

TRANSCENDENTAL FUNCTIONS

MATH 101

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1. EXPONENTIAL AND LOGARITHMIC FUNCTIONS

By definition, let $\ln x = \int_1^x dx/x$ and $\exp x = e^x = \ln^{-1} x$ (the inverse function). If x is a rational number, then e^x does coincide with the x -th power of the number $e = \exp(1)$. Other powers are defined via $a^x = e^{x \ln a}$ (for $a > 0$). The following identities hold:

$$\begin{aligned}(\ln x)' &= 1/x, & (a^x)' &= a^x \ln a, & (ab)^x &= a^x b^x, \\ \ln(xy) &= \ln x + \ln y, & a^{x+y} &= a^x a^y, & (a/b)^x &= a^x / b^x. \\ \ln(x/y) &= \ln x - \ln y, & a^{x-y} &= a^x / a^y, \\ \ln(x^n) &= n \ln x, & a^{nx} &= (a^x)^n,\end{aligned}$$

$\ln x$ is defined for $x > 0$ and takes all real values. e^x is defined for all real x and takes all positive values. Both the functions are increasing. One has

$$\begin{aligned}\ln 0 &= 1, & \lim_{x \rightarrow +\infty} \ln x &= +\infty, & \lim_{x \rightarrow 0^+} \ln x &= -\infty, \\ e^0 &= 1, & \lim_{x \rightarrow +\infty} e^x &= +\infty, & \lim_{x \rightarrow -\infty} e^x &= 0.\end{aligned}$$

See also [integrals involving exponential and hyperbolic functions](#) and [integration by parts](#).

2. LOGARITHMIC DIFFERENTIATION

Sometimes it is easier to find the derivative of the function $\ln f(x)$ and then use the formula $f'(x) = f(x)(\ln f(x))'$.

2.1. Example. Let $f(x) = (x+2)^5(x^2-1)^7(x+3)^9$. Then $\ln f(x) = 5 \ln(x+2) + 7 \ln(x^2-1) + 9 \ln(x+3)$ and

$$f'(x) = (x+2)^5(x^2-1)^7(x+3)^9 \left(\frac{5}{x+2} + \frac{14x}{x^2-1} + \frac{9}{x+3} \right).$$

Remark. Strictly speaking, the calculation is only valid when $x+2 > 0$, $x^2-1 > 0$, and $x+3 > 0$. However, the final result (without logarithms) holds for all values of x .

2.2. Example. Let $f(x) = x^{x^2}$. Then $\ln f(x) = x^2 \ln x$, $(\ln f(x))' = 2x \ln x + x$, and $f'(x) = x^{x^2} (2x \ln x + x)$.

Remark. Logarithmic differentiation is the **only** way to do this one. Do **not** try to use $(x^n)' = nx^{n-1}$ (which assumes $n = \text{const}$) or $(a^x)' = a^x \ln a$ (which assumes $a = \text{const}$)!

3. INVERSE TRIGONOMETRIC FUNCTIONS

The important ones are $\arcsin x = \sin^{-1} x$, $\arccos x = \cos^{-1} x$, and $\arctan x = \tan^{-1} x$. (Do not confuse the inverse function $\sin^{-1} x$ and the reciprocal $(\sin x)^{-1} = 1/\sin x$!) Here are some properties. (I also list the inverse hyperbolic functions here; see below.)

Function	Derivative	Domain	Range	Alternative formula
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$	$[-1, 1]$	$[-\pi/2, \pi/2]$	
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$	$[-1, 1]$	$[0, \pi]$	
$\tan^{-1} x$	$\frac{1}{1+x^2}$	$(-\infty, \infty)$	$(-\pi/2, \pi/2)$	
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$	$(-\infty, \infty)$	$(-\infty, \infty)$	$\ln(x + \sqrt{x^2+1})$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$	$[1, \infty)$	$[0, \infty)$	$\ln(x + \sqrt{x^2-1})$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$	$(-1, 1)$	$(-\infty, \infty)$	$\frac{1}{2} \ln \frac{1+x}{1-x}$

4. HYPERBOLIC FUNCTIONS

The functions are defined via $\sinh x = (e^x - e^{-x})/2$, $\cosh x = (e^x + e^{-x})/2$, $\tanh x = \sinh x / \cosh x$. They share most of the properties of the trigonometric functions, but **beware of the signs!**

Trigonometric identities	Hyperbolic identities
1. $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$	$(\sinh x)' = \cosh x$, $(\cosh x)' = \sinh x$
2. $\sin^2 x + \cos^2 x = 1$	$\cosh^2 x - \sinh^2 x = 1$
3. $1 + \tan^2 x = 1/\cos^2 x$	$1 - \tanh^2 x = 1/\cosh^2 x$
4. $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$	$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
5. $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$	$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
6. $\sin 2x = 2 \sin x \cos x$	$\sinh 2x = 2 \sinh x \cosh x$
7. $\cos 2x = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1$ $= 1 - 2 \sin^2 x$	$\cosh 2x = \cosh^2 x + \sinh^2 x = 2 \cosh^2 x - 1$ $= 1 + 2 \sinh^2 x$
8. $\sin^2 x = (1 - \cos 2x)/2$	$\sinh^2 x = (\cosh 2x - 1)/2$
9. $\cos^2 x = (1 + \cos 2x)/2$	$\cosh^2 x = (\cosh 2x + 1)/2$
10. $\sin x \sin y = \frac{1}{2}(\cos(x+y) - \cos(x-y))$	$\sinh x \sinh y = \frac{1}{2}(\cosh(x+y) - \cosh(x-y))$
11. $\cos x \cos y = \frac{1}{2}(\cos(x+y) + \cos(x-y))$	$\cosh x \cosh y = \frac{1}{2}(\cosh(x+y) + \cosh(x-y))$
12. $\sin x \cos y = \frac{1}{2}(\sin(x+y) + \sin(x-y))$	$\sinh x \cosh y = \frac{1}{2}(\sinh(x+y) + \sinh(x-y))$

4.1. Remark. For those familiar with complex numbers: all the hyperbolic identities can easily be obtained from the corresponding trigonometric ones by a formal substitution. In fact, the two sets of functions are closely related: one has

$$\begin{aligned} \sinh x &= -i \sin(ix), & \cosh x &= \cos(ix), & \tanh x &= -i \tan(ix), \\ \sin x &= -i \sinh(ix), & \cos x &= \cosh(ix), & \tan x &= -i \tanh(ix), \end{aligned}$$

(where $i^2 = -1$. These formulas do make sense, but we will not go too deep into the details.)

Example. Consider the identity $\cos 2y = \cos^2 y - \sin^2 y$ and express the trigonometric functions in terms of their hyperbolic counterparts: $\cosh 2iy = \cosh^2 iy - (-i \sin iy)^2$. Now let $iy = x$ and notice that $i^2 = -1$; we get $\cosh 2x = \cosh^2 x + \sinh^2 x$.

Example. Similarly, from the identity $\sin 2y = 2 \sin y \cos y$ we get $-i \sinh 2x = -2i \sinh x \cosh x$, and it remains to cancel $-i$ to get the hyperbolic identity.

See also integrating [trigonometric](#) and [hyperbolic](#) expressions and [trigonometric/hyperbolic substitutions](#).