

Solutions to Final Exam

Problem 1. Suppose that u is a continuous function and a is a real constant satisfying

$$\int_0^x u(t)e^{a(t^2-x^2)} dt = \sin x$$

for all x . Find a if $u'(\pi/2) = 5$.

SOLUTION: The given expression can be written in the form

$$e^{-ax^2} \int_0^x u(t)e^{at^2} dt = \sin x$$

(as $e^{a(t^2-x^2)} = e^{at^2}e^{-ax^2}$ and e^{-ax^2} can be taken out as a constant). Now, differentiate with respect to x , using the fundamental theorem of calculus, product rule, and chain rule:

$$e^{-ax^2} u(x)e^{ax^2} - 2ax \left[e^{-ax^2} \int_0^x u(t)e^{at^2} dt \right] = \cos x.$$

Notice that, due to the given relation, the expression in brackets is just $\sin x$ (and, of course, $e^{-ax^2}e^{ax^2} = 1$). Thus, we get $u(x) = 2ax \sin x + \cos x$. Differentiate once more and plug in $x = \pi/2$:

$$u'(x) = 2a \sin x + 2ax \cos x - \sin x, \quad 5 = 2a - 1, \quad \boxed{a = 3}.$$

Problem 2. Evaluate

(a) $\int \frac{x}{\sqrt{1+x^2} + \sqrt{(1+x^2)^3}} dx$

(b) $\int x \arctan x dx$ (Just in case: $\arctan x = \tan^{-1} x$ in your textbook notation.)

SOLUTION:

(a)
$$\int \dots = \int \frac{x}{\sqrt{1+x^2} + \sqrt{(1+x^2)^3}} dx = \left[\begin{array}{l} u = \sqrt{1+x^2} \\ du = x dx / \sqrt{1+x^2} \end{array} \right] = \int \frac{du}{\sqrt{1+u}}$$

$$= 2\sqrt{1+u} + C = \boxed{2\sqrt{1+\sqrt{1+x^2}} + C}.$$

(b)
$$\int x \arctan x dx = \left[\begin{array}{ll} u = \arctan x & v = x^2/2 \\ du = dx/(1+x^2) & dv = x dx \end{array} \right] = \frac{x^2}{2} \arctan x - \frac{1}{2} \int \frac{x^2 dx}{1+x^2}$$

$$= \frac{x^2}{2} \arctan x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2} \right) dx = \boxed{\frac{x^2}{2} \arctan x - \frac{x}{2} + \frac{1}{2} \arctan x + C}.$$

Problem 3. Given a tangent to the curve $y = 1/x$, denote by A and B its points of intersection with the x - and y -axis, respectively. Find the tangent(s) for which the segment $[A, B]$ has minimal length.

SOLUTION: The tangent at a point (x_0, y_0) on the curve (where $x_0 \neq 0$ and $y_0 = 1/x_0$) is given by

$$y - \frac{1}{x_0} = -\frac{1}{x_0^2}(x - x_0)$$

Letting $y = 0$ and solving for x , we find $A = (2x_0, 0)$. Letting $x = 0$ and solving for y , we find $B = (0, 2/x_0)$. Thus, the length $|AB|$ in question is

$$\ell(x_0) = 2\sqrt{x_0^2 + \frac{1}{x_0^2}}.$$

Switching back to x , we need to minimize the function $\ell(x)$ above on the domain $x \neq 0$. The derivative

$$\ell'(x) = 2 \left(x - \frac{1}{x^3} \right) / \sqrt{x^2 + \frac{1}{x^2}}$$

has two roots, $x = \pm 1$, and checking the sign one can see that they are both points of minima. Hence, on each of the two intervals $(-\infty, 0)$ and $(0, \infty)$ the function has a unique critical point, which is a minimum and, hence, an absolute minimum. The values $\ell(-1) = \ell(1) = 2\sqrt{2}$ coincide. Finally, the tangents of minimal length are those at $\boxed{(1, 1)}$ and $\boxed{(-1, -1)}$; their length is $\boxed{2\sqrt{2}}$, and their equations are $\boxed{y = -x + 2}$ and $\boxed{y = -x - 2}$, respectively.

Problem 4.

- (a) Find the absolute maximum value of the function $f(x) = x^{1/x}$, $x > 0$.
 (b) Compare e^π and π^e .

SOLUTION: (a) One has

$$\ln f(x) = \frac{1}{x} \ln x, \quad (\ln f(x))' = \frac{1 - \ln x}{x^2}, \quad \text{and} \quad f'(x) = x^{1/x} \frac{1 - \ln x}{x^2}.$$

The only solution to $f'(x) = 0$ is $x = e$. One has $f'(x) > 0$ for $x \in (0, e)$ and $f'(x) < 0$ for $x \in (e, \infty)$. Hence, the function takes on its absolute maximum value at $x = e$; the value is $e^{1/e}$.

(b) According to part (a) one has $e^{1/e} = f(e) > f(\pi) = \pi^{1/\pi}$. Raising both the sides to the power $e\pi$, we obtain $e^\pi > \pi^e$. (*Remark:* the two values are pretty close: $e^\pi \approx 23.14069264$ and $\pi^e \approx 22.45915771$.)

Problem 5. Find the following limits:

- (a) $\lim_{x \rightarrow 0} \left(\frac{1}{x^3} \int_0^x \frac{e^{t^4} - e^{-t^4}}{t^2} dt \right)$
 (b) $\lim_{x \rightarrow 0^+} (\sin(x^2))^{1/\ln x}$

SOLUTION: (a) Use L'Hôpital's rule twice:

$$\lim \dots \stackrel{(1)}{=} \lim_{x \rightarrow 0} \frac{(e^{x^4} - e^{-x^4})/x^2}{3x^2} = \lim_{x \rightarrow 0} \frac{e^{x^4} - e^{-x^4}}{3x^4} \stackrel{(2)}{=} \lim_{x \rightarrow 0} \frac{4x^3 e^{x^4} + 4x^3 e^{-x^4}}{12x^3} = \boxed{\frac{2}{3}}.$$

(Note that L'Hôpital's rule does apply, as the integrand is a continuous function and, hence, $\lim_{x \rightarrow 0} \int_0^x \dots = \int_0^0 \dots = 0$.)

(b) Let $\lim_{x \rightarrow 0^+} (\sin(x^2))^{1/\ln x} = A$. Then

$$\ln A = \lim_{x \rightarrow 0^+} \frac{\ln(\sin(x^2))}{\ln x} \stackrel{\text{L'Hôpital}}{=} \lim_{x \rightarrow 0^+} \frac{2x \cos(x^2)/\sin(x^2)}{1/x} = \lim_{x \rightarrow 0^+} \frac{2x^2 \cos(x^2)}{\sin(x^2)} = 2 \quad \text{and} \quad \boxed{A = e^2}.$$