**Math 220-04 Linear Algebra, Fall 1999**

## Solutions to Midterm 1

**Problem 1.** Find a basis for the solution space of the system

$$\begin{cases} 2x_1 - 6x_2 + 6x_3 + 5x_4 - x_5 = 0 \\ x_1 - 3x_2 + 2x_3 + x_4 = 0 \\ 2x_1 - 6x_2 + 4x_4 = 3 \\ 3x_1 - 9x_2 - 6x_4 - x_5 = -4 \end{cases}$$

Solution: Sorry, the problem was misstated. Certainly, only a homogeneous system may have a solution space. Nevertheless, let's solve the system first. Writing down its augmented matrix and converting it to a reduced row echelon form yields

$$\begin{bmatrix} 1 & -3 & 0 & 0 & 0 & 1/4 \\ 0 & 0 & 1 & 0 & 0 & 17/16 \\ 0 & 0 & 0 & 1 & 0 & 5/8 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}; \qquad \text{hence, the solution is} \quad X = s\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1/4 \\ 0 \\ -7/16 \\ 5/8 \\ 1 \end{bmatrix}.$$

The complete correct answer (5 pts extra credit) would now be something like this: the system is not homogeneous, hence, it has no solution space; however, a basis for the solution space of the corresponding homogeneous system is $\{[3 \quad 1 \quad 0 \quad 0 \quad 0]^{\mathrm{T}}\}$.

**Problem 2.** Prove that there is no $(3 \times 3)$-matrix $A$ with

$$A^3 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad A^7 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(where $A^n$ stands for the $n$-fold product $AA\ldots A$).

Solution: Let $B$ and $C$ be the two given matrices (i.e., the ones that are supposed to be $A^3$ and $A^7$, respectively. Here are at least three ways to solve the problem.

*1st way*: Observe that $BC = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq CB = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, while $A^3A^7 = A^{10} = A^7A^3$ must be equal.

*2nd way*: Observe that $B^7 = 0 \neq C^3 = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, while $(A^3)^7 = A^{21} = (A^7)^3$ must be equal.

*3rd way*: Observe that $B^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and, hence, $A^7 = B^2 * A = \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq C$.

**Problem 3.** Find the transition matrix $P_{S \leftarrow T}$, where $T$ is the standard basis in $\mathbb{R}^n$ and

$$S = \left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} \right\}.$$

Solution: Of course, $P_{S \leftarrow T} = A^{-1}$, where $A$ is the matrix composed of the given vectors. The answer is

$$P_{S \leftarrow T} = \begin{bmatrix} 0 & 2 & -2 & 0 \\ 2 & -1 & 1 & 1 \\ 1 & 3 & 5 & 0 \\ 3 & 4 & 2 & 2 \end{bmatrix}^{-1} = \frac{1}{28}\begin{bmatrix} 30 & 32 & 12 & -16 \\ 5 & -4 & 2 & 2 \\ 9 & -4 & 2 & 2 \\ -46 & -36 & -24 & 32 \end{bmatrix}.$$

**Problem 4.** Find rank $A$ and a basis for the column space of $A$, where

$$A = \begin{bmatrix} 0 & 2 & 3 & 1 & 3 \\ 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 4 & 2 & 2 \\ 2 & 0 & -4 & 3 & 1 \end{bmatrix}.$$

SOLUTION: One should either convert $A^{\mathrm{T}}$ to (reduced) row echelon form and take the transposes of the nontrivial rows of the result, or convert $A$ to row echelon form and take the columns of the **original** matrix $A$ whose numbers correspond to the columns of the row echelon form containing leading entries. The first way yields

$$A^{\mathrm{T}} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \qquad \text{hence, a basis is} \qquad \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\};$$

the second way yields

$$A \sim \begin{bmatrix} 1 & 1 & -1 & -1 & 1 \\ 0 & 2 & 3 & 1 & 3 \\ 0 & 0 & 1 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \qquad \text{hence, a basis is} \qquad \left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 4 \\ -4 \end{bmatrix} \right\}.$$

Both the solutions give rank $A = 3$.

**Problem 5.** Find a basis for the space $V \subset P_4$ of polynomials $p$ of degree up to 4 such that

$$p(5) = \frac{\partial p}{\partial t}(5) = \frac{\partial^2 p}{\partial t^2}(5) = \frac{\partial^3 p}{\partial t^3}(5).$$

SOLUTION: For the basis in $P_4$ take $\{1, (t-5), (t-5)^2, (t-5)^3, (t-5)^4\}$. Then the given condition on a polynomial $p(t) = a_0 + a_1(t-5) + a_2(t-5)^2 + a_3(t-5)^3 + a_4(t-5)^4$ is $a_0 = a_1 = 2a_2 = 6a_3$. Thus, $a_4$ and $a_3$ are arbitrary and one has $a_2 = 3a_3$, $a_1 = 6a_3$, and $a_0 = 6a_3$. Giving $(a_4, a_3)$ the values $(1,0)$ and $(0,1)$, we obtain a basis:

$$\left\{(t-5)^4, \quad (t-5)^3 + 3(t-5)^2 + 6(t-5) + 6\right\}.$$

The dimension of the space is 2.