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Problem Of The Month

October 2023

Problem:

Let $\mathcal{S}$ be the set of all 2023 tuples $(x_1, x_2, \ldots, x_{2023})$, where $x_i \in \{1, 2, \ldots, 100\}$ for each $1 \le i \le 2023$. The subset $T \subset \mathcal{S}$ is said to be down-dense if for each $(x_1, x_2, \ldots, x_{2023}) \in T$ any $(y_1, y_2, \ldots, y_{2023})$ satisfying $y_i \le x_i$ $(1 \le i \le 2023)$ also belongs to T. The subset $T \subset \mathcal{S}$ is said to be up-dense if for each $(x_1, x_2, \ldots, x_{2023}) \in T$ any $(y_1, y_2, \ldots, y_{2023})$ satisfying $y_i \ge x_i$ $(1 \le i \le 2023)$ also belongs to T. Find the minimal possible value of

$$f(A, B) = \frac{|A| \cdot |B|}{|A \cap B|},$$

where A and B are non-empty down-dense and up-dense subsets of $\mathcal{S}$, respectively.

Note: $|T|$ denotes the number of elements of a set T.

Solution: Answer: 100^{2023}.

Let us treat more general case when $\mathcal{S}$ is the set of all n tuples. If $A = B = \mathcal{S}$ then $f(A, B) = 100^n$. We prove that $f(A, B) \ge 100^n$ by induction over n.

$n = 1$. Suppose that $A = \{1, 2, \ldots, a+c\}$, $B = \{100-b-c+1, \ldots, 100\}$. Then $|A \cap B| = c$, $|A| = a + c$, $|B| = b + c$ and

$$f(A, B) = \frac{(a+c)(b+c)}{c} = \frac{(a+b+c)c + ab}{c} = 100 + \frac{ab}{c} \ \ge\ 100.$$

Suppose that the statement is correct for $n - 1$. Let $A = \cup_{i=1}^{100} A_i$, where elements of the set A_i are obtained from elements of A having last entry i by removing this last entry. By definitions $|A| = \sum_{i=1}^{100} |A_i|$ and $A_1 \subset A_2 \subset \cdots \subset A_{100}$. Let $B = \cup_{i=1}^{100} B_i$, where elements of the set B_i are obtained from elements of B having last entry i by removing this last entry. By definitions $|B| = \sum_{i=1}^{100} |B_i|$ and $B_1 \supset B_2 \supset \cdots \supset B_{100}$. Now

$$|A \cap B| = \sum_{i=1}^{100} |A_i \cap B_i| \leq \frac{1}{100^{n-1}} \cdot \sum_{i=1}^{100} |A_i| \cdot |B_i|$$

$$\leq \frac{1}{100^{n-1}} \cdot \frac{1}{100} \cdot \big(\sum_{i=1}^{100} |A_i|\big)\big(\sum_{i=1}^{100} |B_i|\big) = \frac{1}{100^n} \cdot |A| \cdot |B|$$

(The first inequality is valid due to inductive hypothesis, the second inequality is the Chebyshev's rearrangement inequality). We are done.