Bilkent University
Department of Mathematics

Problem Of The Month

May 2022

Problem:

Let x, y, z be three positive real numbers satisfying

$$xyz = 1 \quad \text{ve} \quad \frac{y}{z}(y - x^2) + \frac{z}{x}(z - y^2) + \frac{x}{y}(x - z^2) = 0.$$

Let t_1, t_2 and t_3 be the smallest, the median and the largest of these three numbers, respectively. Find the smallest possible value of

$$\frac{t_1 + t_3}{t_2}.$$

Solution: Answer: $\dfrac{5}{\sqrt[5]{256}}$.

Let $\dfrac{x^2}{y} = a$, $\dfrac{y^2}{z} = b$ and $\dfrac{z^2}{x} = c$. The problem conditions in this new variables take the following form

$$abc = 1, \quad a + b + c = ab + bc + ca.$$

Now we readily get $(a-1)(b-1)(c-1) = 0$. Therefore, at least one of the numbers a, b, c should be equal to 1. Without loss of generality we assume that $a = 1$. Then $y = x^2$ and $z = \dfrac{1}{x^3}$.

If $0 < x \leq 1$ then $x^2 \leq x \leq \dfrac{1}{x^3}$ and if $x > 1$ then $\dfrac{1}{x^3} < x < x^2$. Hence in all cases x is a median: $t_2 = x$. Finally by AM-GM inequality we get

$$\frac{x^2 + \frac{1}{x^3}}{x} = x + \frac{1}{x^4} = \frac{x}{4} + \frac{x}{4} + \frac{x}{4} + \frac{x}{4} + \frac{1}{x^4} \geq 5\sqrt[5]{\frac{1}{4^4}}.$$

The equality holds when $\dfrac{x}{4} = \dfrac{1}{x^4}$ or $x = \sqrt[5]{4}$. In this case $y = x^2 = \sqrt[5]{16}$ and $z = \dfrac{1}{x^3} = \dfrac{1}{\sqrt[5]{64}}$. Thus, $\dfrac{t_1 + t_3}{t_2}$ takes its smallest value $\dfrac{5}{\sqrt[5]{256}}$ at $x = \sqrt[5]{4}$, $y = \sqrt[5]{16}$, and $z = \dfrac{1}{\sqrt[5]{64}}$. Done.