September 2022

Problem:

For a given positive integer \( n \), let \( f(n) \) be the smallest positive number which can be expressed as

\[
\frac{ab}{a + b} - n
\]

where \( a \) and \( b \) are positive integers. Find a closed form for \( f(n) \) as a function of \( n \).

Solution: Answer: \( \frac{1}{(n + 1)^2 + 1} \).

Let us rewrite the condition \( \frac{ab}{a + b} - n > 0 \) as \( ab = (a + b)n + m \), where \( m \) is a positive integer. Equivalently,

\[
(a - n)(b - n) = n^2 + m
\]

which implies that \( a > n, b > n \). Let \( a - n = s \) and \( b - n = t \). Then we have

\[
st = n^2 + m \tag{1}
\]

\[
s + t \leq 1 + st = 1 + n^2 + m \tag{2}
\]

Therefore by using (1) we get

\[
\frac{ab}{a + b} = \frac{n^2 + st + n(s + t)}{2n + s + t} = \frac{n^2 + n^2 + m + n(s + t)}{2n + s + t} = n + \frac{m}{2n + s + t} \tag{3}
\]

and by using (2) in (3) we get

\[
\frac{ab}{a + b} = n + \frac{m}{2n + n^2 + 1 + m}. \tag{4}
\]
Finally, since for all positive integers \( L \) we have \( \frac{m}{L+m} \geq \frac{1}{L+1} \), (4) yields

\[
\frac{a b}{a+b} - n \geq \frac{1}{(n+1)^2 + 1}.
\]

The equality holds at \( a = n + 1, \ b = n^2 + n + 1. \)