Solutions to Midterm II

Problem 1. Graph the function $y = \dfrac{4x}{x^2+4}$; find (and indicate on the graph) the critical points, intervals of increasing and decreasing, inflection points, concavity, x- and y-intercepts, and asymptotes.

SOLUTION: We have

$$y' = -\frac{4(x^2-4)}{(x^2+4)^2} = -\frac{4(x-2)(x+2)}{(x^2+4)^2}, \qquad y'' = \frac{8x(x^2-12)}{(x^2+4)^3} = \frac{8x(x-2\sqrt{3})(x+2\sqrt{3})}{(x^2+4)^3}.$$

Both y' and y'' are defined for all values of x. The solutions to $y'=0$ (critical points) are $x = \pm 2$; the solutions to $y''=0$ are $x=0$ and $x = \pm 2\sqrt{3}$. Determine the signs of y' and y'':

x		$-2\sqrt{3}$		-2		0		2		$2\sqrt{3}$	
y'	$-$		$-$	0	$+$		$+$	0	$-$		$-$
y''	$-$	0	$+$		$+$	0	$-$		$-$	0	$+$
y	$\searrow$ $\frown$	$-\sqrt{3}/2$	$\searrow$ $\smile$	min -1	$\nearrow$ $\smile$		$\nearrow$ $\frown$	max 1	$\searrow$ $\frown$	$\sqrt{3}/2$	$\searrow$ $\smile$

Thus, $x=-2$ is a (local) minimum, $x=2$ is a (local) maximum, and $x=0$ and $\pm 2\sqrt{3}$ are inflection points. Intervals of increasing/decreasing and concavity are shown in the table.

The function is continuous for all x; hence, the graph has no vertical asymptotes. One has $\lim_{x\to\infty} y = 0$; hence, the line $y=0$ (the x-axis) is a horizontal asymptote (both at $x \to +\infty$ and $x \to -\infty$). The graph is straightforward now.

Problem 2. The region bounded below by the parabola $y = x^2$ and above by the line $y=4$ is to be partitioned into two subsections of equal area by cutting across it with the horizontal line $y=c$. Find c and the area of the resulting subsections.

SOLUTION: Let $A(c)$ ($c >= 0$) be the area of the region bounded below by the parabola $y=x^2$ and above by the line $y=c$. Then

$$A(c) = \int_{-\sqrt{c}}^{\sqrt{c}} (c - x^2)dx = (cx - x^3/3)\Big|_{-\sqrt{c}}^{\sqrt{c}} = \frac{4}{3}c^{3/2}.$$

(Limits are found as points of intersection of the parabola and the line, *i.e.*, solutions to $x^2 = c$.) The area of the whole region is $A(4) = 32/3$, and c is found from the equation $A(c) = \frac{1}{2}A(4)$, *i.e.*, $c^{3/2} = 4$. Thus, one has $\boxed{c = 2\sqrt[3]{2}}$ and the area of the subsections is $\boxed{A(2\sqrt[3]{2}) = 16/3}$.

Problem 3. Suppose that x and y are related by the equation

$$x = \int_0^y \frac{1}{\sqrt{1+4t^2}}\,dt.$$

Show that d^2y/dx^2 is proportional to y and find the constant of proportionality.

SOLUTION: Use implicit differentiation (and the fundamental theorem of calculus):

$$\frac{dx}{dx} = y' \cdot \frac{d}{dy}\int_0^y \frac{1}{\sqrt{1+4t^2}}\,dt, \qquad \text{or} \qquad 1 = y' \cdot \frac{1}{\sqrt{1+4y^2}}.$$

Thus, $y' = \sqrt{1+4y^2}$. Differentiate once again:

$$y'' = y' \cdot \frac{d}{dy}\sqrt{1+4y^2} = \sqrt{1+4y^2} \cdot \frac{4y}{\sqrt{1+4y^2}} = \boxed{4y}.$$

Problem 4. The region bounded by the graph $y = \sqrt{x}$, the horizontal line $y = c$, and the vertical lines $x=0$ and $x=4$ is revolved about the line $y=c$ to generate a solid. Find the volume of the resulting solid (as a function of c). Find the value of c that minimizes the volume.

SOLUTION: Let $V(c)$ be the volume of the solid. Using the disk method, one has

$$V(c) = \int_0^4 \pi(\sqrt{x} - c)^2 dx = \int_0^4 \pi(x - 2c\sqrt{x} + c^2) dx = \pi\Big(\frac{x^2}{2} - \frac{4cx^{3/2}}{3} + c^2 x\Big)\Big|_0^4 = \boxed{\frac{4\pi}{3}(3c^2 - 8c + 6)}.$$

(Note that, for the disk method, we do **not** need to know which parts of the parabola lie above/below the axis.) We need to minimize the above function $V(c) = 4\pi(3c^2 - 8c + 6)/3$ on the whole real line. (There are no restrictions on c.) Differentiate to get $V'(c) = 4\pi(6c - 8)/3$. The only critical point is $c = 4/3$, and this is a point of minimum. Hence, the absolute minimum of the volume is $\boxed{V(4/3) = 8\pi/9}$ is taken on at $\boxed{c = 4/3}$.

Problem 5. Evaluate:

(a) $\displaystyle\int_0^{\pi/3} \frac{\tan x}{\sqrt{2 \sec x}}\, dx$

(b) $\displaystyle\int \frac{dx}{1 + \cos x}$

SOLUTION:

$$\text{(a)} \quad \int_0^{\pi/3} \frac{\tan x}{\sqrt{2\sec x}}\, dx = \begin{bmatrix} u = \sec x \\ du = \tan x \sec x \end{bmatrix} = \int_1^2 \frac{du}{\sqrt{2u^3}} = -\frac{\sqrt{2}}{\sqrt{u}}\Big|_1^2 = \boxed{\sqrt{2} - 1}.$$

$$\text{(b)} \quad \int \frac{dx}{1 + \cos x} = \int \frac{dx}{2\cos^2(x/2)} = . \int \sec^2 \frac{x}{2}\, d\Big(\frac{x}{2}\Big) = \boxed{\tan \frac{x}{2} + C}.$$