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Problem Of The Month

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Problem:

In a school having n students any student has exactly 971 friends and if two students are not friends then they have exactly 970 common friends. Find all possible values of n.

Solution: Answer: $n = 972, 974, 980, 1164, 1940$.

Let us reformulate a more general version of the problem in terms of the graph theory: In a regular graph G on n vertices each vertex has a degree $k < n-1$ and if two vertices are not neighbours then they have exactly $k-1$ common neighbours. Find all pairs (n,k) satisfying these conditions.

If $n = 972$ then the complete graph satisfies the conditions. The answer in the remaining cases: all pairs $(n,k) = (2ab, 2a(b-1)+1)$ where $a > 1$ and b are positive integers and also all pairs $(n, n-3)$ (if $n = 4b$ then all pairs $(n, n-3)$ are among pairs given by $(n,k) = (2ab, 2a(b-1)+1)$).

Let w be a vertex with neighbours $v_1, \ldots, v_k$. If some vertex is not a neighbour of w then it is directly connected to $k-1$ neighbours of w and hence it has exactly 1 neighbour among remaining vertices. Therefore, all vertices in the graph $G - \{w, v_1, \ldots, v_k\}$ have degree 1 and consequently the number of vertices not directly connected to w is even. Let us denote them by $u_1, \ldots, u_{2l}$. Since $1 + k + 2l = n$ we get that$2l = n - k - 1$. Without loss of generality for $1 \leq t \leq l$ let u_{2t-1} and u_{2t} be neighbours. When $l = 1$ we get a pair $(n, n-3)$. Let $l \geq 2$. For any $t \geq 3$ the vertices u_1 and u_t are not neighbours and their unique neighbours among vertices u_i are not common. Hence, u_1 and u_t have $k-1$ neighbours among vertices v_i. The same is true for u_2 and u_t. Therefore, all these vertices are connected to the same $k-1$ vertices of w. Without loss of generality, let v_1 be the vertex not connected to u_i, $1 \leq i \leq 2l$. Since the degree of $v_1$is k it is directly connected to each v_i. Therefore, for each $2 \leq i \leq k$ the degree of v_i in $G - \{w, v_1, u_1, \ldots, u_{2l}\}$ is $k - 1 - 1 - 2l = 2k - n - 1$. Then the graph on the vertices $v_2, \ldots, v_k$ satisfies problem conditions with new parameters $(k-1, 2k-n-1)$. Since $(k-1) - (2k-n-1) = n-k$, by repeating the same procedure we get that the graph G contains pieces with $2l+2$ vertices and each piece contains $l+1$ perfectly matching edges and all possible edges between different pieces are drawn. By denoting $l+1$ by a and the number of pieces by b we get the desired answer.

When $k=971$ from $2a(b-1)+1=971$ we get that $a|485=5\cdot 97$. Therefore the possible values for a are $a=5,97,485$ and we get $n=2ab=980,1164,1940$. The pair $(n,n-3)$ yields $n=974$.