Problem Of The Month

November 2022

Problem:

For each pair \((p, a)\), where \(p\) is a prime number and \(a\) is a positive integer the sequences \(\{a_n\}\) and \(\{b_n\}\) are defined as

\[
a_1 = a \quad \text{and} \quad a_{n+1} = a_n + p \lfloor \sqrt[p]{a_n} \rfloor \]

\[
b_n = \sqrt[p]{a_n}
\]

A pair \((p, a)\) is said to be good if the sequence \(\{b_n\}\) contains infinitely many integers.

Find all values of \(p\) such that all pairs \((p, a), \ a = 1, 2, \ldots\) are good.

Solution: Answer: For all primes \(p\) all pairs \((p, a), \ a = 1, 2, \ldots\) are good.

Let \(p\) be a fixed prime number and \(a\) be a given positive integer. Let \(m\) be a positive integer with \(m^p > a\). Since the sequence \(\{a_n\}\) is increasing the exists a smallest term \(a_k\) of the sequence \(\{a_n\}\) satisfying \(a_k > m^p\). By definitions, \(a_{k-1} < m^p\) and

\[
a_k - a_{k-1} = p \lfloor \sqrt[p]{a_{k-1}} \rfloor < pm.
\]

Therefore, \(a_k = m^p + r\), where \(r < pm\). Let us show that along with \(a_k = m^p + r\) the number \((m + 1)^p + r - 1\) is also an element of the sequence \(\{a_n\}\). Indeed, since \(p\) is a prime number the difference

\[
((m + 1)^p + r - 1) - (m^p + r) = \left(\frac{p}{p - 1}\right) m^{p-1} + \left(\frac{p}{p - 2}\right) m^{p-2} + \cdots + \left(\frac{p}{1}\right) m
\]

is a multiple of \(pm\). Therefore, the smallest term \(a_t\) of the sequence \(\{a_n\}\) satisfying \(a_t > (m + 1)^p\) has a form \(a_t = (m + 1)^p + r - 1\). By applying this argument repeatedly one can show that the sequence \(\{a_n\}\) contains terms \((m + 2)^p + r - 2\), \((m + 3)^p + r - 3\), \ldots, \((m + r - 1)^p + 1\) and finally the term \(a_t = (m + r)^p\). Thus, \(b_t = m + r\) is an integer. By choosing sparsely located distinct numbers \(m\) we can get infinitely many integer terms of \(\{b_n\}\).