Bilkent University
Department of Mathematics

Problem Of The Month

April 2024

Problem:

Let $n \geq 3$ be an integer and $a_1, a_2, \ldots, a_n$ be real numbers. For each $1 \leq k \leq n$ the real numbers $b_1, b_2, \ldots, b_{n+1}$ are defined by

$$b_k = \frac{a_k + \max\{a_{k+1}, a_{k+2}\}}{2}$$

and $b_{n+1} = b_1$ ($a_{n+1} = a_1$ and $a_{n+2} = a_2$). Show that the inequality

$$\sum_{i=1}^{n} (a_i - a_{i+1})^2 \geq \sum_{i=1}^{n} (b_i - b_{i+1})^2$$

is held for each $n \geq 3$ and all real numbers $a_1, a_2 \ldots, a_n$.

Solution:

Let $2b_i = c_i$ and $a_i - a_{i+1} = x_i$. Since for all real numbers x and y

$$\max\{x, y\} = \frac{x+y}{2} + |\frac{x-y}{2}|$$

we get that

$$c_i - c_{i+1} = x_i + \frac{x_{i+1}}{2} + \frac{x_{i+2}}{2} + |\frac{x_{i+1}}{2}| - |\frac{x_{i+2}}{2}| \qquad (1)$$

For each index i let

$$y_i = \frac{x_i}{2} + |\frac{x_i}{2}| \quad \text{and} \quad z_i = \frac{x_i}{2} - |\frac{x_i}{2}|.$$

Then, $y_i = x_i$, $z_i = 0$ for $x_i \geq 0$ and $y_i = 0$, $z_i = x_i$ for $x_i < 0$. Inserting y_i and z_i to (1) we get

$$(c_i - c_{i+1})^2 = (x_i + y_{i+1} + z_{i+2})^2 \leq 2x_i^2 + 2(y_{i+1} + z_{i+2})^2 = 2x_i^2 + 2y_{i+1}^2 + 2z_{i+2}^2 + 4y_{i+1}z_{i+2}$$

(above we used the inequality $(t+s)^2 \leq 2t^2 + 2s^2$). Since $y_i \geq 0$ and $z_i \leq 0$ we get that $y_{i+1}z_{i+2} \leq 0$. Hence we get

$$(c_i - c_{i+1})^2 \leq 2x_i^2 + 2y_{i+1}^2 + z_{i+2}^2 \qquad (2)$$

Finally by using of the inequality (2) and equality $y_i^2 + z_i^2 = x_i^2$ in the problem statement

$$\sum_{i=1}^{n} x_i^2 \geq \sum_{i=1}^{n} \frac{1}{4}(c_i - c_{i+1})^2$$

we complete the solution.