Problem: Let $S$ be the set of all 2023 tuples $(x_1, x_2, \ldots, x_{2023})$, where $x_i \in \{1, 2, \ldots, 100\}$ for each $1 \leq i \leq 2023$. The subset $T \subset S$ is said to be down-dense if for each $(x_1, x_2, \ldots, x_{2023}) \in T$ any $(y_1, y_2, \ldots, y_{2023})$ satisfying $y_i \leq x_i$ ($1 \leq i \leq 2023$) also belongs to $T$. The subset $T \subset S$ is said to be up-dense if for each $(x_1, x_2, \ldots, x_{2023}) \in T$ any $(y_1, y_2, \ldots, y_{2023})$ satisfying $y_i \geq x_i$ ($1 \leq i \leq 2023$) also belongs to $T$. Find the minimal possible value of

$$f(A, B) = \frac{|A| \cdot |B|}{|A \cap B|},$$

where $A$ and $B$ are non-empty down-dense and up-dense subsets of $S$, respectively.

Note: $|T|$ denotes the number of elements of a set $T$.

Solution: Answer: $100^{2023}$.

Let us treat more general case when $S$ is the set of all $n$ tuples. If $A = B = S$ then $f(A, B) = 100^n$. We prove that $f(A, B) \geq 100^n$ by induction over $n$.

$n = 1$. Suppose that $A = \{1, 2, \ldots, a+c\}$, $B = \{100-b-c+1, \ldots, 100\}$. Then $|A \cap B| = c$, $|A| = a + c$, $|B| = b + c$ and

$$f(A, B) = \frac{(a + c)(b + c)}{c} = \frac{(a + b + c)c + ab}{c} = 100 + \frac{ab}{c} \geq 100.$$

Suppose that the statement is correct for $n - 1$. Let $A = \cup_{i=1}^{100} A_i$, where elements of the set $A_i$ are obtained from elements of $A$ having last entry $i$ by removing this last entry. By definitions $|A| = \sum_{i=1}^{100} |A_i|$ and $A_1 \subset A_2 \subset \cdots \subset A_{100}$. Let $B = \cup_{i=1}^{100} B_i$, where elements
of the set $B_i$ are obtained from elements of $B$ having last entry $i$ by removing this last entry. By definitions $|B| = \sum_{i=1}^{100} |B_i|$ and $B_1 \supset B_2 \supset \cdots \supset B_{100}$. Now

$$|A \cap B| = \sum_{i=1}^{100} |A_i \cap B_i| \leq \frac{1}{100^{n-1}} \cdot \sum_{i=1}^{100} |A_i| \cdot |B_i|$$

$$\leq \frac{1}{100^{n-1}} \cdot \frac{1}{100} \cdot \left( \sum_{i=1}^{100} |A_i| \right) \left( \sum_{i=1}^{100} |B_i| \right) = \frac{1}{100^n} \cdot |A| \cdot |B|$$

(The first inequality is valid due to inductive hypothesis, the second inequality is the Chebyshev’s rearrangement inequality). We are done.