Midterm II Topics[1]

1. Anti-derivatives. A function $F(x)$ is called an *anti-derivative* of $f(x)$ if $F'(x) = f(x)$. Any continuous function has anti-derivatives. Any two anti-derivatives of a given function differ by a constant. The set of all anti-derivatives of $f(x)$ is called the *indefinite integral* of $f(x)$; the typical notation is $\int f(x)\,dx = F(x) + C$, where $F(x)$ is one of the anti-derivatives.

> **Important Remark:** As in the case of derivatives, the variable of integration (indicated by the dx pattern) is **very** important! In an expression like $\int(ax + x^2)da$ the letter x should be treated as a constant; thus, $\int(ax + x^2)da = \frac{1}{2}a^2x + x^2a + C$.

2. Riemann sums, definite integrals. The expression $\sum_{i=1}^{n} f(c_i)\Delta x_1$ is called the *Riemann sum* of $f(x)$. (Here $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ is a *partition* of a segment $[a, b]$, $c_i \in [x_{i-1}, x_i]$, and $\Delta x_i = x_i - x_{i-1}$.) The limit $\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_1$ is called the *definite integral* of $f(x)$ from a to b and is denoted by $\int_a^b f(x)\,dx$. The variable of integration (= *dummy variable*) is important (see the remark in §1), although it disappears in the result: the result of evaluation of a definite integral in respect to x must **not** contain x! If f is continuous on $[a, b]$, the integral exists. (See also *Improper integrals*.) Geometrically, the integral is the (signed) area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$.

Main properties:.

(1) $\int_a^a f(x)\,dx = 0$ (definition);
(2) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ (definition);
(3) $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$;
(4) $\int_a^b c_1f_1(x) + c_2f_2(x))dx = c_1\int_a^b f_1(x)\,dx + c_2\int_a^b f_2(x)\,dx$, where $c_1, c_2 = \text{const}$;
(5) if $f(x) \geqslant g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$;
(6) if $a \leqslant b$ and $m \leqslant f(x) \leqslant M$ for all $x \in [a, b]$, then $m(b - a) \leqslant \int_a^b f(x)\,dx \leqslant M(b - a)$;
(7) if f is continuous on $[a, b]$, then there is a point $c \in [a, b]$ such that $\int_a^b f(x)\,dx = f(c)(b - a)$ (*the mean value theorem*).

3. The fundamental theorem of integral calculus. *If $f(x)$ is continuous, then*

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x),$$

i.e., the integral with variable upper limit is an anti-derivative of f.

Corollary (differentiation of an integral with variable limits).

$$\frac{d}{dx}\int_{\varphi(x)}^{\psi(x)} f(t)\,dt = f(\psi(x))\psi'(x) - f(\varphi(x))\varphi'(x).$$

Corollary (calculation of definite integrals).

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b, \quad \text{where } F \text{ is any anti-derivative of } f.$$

4. Application of definite integrals. Assume that we want to calculate a quantity S given by the naïve law $S = Ax$ (which holds whenever A does not depend on x). If A does depend on x, $A = A(x)$, then we proceed as follows: divide the interval $[a, b]$ where x changes into small subintervals. Within each small subinterval Δx_i one can assume that A does not change much and, hence, $\Delta S_i = A(x_i)\Delta x_i$. Passing to the limit, one gets $S = \int_a^b A(x)\,dx$. More precisely, for the integral formula to hold the error in the approximate formula $\Delta S_i \approx A(x_i)\Delta x_i$ must be 'much smaller' than Δx_i (i.e., of order $(\Delta x_i)^2$ or higher).

Below are some particular formulas.

Area of a plane region. Subdivide the region to represent it as the union/difference of simple regions, each bounded by the graphs of two functions $f(x) \geqslant g(x)$ and two vertical lines $x = a$ and $x = b$. The area of one such simple region is $\int_a^b(f(x) - g(x))dx$. Sometimes it is easier to use regions bounded by two curves $x = f(y)$ and $x = g(y)$, $f(y) \geqslant g(y)$, and two horizontal lines $y = a$ and $y = b$. Then the area is $\int_a^b(f(y) - g(y))dy$ (see the remark in §1). Both approaches can be combined with formulas from elementary geometry (when some of the regions are triangles or rectangulars).

[1]See also Midterm I Topics and Integration

Important Remark: Draw a picture to visualize the region! The condition $f(x) \geqslant g(x)$ in the integral formulas is important: otherwise some parts of the region will contribute to the area with the minus sign.

Arc length. The length of the curve $y = f(x)$, $a \leqslant x \leqslant b$ is given by $L = \int_a^b \sqrt{1+(f')^2}dx$.

Volume of a solid of revolution. The volume of the solid obtained by revolving the region bounded by the graph of a function $f(x) \geqslant 0$, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by:
$V = \int_a^b \pi f^2(x)\,dx$ (rotation about the x-axis) or
$V = \int_a^b 2\pi x f(x)\,dx$ (rotation about the y-axis).
A more complicated region should be subdivided into simple ones as when calculating the area.

Area of a surface of revolution. The area of the surface obtained by revolving the curve $y = f(x)$, $a \leqslant x \leqslant b$ is given by
$S = \int_a^b 2\pi f(x)\sqrt{1+(f')^2}dx$ (rotation about the x-axis) or
$S = \int_a^b 2\pi x\sqrt{1+(f')^2}dx$ (rotation about the y-axis).

General volume formula. The volume of a solid is given by $V = \int_a^b S(x)\,dx$, where $S(x)$ is the area of the cross-section through x perpendicular to the x-axis and a and b are the x-coordinates of, respectively, the leftmost and the rightmost points of the solid.

Work. The work of a force $F(x)$ directed along the x-axis on the segment $a \leqslant x \leqslant b$ is $W = \int_a^b F(x)\,dx$.

Fluid force. The total force of a fluid of density w against one side of a submerged vertical plate running from depth $y = a$ to depth $y = b$ is $F = \int_a^b wyL(y)\,dy$, where $L(y)$ is the horizontal extent of the plate (measured along the plate) at depth y. (In this formula the y-axis is assumed to point downwards.)

Moments, mass, center of gravity. The coordinates (x_c, y_c) of the center of gravity of a system are given by $x_c = M_x/M$, $y_c = M_y/M$, where M is the mass of the system and M_x, M_y its moments about the y-axis and x-axis, respectively. Here are some special cases:

Thin rod along the x-axis of linear density $\delta(x)$.
$M = \int_a^b \delta(x)\,dx$, $M_x = \int_a^b x\delta(x)\,dx$, $M_y = 0$.

Thin wire along the graph $y = f(x)$, $a \leqslant x \leqslant b$, of linear density $\delta(x)$.
$M = \int_a^b \delta(x)\sqrt{1+(f')^2}dx$, $M_x = \int_a^b x\delta(x)\sqrt{1+(f')^2}dx$, $M_y = \int_a^b f(x)\delta(x)\sqrt{1+(f')^2}dx$.

Thin flat plate bounded by the graphs of $f(x) \geqslant g(x)$ and the vertical lines $x = a$, $x = b$, of surface density $\delta(x, y)$.
$M = \int_a^b\big(\int_{g(x)}^{f(x)} \delta(x,y)\,dy\big)dx$, $M_x = \int_a^b\big(\int_{g(x)}^{f(x)} x\delta(x,y)\,dy\big)dx$, $M_y = \int_a^b\big(\int_{g(x)}^{f(x)} y\delta(x,y)\,dy\big)dx$.

Thin flat plate bounded by the curves $x = f(y)$, $x = g(y)$, $f(y) \geqslant g(y)$, and the horizontal lines $y = a$, $y = b$, of surface density $\delta(x, y)$.
$M = \int_a^b\big(\int_{g(y)}^{f(y)} \delta(x,y)\,dx\big)dy$, $M_x = \int_a^b\big(\int_{g(y)}^{f(y)} x\delta(x,y)\,dx\big)dy$, $M_y = \int_a^b\big(\int_{g(y)}^{f(y)} y\delta(x,y)\,dx\big)dy$.

Important Remark: In the last two cases be **very** careful about the variable in respect to which you integrate! (See remark in §1.) Say, in the last case, after the inner integral (with respect to x, with y treated as a constant) is evaluated, the expression must no longer contain x, and the outer integration is in respect to y.

More complicated regions should be subdivided into simple ones of one of the two above forms. Mass and moments are additive. Keep in mind that mass must always be positive, while moments can take negative values as well.

Use symmetry whenever possible! If both the plate and the density are symmetric with respect to an axis, the corresponding moment is 0.

5. Transcendental functions.

Exponential and logarithmic functions. By definition, let $\ln x = \int_1^x dx/x$ and $\exp x = e^x = \ln^{-1} x$ (the inverse function). If x is a rational number, then e^x does coincide with the x-th power of $e = \exp 1$. Other powers are defined via $a^x = e^{x \ln a}$ (for $a > 0$). The following identities hold:

$$
\begin{array}{lll}
(\ln x)' = 1/x, & (a^x)' = a^x \ln a, & (ab)^x = a^x b^x, \\
\ln(xy) = \ln x + \ln y, & a^{x+y} = a^x a^y, & (a/b)^x = a^x/b^x. \\
\ln(x/y) = \ln x - \ln y, & a^{x-y} = a^x/a^y, & \\
\ln(x^n) = n \ln x, & a^{nx} = (a^x)^n, &
\end{array}
$$

$\ln x$ is defined for $x > 0$ and takes all real values. e^x is defined for all real x and takes all positive values. Both the functions are increasing. One has

$$\ln 0 = 1, \qquad \lim_{x\to+\infty} \ln x = +\infty, \qquad \lim_{x\to 0^+} \ln x = -\infty,$$
$$e^0 = 1, \qquad \lim_{x\to+\infty} e^x = +\infty, \qquad \lim_{x\to-\infty} e^x = 0.$$

Logarithmic differentiation. Sometimes it is easier to find the derivative of the function $\ln f(x)$ and then use the formula $f'(x) = f(x)(\ln f(x))'$.

Example 1. Let $f(x) = (x+2)^5(x^2-1)^7(x+3)^9$. Then $\ln f(x) = 5\ln(x+2) + 7\ln(x^2-1) + 9\ln(x+3)$ and

$$f'(x) = (x+2)^5(x^2-1)^7(x+3)^9\Big(\frac{5}{x+2} + \frac{14x}{x^2-1} + \frac{9}{x+3}\Big).$$

Remark. Strictly speaking, the calculation is only valid when $x+2 > 0$, $x^2-1 > 0$, and $x+3 > 0$. However, the final result (without logarithms) holds for all values of x.

Example 2. Let $f(x) = x^{x^2}$. Then $\ln f(x) = x^2 \ln x$, $(\ln f(x))' = 2x\ln x + x$, and $f'(x) = x^{x^2}(2x\ln x + x)$.

Inverse trigonometric functions. The important ones are $\arcsin x = \sin^{-1} x$, $\arccos x = \cos^{-1} x$, and $\arctan x = \tan^{-1} x$. (Do not confuse the inverse function $\sin^{-1} x$ and the reciprocal $(\sin x)^{-1} = 1/\sin x$!) Here are some properties. (I also list the inverse hyperbolic functions here; see below.)

Function	Derivative	Domain	Range	Alternative formula
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$	$[-1,1]$	$[-\pi/2, \pi/2]$	
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$	$[-1,1]$	$[0,\pi]$	
$\tan^{-1} x$	$\frac{1}{1+x^2}$	$(-\infty,\infty)$	$(-\pi/2, \pi/2)$	
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$	$(-\infty,\infty)$	$(-\infty,\infty)$	$\ln(x+\sqrt{x^2+1})$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$	$[1,\infty)$	$[0,\infty)$	$\ln(x+\sqrt{x^2-1})$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$	$(-1,1)$	$(-\infty,\infty)$	$\frac{1}{2}\ln\frac{1+x}{1-x}$

Hyperbolic functions. The functions are defined via $\sinh x = (e^x - e^{-x})/2$, $\cosh x = (e^x + e^{-x})/2$, $\tanh x = \sinh x/\cosh x$. They share most of the properties of the trigonometric functions, but **beware of the signs!**

.	Trigonometric identities	Hyperbolic identities
1.	$(\sin x)' = \cos x, \quad (\cos x)' = -\sin x$	$(\sinh x)' = \cosh x, \quad (\cosh x)' = \sinh x$
2.	$\sin^2 x + \cos^2 x = 1$	$\cosh^2 x - \sinh^2 x = 1$
3.	$1 + \tan^2 x = 1/\cos^2 x$	$\tanh^2 x = 1 - \cosh^2 x$
4.	$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$	$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
5.	$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$	$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
6.	$\sin 2x = 2\sin x \cos x$	$\sinh 2x = 2\sinh x \cosh x$
7.	$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$	$\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1$
	$= 1 - 2\sin^2 x$	$= 1 + 2\sinh^2 x$
8.	$\sin^2 x = (1 - \cos 2x)/2$	$\sinh^2 x = (\cosh 2x - 1)/2$
9.	$\cos^2 x = (1 + \cos 2x)/2$	$\cosh^2 x = (\cosh 2x + 1)/2$
10.	$\sin x \sin y = -\frac{1}{2}(\cos(x+y) - \cos(x-y))$	$\sinh x \sinh y = \frac{1}{2}(\cosh(x+y) - \cosh(x-y))$
11.	$\cos x \cos y = \frac{1}{2}(\cos(x+y) + \cos(x-y))$	$\cosh x \cosh y = \frac{1}{2}(\cosh(x+y) + \cosh(x-y))$
12.	$\sin x \cos y = \frac{1}{2}(\sin(x+y) + \sin(x-y))$	$\sinh x \cosh y = \frac{1}{2}(\sinh(x+y) + \sinh(x-y))$

Remark. For those familiar with complex numbers: all the hyperbolic identities can easily be obtained from the corresponding trigonometric ones by a formal substitution. In fact, the two sets of functions are closely related: one has $\sinh x = -i\sin(ix)$, $\cosh x = \cos(ix)$, and $\tanh x = -i\tan(ix)$ (where $i^2 = -1$. These formulas do make sense, but we will not go too deep into the details.)

L'Hôpital's rule. *Let a be either a number or ∞. Assume that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ are either both 0 or both ∞ and that the limit $\lim_{x\to a} f'(x)/g'(x)$ exists. Then the limit $\lim_{x\to a} f(x)/g(x)$ also exists and one has*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Important Remark: (1) L'Hôpital's rule **only** applies if the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ are either both 0 or both ∞. Otherwise it will give a wrong result. (2) If the limit $\lim_{x\to a} f'(x)/g'(x)$ does not exist, this does **not** imply that the original limit does not exist. (3) Do not confuse $f'(x)/g'(x)$ and $(f(x)/g(x))'$!

Of course, l'Hôpital's rule can be iterated and combined with other rules and known limits. Below are some limits that are worth remembering.

$$\lim_{x\to 0} \frac{\sin x}{x} = 1, \quad \lim_{x\to 0} \frac{e^x - 1}{x} = 1, \quad \lim_{x\to 0} \frac{\ln(1+x)}{x} = 1, \quad \lim_{x\to 0} \frac{(1+x)^n - 1}{x} = n, \quad \lim_{x\to 0} (1+x)^{1/x} = e.$$

L'Hôpital's rule lets us handle the indefinite forms $[0/0]$ and $[\infty/\infty]$. The other five indefinite forms can be reduced to these two as follows.

$$\begin{aligned}
[0\cdot\infty]: &\qquad \lim f(x)g(x) = \lim \frac{f(x)}{1/g(x)} = \left[\frac{0}{0}\right] = \lim \frac{g(x)}{1/f(x)} = \left[\frac{\infty}{\infty}\right];\\
[\infty-\infty]: &\qquad \lim(f(x)-g(x)) = \lim\bigl(f(x)g(x)\bigr)\Bigl(\frac{1}{g(x)} - \frac{1}{f(x)}\Bigr) = [0\cdot\infty];\\
[1^\infty]: &\qquad \lim f(x)^{g(x)} = e^{\lim\left(g(x)\ln f(x)\right)} = e^{[\infty\cdot 0]};\\
[0^0]: &\qquad \lim f(x)^{g(x)} = e^{\lim\left(g(x)\ln f(x)\right)} = e^{[0\cdot\infty]};\\
[\infty^0]: &\qquad \lim f(x)^{g(x)} = e^{\lim\left(g(x)\ln f(x)\right)} = e^{[0\cdot\infty]} \quad \text{(must have } \lim f(x) = +\infty \text{ here).}
\end{aligned}$$

6. Techniques of integration. See `integration.pdf`.

7. Improper integrals. A definite integral is called *improper* if either it has infinite limits or the integrand is discontinuous (or both). To evaluate an improper integral, split the interval of integration into subintervals so that each subinterval has at most one singularity (i.e., an infinite endpoint or a point of discontinuity of the integrand, which must coincide with one of the endpoints of the subinterval).

Example. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x}$ should be split into $\displaystyle\int_{-\infty}^{-1} \frac{dx}{x} + \int_{-1}^{0} \frac{dx}{x} + \int_{0}^{1} \frac{dx}{x} + \int_{1}^{\infty} \frac{dx}{x}$. The subdivision points $-1 \in (-\infty, 0)$ and $1 \in (0, \infty)$ are chosen arbitrarily.

Now an integral with one singularity is defined as follows:

$$\begin{aligned}
\int_a^\infty f(x)\,dx &= \lim_{b\to\infty} \int_a^b f(x)\,dx && \text{(infinite upper limit)},\\
\int_{-\infty}^b f(x)\,dx &= \lim_{a\to-\infty} \int_a^b f(x)\,dx && \text{(infinite lower limit)},\\
\int_a^b f(x)\,dx &= \lim_{b'\to b^-} \int_a^{b'} f(x)\,dx && (f(x) \text{ is discontinuous at } b),\\
\int_a^b f(x)\,dx &= \lim_{a'\to a^+} \int_{a'}^{b} f(x)\,dx && (f(x) \text{ is discontinuous at } a).
\end{aligned}$$

If the limit exists, the integral is said to *converge*; otherwise it is said to *diverge*. An integral with several singularities converges if and only if so does **each** of the integrals over the subintervals.

Example. $\displaystyle\int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty} -\frac{1}{x}\Big|_1^b = \lim_{b\to\infty}\Bigl(1 - \frac{1}{b}\Bigr) = 1$ converges; $\displaystyle\int_0^1 \frac{dx}{x^2} = \lim_{a\to 0^+} -\frac{1}{x}\Big|_a^1 = \infty$ diverges (**!!!**).

Important Remark: Do **not** try to apply the Newton-Leibniz formula $\int_a^b f(x)\,dx = F(x)\big|_a^b$ to discontinuous functions! First, it doesn't make sense, second, it may give a wrong result. For example, $\int_{-1}^1 dx/x = \int_{-1}^0 dx/x + \int_0^1 dx/x$ diverges (as $\lim_{x\to 0^+} \ln x = -\infty$), while the Newton-Leibniz formula would give 0.

In many cases it is only important to know whether the integral converges or not. The following remarks/tests may help to decide this. (For simplicity we consider singularity at infinity; the limit ∞ below can be replaced with a point of discontinuity of f.)

Independence of the limits. The integrals $\int_a \infty f(x)\,dx$ and $\int_a \infty f(x)\,dx$ converge or diverge simultaneously provided that f is continuous on $[a, b]$. (Of course, the **value** of the integral **does** depend on the limits.)

The domination test. Assume that $0 \leqslant f(x) \leqslant g(x)$ for all sufficiently large x. Then
(1) if $\int_a^\infty g(x)\,dx$ converges, so does $\int_a^\infty f(x)\,dx$, and
(2) if $\int_a^\infty f(x)\,dx$ diverges, so does $\int_a^\infty g(x)\,dx$.

The limit comparison test. Assume that $f(x)$ and $g(x)$ are positive functions and $\lim_{x\to+\infty} f(x)/g(x) = L$, $0 < L < \infty$. Then $\int_a^\infty g(x)\,dx$ and $\int_a^\infty f(x)\,dx$ converge or diverge simultaneously.
The following integrals are useful for comparison:

$$\int_1^\infty \frac{dx}{x^p} \qquad \text{converges for } p > 1 \text{ and diverges for } p \leqslant 1,$$

$$\int_0^1 \frac{dx}{x^p} \qquad \text{converges for } p < 1 \text{ and diverges for } p \geqslant 1,$$

$$\int_0^\infty e^{ax}\,dx \qquad \text{converges for } a < 0 \text{ and diverges for } a \geqslant 0.$$