Solutions to Midterm II

Problem 1. Evaluate $\displaystyle\int \frac{dx}{\cos x(1+\sin x)}$.

SOLUTION: One has (substituting $t = \sin x$):

$$\begin{aligned}\int \frac{\cos x\,dx}{\cos^2 x(1+\sin x)} &= \int \frac{d(\sin x)}{(1-\sin^2)(\sin x+1)} = \int \frac{dt}{(1-t)(1+t)^2}\\ &= \int \left[\frac{1}{4}\frac{1}{t+1} - \frac{1}{4}\frac{1}{t-1} - \frac{1}{2}\frac{1}{(t+1)^2}\right] dt = \frac{1}{2}\frac{1}{t+1} + \frac{1}{4}\ln\left|\frac{1+t}{1-t}\right| + C\\ &= \frac{1}{2}\frac{1}{\sin x+1} + \frac{1}{4}\ln\left|\frac{(1+\sin x)^2}{1-\sin^2 x}\right| + C = \boxed{\frac{1}{2}\frac{1}{\sin x+1} + \frac{1}{2}\ln\left|\frac{1+\sin x}{\cos x}\right| + C}.\end{aligned}$$

(The decomposition into partial fractions is straightforward and therefore is omitted. To get simple equations for the coefficients, one can let $t = \pm 1$ or 0.)

Problem 2. Evaluate (or show that the integral diverges):

(a) $\displaystyle\int \frac{dx}{x+\sqrt{x^2+1}}$.

(b) $\displaystyle\int_0^\infty \frac{(x^2+x+1)^2}{(x+1)(x^2+1)^2}\,dx$.

SOLUTION: (a) Let $x = \sinh t$, so that $t = \ln(x+\sqrt{x^2+1})$. Then the integral is

$$\begin{aligned}\int \frac{\cosh t\,dt}{\sinh t+\cosh t} &= \int \frac{e^t+e^{-t}}{2e^t}\,dt = \frac{1}{2}\int (1+e^{-2t})dt = \frac{t}{2} - \frac{1}{4}e^{-2t} + C\\ &= \frac{1}{2}\ln(x+\sqrt{x^2+1}) - \frac{1}{4(x+\sqrt{x^2+1})^2} + C = \boxed{\frac{1}{2}\ln(x+\sqrt{x^2+1}) - \frac{1}{2}x^2 + \frac{1}{2}x\sqrt{x^2+1} + C_1}.\end{aligned}$$

(Note that $1/(x+\sqrt{x^2+1}) = \sqrt{x^2+1}-x$, as $(\sqrt{x^2+1}-x)(\sqrt{x^2+1}+x) = x^2+1-x^2 = 1$.)

Alternatively, one can let $x = \tan t$, $-\pi/2 < t < \pi/2$, so that $\sin t = x/\sqrt{x^2+1}$ and $\cos t = 1/\sqrt{x^2+1}$. Then the integral is

$$\begin{aligned}\int \frac{\sec^2 t\,dt}{\tan t+\sec t} &= \int \frac{dt}{\cos t(\sin t+1)} = \text{[see Problem 4!]} = \frac{1}{2}\frac{1}{\sin t+1} + \frac{1}{2}\ln\left|\frac{1+\sin t}{\cos t}\right| + C\\ &= \frac{1}{2}\frac{\sqrt{x^2+1}}{x+\sqrt{x^2+1}} + \frac{1}{2}\ln(x+\sqrt{x^2+1}) + C = \boxed{-\frac{1}{2}x^2 + \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln(x+\sqrt{x^2+1}) + C_1}.\end{aligned}$$

(As above, note that $1/(x+\sqrt{x^2+1}) = \sqrt{x^2+1}-x$.)

(b) Denote the integrant by $f(x)$ and observe that it decreases as $1/x$ when $x\to\infty$. (Formally speaking, one has $\displaystyle\lim_{x\to\infty}\frac{f(x)}{1/x} = 1$.) Hence, by the **limit comparison test**, the integral in question $\boxed{\text{diverges}}$, as so does $\int_1^\infty dx/x$. (Note that a direct computation of the integral is doable, but tedious.)

Problem 3. Evaluate (or show that the sequence diverges):

(a) $\displaystyle\lim_{n\to\infty} n\sin\frac{1}{n}\sin\frac{n\pi}{2}$.

(b) $\displaystyle\lim_{n\to\infty}\left(\frac{3n+1}{3n-1}\right)^n$.

SOLUTION: (a) First, note that $\lim_{n\to\infty} n\sin\frac{1}{n} = \lim_{x\to 0}\frac{\sin x}{x} = 1$ (where $x = 1/n$). Now, if $n = 2k$ is even, then $\sin(n\pi/2) = 0$ and the limit is 0. If $n = 4l \pm 1$ is odd, then $\sin(n\pi/2) = \pm 1$ and the limit is ± 1. Since three subsequences converge to three distinct limits, the original sequence $\boxed{\text{diverges}}$.

(b) Denote the limit in question by A. Then

$$\ln A = \lim_{n\to\infty} n\ln\frac{3n+1}{3n-1} = \lim_{x\to 0}\frac{\ln\frac{3+x}{3-x}}{x} = \lim_{x\to 0}\frac{3-x}{3+x}\left(\frac{3+x}{3-x}\right)' = \frac{2}{3}.$$

(Here, $x = 1/n \to 0$.) Thus, $\boxed{A = e^{2/3}}$. Alternatively, one has

$$\lim_{n\to\infty}\left(\frac{3n+1}{3n-1}\right)^n = \lim_{n\to\infty}\left[\left(1+\frac{2}{3n-1}\right)^{(3n-2)/2}\right]^{2n/(3n-2)} = e^{\lim_{n\to\infty}\frac{2n}{3n-2}} = \boxed{e^{2/3}}.$$

Problem 4. Determine whether each of the following series is convergent or divergent. State clearly the name and the conditions of the test you are using. **Show all your work!**

(a1) $\displaystyle\sum_{n=1}^{\infty}\frac{2^n n!}{n^n}$; (a2) $\displaystyle\sum_{n=1}^{\infty}\frac{3^n n!}{n^n}$; (a3) generalize.

(b) $\displaystyle\sum_{n=1}^{\infty} a_n$, where $a_1 = 1$ and $a_{n+1} = \dfrac{1}{1+a_n}$ for $n \geq 1$.

SOLUTION: (a) Apply the **ratio test** to $\displaystyle\sum_{n=1}^{\infty}\frac{c^n n!}{n^n}$, where $c = \text{const}$:

$$\frac{a_{n+1}}{a_n} = \frac{c^{n+1}(n+1)!n^n}{c^n n!(n+1)^{n+1}} = c\left(\frac{n}{n+1}\right)^n = c\left(1+\frac{1}{n}\right)^{-n} \xrightarrow[n\to\infty]{} \frac{c}{e}.$$

Thus, the series $\boxed{\text{converges for } c < e \text{ and diverges for } c > e}$. (We leave the border case $c = e$ open.) In particular, $\boxed{\text{(a1) converges}}$ and $\boxed{\text{(a2) diverges}}$.

(b) If the series were to converge, one would have $\lim_{n\to\infty} a_n = 0$ (**the n-th term test**). But then, passing to the limit in the recursive relation, one would get the contradiction $0 = 1/(1+0) = 1$. Thus, the series $\boxed{\text{diverges}}$.

Problem 5. Determine whether each of the following series is convergent or divergent. State clearly the name and the conditions of the test you are using. **Show all your work!**

(a) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\ln n\ln(\ln n)}$.

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{(n!)^n}{n^{(n^2)}}$.

SOLUTION: (a) We apply the **integral test** to the function $f(x) = 1/(x\ln x\ln(\ln x))$ (note that the integration starts at $x = 3$ as $f(x)$ has a point of discontinuity at $x = e$):

$$\int_3^{\infty}\frac{dx}{x\ln x\ln(\ln x)} = \int_3^{\infty}\frac{d(\ln x)}{\ln x\ln(\ln x)} = \int_3^{\infty}\frac{d(\ln(\ln x))}{\ln(\ln x)} = \ln(\ln(\ln x))\Big|_3^{\infty} = \infty.$$

Thus, the series $\boxed{\text{diverges}}$.

(b) The series $\boxed{\text{converges}}$ by the **root test**: $\sqrt[n]{a_n} = \dfrac{n!}{n^n} < \dfrac{1}{n} \xrightarrow[n\to\infty]{} 0.$