by the vector corresponding to $\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$. Hence, $\dim W^{\perp} = 1$ and $\dim W = \dim M_{2,2} - \dim W^{\perp} = 3$. (In particular, this means that the given vectors are linearly dependent. However, there is no need to prove this directly or construct an explicit basis for W.)