Bilkent University
Department of Mathematics

Problem Of The Month

March 2024

Problem:

For each positive integer n, let $\varphi(n)$ be the total number of positive integers not exceeding n and co-prime with n. Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ satisfying the following two conditions:

$$a - b \mid f(a) - f(b) \quad \text{and}$$

$$f(\varphi(a)) = \varphi(f(a)).$$

Solution: Answer: $f(x) \equiv 1$ and $f(x) = x$.

By putting $a = 1$ to $f(\varphi(a)) = \varphi(f(a))$ we get $f(1) = \varphi(f(1))$ and hence $f(1) = 1$. By putting $a = 2$ we get $f(1) = \varphi(f(2))$ and since $f(1) = 1$ we get that either $f(2)$ is either 1 or 2.

Case 1: $f(2) = 1$. Since $1 = f(2) = f(\varphi(3)) = \varphi(f(3))$ we get that $f(3)$ is either 1 or 2. But since $(3-1) \mid f(3) - f(1)$ we get that $f(3) = 1$. By induction we show that $f(n) = 1$ for all positive integers n. Assume that $f(a) = 1$ for all $a = 1, 2, \ldots, n-1$. Then since $f(\varphi(n)) = \varphi(f(n))$ and $\varphi(n) \leq n-1$ we get $\varphi(f(n)) = 1$ and hence $f(n)$ is either 1 or 2. Finally again since $n-(n-2) \mid f(n) - f(n-2)$ we get that $f(n) = 1$. Done.

Case 2: $f(2) = 2$. By induction we show that $f(n) = n$ for all positive integers a. Assume that $f(a) = a$ for all $a = 1, 2, \ldots, n-1$. By the first condition and by inductive hypothesis for each $1 \leq a \leq n-1$ we have $n - a \mid f(n) - a$ or equivalently $n - a \mid f(n) - n$. Therefore, for each integer $1 \leq m \leq n-1$ we have $m \mid f(n) - n$. Hence the common divisor M of all integers $1 \leq m \leq n-1$ also divides $f(n) - n :\ f(n) = Ms + n$, where s is integer. Then since $\varphi(n) < n$ by the second condition and by inductive hypothesis $\varphi(n) = \varphi(Ms + n)$. On the other hand, since each integer k with $(n, k) = 1$ and $k < n$ divides M we have $(k, Ms + n) = 1$. Therefore, $\varphi(n) \leq \varphi(Ms + n)$. If $s \neq 0$ then $(Ms + n - 1, Ms + n) = 1$ and hence we get that $\varphi(n) < \varphi(Ms + n)$. This contradiction shows that $s = 0$ and $f(n) = n$. Done.