Bilkent University
Department of Mathematics

Problem Of The Month

September 2005

Problem: Suppose that $0 \leq x_1, x_2, \ldots, x_{10} \leq \frac{\pi}{2}$ and

$$\sin^2 x_1 + \sin^2 x_2 + \cdots + \sin^2 x_{10} = 1.$$

Prove that

$$\frac{\cos x_1 + \cos x_2 + \cdots + \cos x_{10}}{\sin x_1 + \sin x_2 + \cdots + \sin x_{10}} \geq 3.$$

Solution: Denote $\sin x_1 + \sin x_2 + \cdots + \sin x_{10}$ by A. Note that

$$\frac{\cos x_i}{3} = \frac{\sqrt{1 - \sin^2 x_i}}{3} = \sqrt{\frac{\sin^2 x_1 + \cdots + \sin^2 x_{i-1} + \sin^2 x_{i+1} + \cdots \sin^2 x_{10}}{9}}.$$

And by using of the quadratic mean – arithmetic mean inequality, we get

$$\frac{\cos x_i}{3} \geq \frac{\sin x_1 + \cdots + \sin x_{i-1} + \sin x_{i+1} + \cdots \sin x_{10}}{9} = \frac{A - \sin x_i}{9}.$$

By summing of the obtained inequalities

$$3\cos x_i \geq A - \sin x_i \qquad \text{for } i = 1, \ldots, 10,$$

we get

$$\begin{aligned} 3(\cos x_1 + \cos x_2 + \cdots \cos x_{10}) &\geq 10A - \sin x_1 - \sin x_2 - \cdots - \sin x_{10} \\ &= 9(\sin x_1 + \sin x_2 + \cdots + \sin x_{10}). \end{aligned}$$

Done.