Problem Of The Month

December 2023

Problem:

There are several red and several white boxes on the table, each of these boxes contains at least one ball. A positive integer number not exceeding 1111 is written on each of these balls.

† Any two boxes contain different number of balls.

†† No box contains two balls with the same number.

††† For each $1 \leq i \leq 1111$ there is at most one red box containing ball with number $i$.

†††† For each $1 \leq i \leq 1111$ there is at most one white box containing ball with number $i$.

Find the maximal possible number of boxes on the table.

Solution: Answer: 66.

Suppose that the total number of boxes is $n$ and these boxes contain $a_1 < a_2 < \cdots < a_n$ balls. Then

$$2 \cdot 1111 \geq \sum_{i=1}^{n} a_i \geq \sum_{i=1}^{n} i = \frac{n(n + 1)}{2}$$

The largest integer $n$ satisfying the above inequality is $n = 66$. Now one can construct a proper example for $n = 66$ when there are 46 red boxes containing 1, 2, \ldots, 15, 16, 18, 19, \ldots, 46, 47 balls and 20 white boxes containing 17, 48, 49, \ldots, 46, 47 balls, respectively since here the total number of balls in red boxes is 1111 the and the total number of balls in white boxes is 1100.