Problem:

Let $P$ be a polynomial of degree $n$ with real coefficients such that at least $n$ of its coefficients coincide and at most one of its coefficients is zero. Given that all $n$ roots of $P$ are real numbers find the maximal possible value of $n$.

Solution: Answer: The maximal possible value is $n = 4$.

The polynomial

$$P(x) = x^4 + x^3 - 4x^2 + x + 1 = (x - 1)(x - 1)(x + \frac{3 + \sqrt{5}}{2})(x + \frac{3 - \sqrt{5}}{2})$$

satisfies problem conditions. Let us show that there is no polynomial of degree $n \geq 5$ satisfying the conditions. Without loss of generality we assume that at least $n$ coefficients of the polynomial are 1.

Let $x_1, x_2, \ldots, x_n$ be the roots and $S_k$ be the sum of k-wise sums of the roots:

$$S_1 = x_1 + \cdots + x_n, S_2 = x_1x_2 + \cdots + x_{n-1}x_n, \ldots, S_n = x_1x_2\cdots x_n$$

By Vieta theorem we have

$$S_1^2 - 2S_2 = \sum_{i=1}^{n} x_i^2 \geq 0.$$ 

If the first three coefficients of $P$ are 1 then we get a contradiction with the last inequality:

$$S_1 = -1, S_2 = 1 \text{ and } S_1^2 - 2S_2 = -1 < 0.$$ 

Therefore, one of the first three coefficients is not 1. Then the free coefficient is 1 and 0 is not a root of $P$. In this case we have
\[
\left( \frac{S_{n-1}}{S_n} \right)^2 - 2 \left( \frac{S_{n-2}}{S_n} \right) = \sum_{i=1}^{n} \frac{1}{x_i^2} > 0.
\]

If the last three coefficients of \( P \) are 1 then we get a contradiction with the last inequality:

\[
S_{n-2} = -S_{n-1} = S_n \quad \text{ve} \quad \left( \frac{S_{n-1}}{S_n} \right)^2 - 2 \left( \frac{S_{n-2}}{S_n} \right) < 0.
\]

Since at least \( n \) coefficients of \( P \) out of \( n + 1 \) are 1 when \( n \geq 5 \) either first three or last three coefficients of \( P \) should be 1, a contradiction.