Problem Of The Month

April 2024

Problem:

Let \( n \geq 3 \) be an integer and \( a_1, a_2, \ldots, a_n \) be real numbers. For each \( 1 \leq k \leq n \) the real numbers \( b_1, b_2, \ldots, b_{n+1} \) are defined by

\[
b_k = \frac{a_k + \max\{a_{k+1}, a_{k+2}\}}{2}
\]

and \( b_{n+1} = b_1 \) (\( a_{n+1} = a_1 \) and \( a_{n+2} = a_2 \)). Show that the inequality

\[
\sum_{i=1}^{n} (a_i - a_{i+1})^2 \geq \sum_{i=1}^{n} (b_i - b_{i+1})^2
\]

is held for each \( n \geq 3 \) and all real numbers \( a_1, a_2 \ldots, a_n \).

Solution:

Let \( 2b_i = c_i \) and \( a_i - a_{i+1} = x_i \). Since for all real numbers \( x \) and \( y \)

\[
\max\{x, y\} = \frac{x + y}{2} + \left| \frac{x - y}{2} \right|
\]

we get that

\[
c_i - c_{i+1} = x_i + \frac{x_{i+1}}{2} + \frac{x_{i+2}}{2} + \left| \frac{x_{i+1}}{2} \right| - \left| \frac{x_{i+2}}{2} \right|
\]

For each index \( i \) let

\[
y_i = \frac{x_i}{2} + \left| \frac{x_i}{2} \right| \quad \text{and} \quad z_i = \frac{x_i}{2} - \left| \frac{x_i}{2} \right|.
\]

Then, \( y_i = x_i \), \( z_i = 0 \) for \( x_i \geq 0 \) and \( y_i = 0 \), \( z_i = x_i \) for \( x_i < 0 \). Inserting \( y_i \) and \( z_i \) to (1) we get

\[
(c_i - c_{i+1})^2 = (x_i + y_{i+1} + z_{i+2})^2 \leq 2x_i^2 + 2(y_{i+1} + z_{i+2})^2 = 2x_i^2 + 2y_{i+1}^2 + 2z_{i+2}^2 + 4y_{i+1}z_{i+2}
\]
(above we used the inequality \((t + s)^2 \leq 2t^2 + 2s^2\)). Since \(y_i \geq 0\) and \(z_i \leq 0\) we get that \(y_{i+1}z_{i+2} \leq 0\). Hence we get

\[
(c_i - c_{i+1})^2 \leq 2x_i^2 + 2y_{i+1}^2 + z_{i+2}^2 \quad (2)
\]

Finally by using of the inequality (2) and equality \(y_i^2 + z_i^2 = x_i^2\) in the problem statement

\[
\sum_{i=1}^{n} x_i^2 \geq \sum_{i=1}^{n} \frac{1}{4} (c_i - c_{i+1})^2
\]

we complete the solution.