Bilkent University
Department of Mathematics

Problem Of The Month

July-August 2022

Problem:

Find the smallest value of

$$xy + yz + zx + \frac{1}{x} + \frac{2}{y} + \frac{5}{z} \ ,$$

where x, y, z are positive real numbers.

Solution: Answer: $3 \cdot \sqrt[3]{36}$.

Using AM-GM inequality, we get

$$\begin{aligned} xy + \frac{1}{3x} + \frac{1}{2y} \ &\geq \ 3\sqrt[3]{\frac{1}{6}}, \\ yz + \frac{3}{2y} + \frac{3}{z} \ &\geq \ 3\sqrt[3]{\frac{9}{2}}, \\ zx + \frac{2}{z} + \frac{2}{3x} \ &\geq \ 3\sqrt[3]{\frac{4}{3}}. \end{aligned}$$

Side by side sum of these three inequalities gives the following inequality:

$$xy + yz + zx + \frac{1}{x} + \frac{2}{y} + \frac{5}{z} \ \geq \ 3 \cdot \sqrt[3]{36}.$$

The equality holds when $(x, y, z) = \left(\frac{\sqrt[3]{6}}{3}, \ \frac{\sqrt[3]{6}}{2}, \ \sqrt[3]{6} \right)$.