Bilkent University
Department of Mathematics

Problem Of The Month

September 2022

Problem:

For a given positive integer n, let $f(n)$ be the smallest *positive* number which can be expressed as

$$\frac{ab}{a+b} - n$$

where a and b are positive integers. Find a closed form for $f(n)$ as a function of n.

Solution: Answer: $\dfrac{1}{(n+1)^2+1}$.

Let us rewrite the condition $\dfrac{ab}{a+b} - n > 0$ as $ab = (a+b)n + m$, where m is a positive integer. Equivalently,

$$(a-n)(b-n) = n^2 + m$$

which implies that $a > n$, $b > n$. Let $a - n = s$ and $b - n = t$. Then we have

$$st = n^2 + m \tag{1}$$

$$s + t \leq 1 + st = 1 + n^2 + m \tag{2}$$

Therefore by using (1) we get

$$\frac{ab}{a+b} = \frac{n^2 + st + n(s+t)}{2n+s+t} = \frac{n^2 + n^2 + m + n(s+t)}{2n+s+t} = n + \frac{m}{2n+s+t} \tag{3}$$

and by using (2) in (3) we get

$$\frac{ab}{a+b} = n + \frac{m}{2n + n^2 + 1 + m}. \tag{4}$$

Finally, since for all positive integers L we have $\dfrac{m}{L+m} \geq \dfrac{1}{L+1}$, (4) yields

$$\frac{ab}{a+b} - n \ \geq \ \frac{1}{(n+1)^2+1}.$$

The equality holds at $a = n+1,\ b = n^2+n+1$.