Solutions to Final Exam

Problem 1. Suppose that u is a continuous function and a is a real constant satisfying

$$\int_0^x u(t)e^{a(t^2-x^2)}\,dt = \sin x$$

for all x. Find a if $u'(\pi/2) = 5$.

SOLUTION: The given expression can be written in the form

$$e^{-ax^2}\int_0^x u(t)e^{at^2}\,dt = \sin x$$

(as $e^{a(t^2-x^2)} = e^{at^2}e^{-ax^2}$ and e^{-ax^2} can be taken out as a constant). Now, differentiate with respect to x, using the fundamental theorem of calculus, product rule, and chain rule:

$$e^{-ax^2}u(x)e^{ax^2} - 2ax\left[e^{-ax^2}\int_0^x u(t)e^{at^2}\,dt\right] = \cos x.$$

Notice that, due to the given relation, the expression in brackets is just $\sin x$ (and, of course, $e^{-ax^2}e^{ax^2} = 1$). Thus, we get $u(x) = 2ax\sin x + \cos x$. Differentiate once more and plug in $x = \pi/2$:

$$u'(x) = 2a\sin x + 2ax\cos x - \sin x, \qquad 5 = 2a - 1, \qquad \boxed{a = 3}.$$

Problem 2. Evaluate

(a) $\displaystyle\int \frac{x}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}}\,dx$

(b) $\displaystyle\int x\arctan x\,dx$ (Just in case: $\arctan x = \tan^{-1}x$ in your textbook notation.)

SOLUTION:

(a)
$$\int\ldots = \int \frac{x}{\sqrt{1+x^2}\sqrt{1+\sqrt{(1+x^2)}}}\,dx = \begin{bmatrix} u = \sqrt{1+x^2} \\ du = x\,dx/\sqrt{1+x^2}\end{bmatrix} = \int\frac{du}{\sqrt{1+u}}$$

$$= 2\sqrt{1+u} + C = \boxed{2\sqrt{1+\sqrt{1+x^2}} + C}.$$

(b)
$$\int x\arctan x\,dx = \begin{bmatrix} u = \arctan x & v = x^2/2 \\ du = dx/(1+x^2) & dv = x\,dx\end{bmatrix} = \frac{x^2}{2}\arctan x - \frac{1}{2}\int\frac{x^2\,dx}{1+x^2}$$

$$= \frac{x^2}{2}\arctan x - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx = \boxed{\frac{x^2}{2}\arctan x - \frac{x}{2} + \frac{1}{2}\arctan x + C}.$$

Problem 3. Given a tangent to the curve $y = 1/x$, denote by A and B its points of intersection with the x- and y-axis, respectively. Find the tangent(s) for which the segment $[A, B]$ has minimal length.

SOLUTION: The tangent at a point (x_0, y_0) on the curve (where $x_0 \neq 0$ and $y_0 = 1/x_0$) is given by

$$y - \frac{1}{x_0} = -\frac{1}{x_0^2}(x - x_0)$$

Letting $y = 0$ and solving for x, we find $A = (2x_0, 0)$. Letting $x = 0$ and solving for y, we find $B = (0, 2/x_0)$. Thus, the length $|AB|$ in question is

$$\ell(x_0) = 2\sqrt{x_0^2 + \frac{1}{x_0^2}}.$$

Switching back to x, we need to minimize the function $\ell(x)$ above on the domain $x \neq 0$. The derivative

$$\ell'(x) = 2\left(x - \frac{1}{x^3}\right)\Big/\sqrt{x^2 + \frac{1}{x^2}}$$

has two roots, $x = \pm 1$, and checking the sign one can see that they are both points of minima. Hence, on each of the two intervals $(-\infty, 0)$ and $(0, \infty)$ the function has a unique critical point, which is a minimum and, hence, an absolute minimum. The values $\ell(-1) = \ell(1) = 2\sqrt{2}$ coincide. Finally, the tangents of minimal length are those at $\boxed{(1,1)}$ and $\boxed{(-1,-1)}$; their length is $\boxed{2\sqrt{2}}$, and their equations are $\boxed{y = -x + 2}$ and $\boxed{y = -x - 2}$, respectively.

Problem 4.
(a) Find the absolute maximum value of the function $f(x) = x^{1/x}$, $x > 0$.
(b) Compare e^π and π^e.

SOLUTION: (a) One has

$$\ln f(x) = \frac{1}{x}\ln x, \qquad \big(\ln f(x)\big)' = \frac{1-\ln x}{x^2}, \qquad \text{and} \qquad f'(x) = x^{1/x}\frac{1-\ln x}{x^2}.$$

The only solution to $f'(x) = 0$ is $x = e$. One has $f'(x) > 0$ for $x \in (0, e)$ and $f'(x) < 0$ for $x \in (e, \infty)$. Hence, the function takes on its absolute maximum value at $\boxed{x = e}$; the value is $\boxed{e^{1/e}}$.

(b) According to part (a) one has $e^{1/e} = f(e) > f(\pi) = \pi^{1/\pi}$. Raising both the sides to the power $e\pi$, we obtain $\boxed{e^\pi > \pi^e}$. (*Remark*: the two values are pretty close: $e^\pi \approx 23.14069264$ and $\pi^e \approx 22.45915771$.)

Problem 5. Find the following limits:

(a) $\displaystyle\lim_{x\to 0}\left(\frac{1}{x^3}\int_0^x \frac{e^{t^4} - e^{-t^4}}{t^2}\,dt\right)$

(b) $\displaystyle\lim_{x\to 0^+}\big(\sin(x^2)\big)^{1/\ln x}$

SOLUTION: (a) Use L'Hôpital's rule twice:

$$\lim \ldots \overset{(1)}{=} \lim_{x\to 0}\frac{\big(e^{x^4} - e^{-x^4}\big)/x^2}{3x^2} = \lim_{x\to 0}\frac{e^{x^4} - e^{-x^4}}{3x^4} \overset{(2)}{=} \lim_{x\to 0}\frac{4x^3e^{x^4} + 4x^3e^{-x^4}}{12x^3} = \boxed{\frac{2}{3}}.$$

(Note that L'Hôpital's rule does apply, as the integrand is a continuous function and, hence, $\lim_{x\to 0}\int_0^x \ldots = \int_0^0 \ldots = 0$.)

(b) Let $\lim_{x\to 0^+}\big(\sin(x^2)\big)^{1/\ln x} = A$. Then

$$\ln A = \lim_{x\to 0^+}\frac{\ln\big(\sin(x^2)\big)}{\ln x} \overset{\text{L'Hôpital}}{=} \lim_{x\to 0^+}\frac{2x\cos(x^2)/\sin(x^2)}{1/x} = \lim_{x\to 0^+}\frac{2x^2\cos(x^2)}{\sin(x^2)} = 2 \quad \text{and} \quad \boxed{A = e^2}.$$