Problem Of The Month

December 2021

Problem:

Find all primes \( p \) for which there exist an odd integer \( n \) and a polynomial \( Q(x) \) with integer coefficients such that the polynomial

\[
1 + pn^2 + \prod_{i=1}^{2p-2} Q(x^i)
\]

has at least one integer root.

Solution: Answer: \( p = 2 \).

Let \( P(x) = 1 + pn^2 + \prod_{i=1}^{2p-2} Q(x^i) \). For \( p = 2, \ n = 1 \) and \( Q(x) = 2x + 1 \) are suitable, since the corresponding polynomial has a root -1:

\[
1 + 2 \cdot 1^2 + (2 \cdot (-1) + 1)(2 \cdot (-1^2) + 1) = 0.
\]

Let us show that for all primes \( p \geq 3 \) no suitable \( n \) and \( Q(x) \) exist. By Fermat’s little theorem \( x^i = x^{i+p-1} \) and hence \( Q(x^i) = Q(x^{i+p-1}) \) for all \( 1 \leq i \leq p - 1 \). Therefore, \( P(x) = 0 \) in modulo \( p \) leads to

\[
0 \equiv 1 + pn^2 + \prod_{i=1}^{2p-2} Q(x^i) \equiv 1 + (\prod_{i=1}^{p-1} Q(x^i))^2
\]

Thus -1 is a quadratic residue modulo \( p \) and hence \( p \equiv 1 \) (mod 4). Then \( P(x) = 0 \) in modulo 4 leads to

\[
0 \equiv 1 + 1 + \prod_{i=1}^{2p-2} Q(x^i) \equiv 2 + (\prod_{i=1}^{p-1} Q(x^i))^2 \quad (1)
\]
Note that for integer $x$ and positive $i, j$ the values $Q(x^i)$ and $Q(x^j)$ have the same parity. Therefore, both cases when all $Q(x^i)$ are odd and all $Q(x^i)$ are even we get a contradiction with (1). Done.