Bilkent University
Department of Mathematics

Problem Of The Month

February 2023

Problem:

We say that two monic quadratic polynomials $P(x) = x^2 + ax + b$ and $Q(x) = x^2 + cx + d$ with integer coefficients are disjoint if $P(i) \neq Q(j)$ for any two integers i and j. Find the maximal number of pairwise disjoint monic quadratic polynomials with integer coefficients.

Solution: Answer: 2.

Note that for any integer C the change of variables $x \mapsto x + C$ leaves unchanged the range of any polynomial. Therefore, we can transfer each polynomial $P(x) = x^2 + ax + b$ to

$$P(x - \lfloor\frac{a}{2}\rfloor) = x^2 + x(a - 2\lfloor\frac{a}{2}\rfloor) + (\lfloor\frac{a}{2}\rfloor)^2 - a(\lfloor\frac{a}{2}\rfloor) + b$$

by using of the change $x \mapsto x - \lfloor\frac{a}{2}\rfloor$. As a result we will get polynomials of the forms either $P(x) = x^2 + b$ or $P(x) = x^2 + x + b$. Any two polynomials of different forms are not disjoint: For any two polynomial $P_1(x) = x^2 + b_1$ and $P_2(x) = x^2 + x + b_2$ we have $P_1(b_1 - b_2) = P_2(b_1 - b_2)$. Therefore, the set of pairwise disjoint polynomials may contain only polynomials of the same form.

Let $P_1(x) = x^2 + b_1$ and $P_2(x) = x^2 + b_2$. If $b_1 - b_2 = 2k + 1$ then $P_1(k) = P_2(k + 1)$. If $b_1 - b_2 = 4k$ then $P_1(k - 1) = P_2(k + 1)$. Therefore, if $P_1(x) = x^2 + b_1$ and $P_2(x) = x^2 + b_2$ are disjoint then $b_1 - b_2 = 1, 3 \pmod 4$. Thus there are at most two disjoint polynomials of the form $P(x) = x^2 + b$.

Let $P_1(x) = x^2 + x + b_1$ and $P_2(x) = x^2 + x + b_2$. If $b_1 - b_2 = 2k$ then $P_1(k - 1) = P_2(k)$. Therefore, if $P_1(x) = x^2 + b_1$ and $P_2(x) = x^2 + b_2$ are disjoint then $b_1 - b_2 = 1 \pmod 2$. Thus there are at most two disjoint polynomials of the form $P(x) = x^2 + x + b$.

Therefore, the set of pairwise disjoint polynomials may contain only two polynomials.

The polynomials $P_1(x) = x^2$ and $P_2 = x^2 + 2$, also the polynomials $P_1(x) = x^2 + x$ and $P_2 = x^2 + x + 1$ are disjoint. We are done.