**Bilkent University**
**Department of Mathematics**

# Problem Of The Month

May 2021

**Problem:**

Find the greatest real number $T$ satisfying

$$T \leq \frac{a^3 + b^3 + c^3 - 3abc}{ab^2 + bc^2 + ca^2 - 3abc}$$

for all positive real numbers $a$, $b$, $c$.

**Solution:** Answer: $T = \dfrac{3}{\sqrt[3]{4}}$.

Since the inequality is cyclic w.l.o.g. we assume that $\min\{a, b, c\} = c$. Then for some non-negative $x$ and $y$ we have $a = c + x$, $b = c + y$. In the new variables $c, x, y$

$$a^3 + b^3 + c^3 - 3abc = (c+x)^3 + (c+y)^3 + c^3 - 3(c+x)(c+y)c = (3c+x+y)(x^2 - xy + y^2)$$

$$ab^2 + bc^2 + ca^2 - 3abc = (c+x)(c+y)^2 + (c+y)c^2 + c(c+x)^2 - 3(c+x)(c+y)c = (x^2 - xy + y^2)c + xy^2$$

Therefore, the inequality transfer to

$$(3 - T)(x^2 - xy + y^2)c + x^3 + y^3 - Txy^2 \geq 0 \tag{1}$$

for all positive $c, x, y$.

For $x = 1$, $y = \sqrt[3]{2}$ and any $c > 0$ the inequality (1) becomes

$$(3 - T)(1 - \sqrt[3]{2} + \sqrt[3]{4})c + 3 - \sqrt[3]{4}T \geq 0. \tag{2}$$

Let us show that if $T$ satisfies (2) then $T \leq \dfrac{3}{\sqrt[3]{4}}$. On the contrary, suppose that $T > \dfrac{3}{\sqrt[3]{4}}$.

If $T > 3$ then (2) is not held for any $c > 0$. If $T < 3$ then for

$$c < \frac{\sqrt[3]{4}T - 3}{(3-T)(1-\sqrt[3]{2}+\sqrt[3]{4})}$$

(2) is not held. Thus, $T \leq \frac{3}{\sqrt[3]{4}}$.

Now let us show that for $T = \frac{3}{\sqrt[3]{4}}$ the inequality (1) holds. Since $T < 3$ we have $(3-T)(x^2 - xy + y^2)c \geq 0$ and the required inequality will follow from $x^3 + y^3 \geq Txy^2$ which in turn is a consequence of AM-GM inequality:

$$x^3 + y^3 = x^3 + \frac{y^3}{2} + \frac{y^3}{2} \geq \frac{3}{\sqrt[3]{4}}xy^2 = Txy^2.$$

We are done.