Solutions to Midterm II

Problem 1. Prove that

$$\int_{\cot x}^{\tan x} \frac{dt}{t^2+1} = 2x - \frac{\pi}{2}.$$

SOLUTION: Denote the left hand side by $F(x)$. The fundamental theorem of calculus yields

$$F' = \frac{\sec^2 x}{\tan^2 x + 1} - \frac{-\csc^2 x}{\cot^2 x + 1} = \frac{\sec^2 x}{\sec^2 x} + \frac{\csc^2 x}{\csc^2 x} = 2.$$

Hence, $F(x) = 2x + C$ for some constant C. Plugging in $x = \pi/4$ and observing that $F(\pi/4) = \int_1^1 dt/(t^2+1) = 0$, one finds $C = -\pi/2$. **The last step is important!!!** Otherwise, how would you explain $-\pi/2$ (and not $+25$)?

Problem 2. Evaluate:

(a) $\displaystyle\int_{\pi^2/36}^{\pi^2/4} \frac{\cos\sqrt{t}}{\sqrt{t}\sin\sqrt{t}}\,dt$

(b) $\displaystyle\int \frac{dx}{1-\cos x}$

SOLUTION:

(a) $$\int_{\pi^2/36}^{\pi^2/4} \frac{\cos\sqrt{t}}{\sqrt{t}\sin\sqrt{t}}\,dt = \begin{bmatrix} u = \sin\sqrt{t} \\ du = (1/2\sqrt{t})\cos\sqrt{t}\,dt \end{bmatrix} = 2\int_{1/2}^{1} \frac{du}{\sqrt{u}} = 4\sqrt{u}\Big|_{1/2}^{1} = \boxed{4 - 2\sqrt{2}}.$$

(b) $$\int \frac{dx}{1-\cos x} = \int \frac{dx}{2\sin^2(x/2)} = \int \csc^2\frac{x}{2}\, d\Big(\frac{x}{2}\Big) = \boxed{-\cot\frac{x}{2} + C}.$$

Problem 3. Find the smallest value of the positive constant m that will make the expression $f(x) = mx - 1 + \dfrac{1}{x}$ greater than or equal to zero for all positive x.

SOLUTION: *1st way.* We need to minimize $f(x)$ on $(0,\infty)$. One has $f'(x) = m - (1/x^2)$. Hence, the only positive critical point is $x = 1/\sqrt{m}$. (Recall that $m > 0$.) At this point $f''(1/\sqrt{m}) = 2\sqrt{m^3} > 0$; hence, the point is a (local) minimum. As it is the only critical point, it is the global minimum as well. (**Do not forget this step!!!** Just finding a critical point is not enough.) Thus, the minimal value of $f(x)$ on $(0,\infty)$ is $f(1/\sqrt{m}) = 2\sqrt{m} - 1$, and the positivity condition is satisfied if and only if this value is nonnegative: $2\sqrt{m} - 1 \geq 0$. Thus, $m \geq 1/4$, and the smallest value is $\boxed{m = 1/4}$.

2nd way. Since $x > 0$, we can rewrite the inequality $f(x) \geq 0$ in the form $m \geq (x-1)/x^2$ for all $x > 0$. Thus, m should be greater than or equal to the **maximal** value of the function $g(x) = (x-1)/x^2$ on the interval $(0,\infty)$. One has $g'(x) = (2-x)/x^3$. Hence, $x = 2$ is the only critical point in $(0,\infty)$. At this point, g' changes sign from $+$ to $-$; hence, $x = 2$ is the global maximum and $m \geq g(2) = 1/4$.

3rd way (probably, the easiest one). Rewrite the inequality $f(x) \geq 0$ in the form $(mx^2 - x + 1)/x \geq 0$. Since $x > 0$, this is equivalent to $mx^2 - x + 1 \geq 0$. Since $m > 0$, the quadratic trinomial $mx^2 - x + 1$ takes its minimal value at the **positive** point $x_0 = 1/2m$, and this minimal value is non-negative if and only if the discriminant $D \leq 0$. Thus, $D = (1)^2 - 4 \cdot m \cdot 1 \leq 0$, and $m \geq 1/4$. (Note that just requiring that $D \leq 0$ is not sufficient, as *a priori* it could still happen that the function is positive for all **positive** values of x, even though it does take some negative values at negative x.)

Remark. Those familiar with the arithmetic/geometric mean inequality could observe that, if both x and m are positive, one has $\frac{1}{2}(mx + (1/x)) \geq \sqrt{mx \cdot (1/x)} = \sqrt{m}$, and this value must be at least $1/2$.

Problem 4. Let $R > r > 0$. A ball of radius R is cut by a right circular cylinder of radius r whose axis passes through the center of the ball, and the interior of the cylinder is removed. Prove that the volume of the remaining part of the ball equals that of a ball of radius $\sqrt{R^2 - r^2}$.

SOLUTION: Let us just find the volume and see that it equals $(4/3)\pi(\sqrt{R^2-r^2})^3$, *i.e.*, the volume of the second ball mentioned in the problem.

Both the ball and the cylinder are solids of revolution. Hence, the solid in question can be obtained by revolving about the x-axis the disk segment bounded by the circle $y=\sqrt{R^2-x^2}$ (from above) and the horizontal line $y=r$ (from below). The endpoints of the segment correspond to the values $x=\pm\sqrt{R^2-r^2}$. Hence, the volume of the resulting solid is (*via* the disk method)

$$\pi\int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}}\big((R^2-x^2)-r^2\big)dx=\pi\Big((R^2-r^2)x-\frac{x^3}{3}\Big)\Big|_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}}=\frac{4}{3}\pi(R^2-r^2)^{3/2},$$

as required.

Problem 5. Graph the function $y=x\sqrt{4-x^2}$; find (and indicate on the graph) the critical points, intervals of increasing and decreasing, points of inflection, concavity, x- and y-intercepts, and asymptotes.

SOLUTION: First, find the domain. For the radical to make sense, one must have $4-x^2\geq 0$. Hence, the domain is $[-2,2]$.

Find the derivatives:

$$y'=\frac{4-2x^2}{\sqrt{4-x^2}},\qquad y''=\frac{2x(x^2-6)}{(4-x^2)^{3/2}}.$$

Thus, the critical points are $x=\pm\sqrt{2}$ (with $y'=0$) and $x=\pm2$ (with $y'=\infty$). The zeroes of the expression for the second derivative are $x=0$ and $x=\pm\sqrt{6}$; however, **the latter two values are not in the domain!!!**

Determine the signs of y' and y'':

x	-2		$-\sqrt{2}$		0		$\sqrt{2}$		2
y'	∞	$-$	0	$+$		$+$	0	$-$	∞
y''		$+$		$+$	0	$-$		$-$	
y	0	$\searrow$ $\smile$	min -2	$\nearrow$ $\smile$	infl.pt 0	$\nearrow$ $\frown$	max 2	$\searrow$ $\frown$	0

(Note that we cannot speak about the sign of y' or y'' 'at infinity', as the domain of the function is bounded. Hence, we should start with determining the sign in one of the intervals, say, $(\sqrt{2},2)$.)

Since the function is continuous, the graph has no vertical asymptotes. Oblique asymptotes are out of question as the domain is bounded. The y-intercept is the point $(0,0)$. To find the x-intercepts, solve the equation $x\sqrt{4-x^2}=0$. This gives us three points: $(0,0)$ and $(\pm2,0)$.

Finally, here is the graph (note that at $x=\pm\sqrt{2}$ the tangents are vertical):