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Problem Of The Month

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Problem:

There are several red and several white boxes on the table, each of these boxes contains at least one ball. A positive integer number not exceeding 1111 is written on each of these balls.

† Any two boxes contain different number of balls.

†† No box contains two balls with the same number.

††† For each $1 \leq i \leq 1111$ there is at most one red box containing ball number i.

†††† For each $1 \leq i \leq 1111$ there is at most one white box containing ball number i.

††††† For any two balls with numbers i and j, where $1 \leq i \leq 1111$, $1 \leq j \leq 1111$ and $i \neq j$ there is at most one box containing these two balls.

Find the maximal possible number of boxes on the table.

Solution: Answer: 57.

Let us give an example for 38 red boxes $R_1, R_2, \ldots, R_{38}$ and 19 white boxes $W_{39}, W_{40}, \ldots, W_{57}$ satisfying the conditions. We take 1102 balls numbered by $1, 2, \ldots, 1002$ and divide them into groups A and B of sizes 741 and 361, respectively. Balls from A will be labeled by different pairs of positive indices (i, j):

$$A' = \{(i, j) : i, j \in \mathbb{Z}_+, \ \ i + j \leq 39\}$$

Balls from B will be labeled by different pairs of positive indices (i, j):

$$B' = \{(i, j) : i, j \in \mathbb{Z}_+, \ \ i \leq 19, j \leq 19\}.$$

For each $1 \leq k \leq 38$ to the red box R_k we put k balls from A with $i = 39 - k$ and for each $39 \leq k \leq 57$ to the white box W_k we put k balls from A and B with $j = 58 - k$. It can be readily seen that all conditions of the problem are satisfied: each ball from A is in exactly one red box and in at most one white box and each ball from B is in exactly one white box.

Now let us show that if n boxes satisfy the conditions then $n \leq 57$. Let us consider 38 boxes with largest numbers of balls. Without loss of generality suppose that these boxes are boxes $R_1, R_2, \ldots, R_m$ and $W_{m+1}, \ldots, W_{38}$. Let $P(i)$ and $Q(j)$ be the set of balls in boxes R_i and W_j. By conditions $|P_i \cap Q_j| \leq 1$ and for $i \neq j$ we have $|P_i \cap P_j| = 0$ and $|Q_i \cap Q_j| = 0$. Therefore,

$$\sum |P_i| + \sum |Q_i|$$

$$= |P_1 \cup \ldots \cup P_k \cup Q_{m+1} \cup \ldots \cup Q_{38}| + \sum |P_i \cap Q_j|$$

$$\leq 1111 + k(38 - k) \leq 1472.$$

On the other hand, since all boxes contain different numbers of balls, we get

$$\sum |P_i| + \sum |Q_i| \geq n + (n-1) + \ldots + (n-37) = (2n - 37) \cdot 19$$

Therefore, $2n - 37 < 78$ and hence $n \leq 57$.

Note: In the general case when there are l balls the answer is a maximal integer satisfying the inequality $n(n+1) \leq 3l$.