Bilkent University
Department of Mathematics

Problem Of The Month

April 2021

Problem:

There are two boxes one coloured red and other coloured white. At the beginning each box contains 500 beads and there are infinitely many spare beads. Alice and Bob play a game alternatively making moves, Alice making first move. A player making a move performs one of the four actions: adds a bead to the red box (R^+), removes a bead from the red box (R^-), adds a bead to the white box (W^+), removes a bead from the white box (W^-) depending on the last move of her/his opponent and according to the following table: (at the first move Alice can choose any one of the four actions):

Last move	R^+	R^-	W^+	W^-
Alice's move	R^+ or W^+	R^- or W^-	W^+ or R^-	W^- or R^+
Bob's move	R^+ or W^-	R^- or W^+	W^+ or R^+	W^- or R^-

A player making 1001 or 0 beads in any box loses the game. Who wins?

Note: Due to the table after Bob's move R^+ Alice can move either R^+ or W^+, after Alice's move R^+ Bob can move either R^+ or W^-, ...

Solution: Answer: Alice wins.

Let us consider the n-game when at the beginning each box contains n beads and a player making $2n+1$ or 0 beads in any box loses the game. We will show that Alice wins the n-game for any n.

The position when the white and red boxes contain a and b beads respectively will be denoted by (a, b). The positions $(1, p)$, $(q, 1)$, $(2n, r)$ and $(s, 2n)$ will be called *boundary* positions. Note that Alice always makes moves from positions (a, b) with $a + b = 0 \pmod 2$ and Bob always makes moves from positions (a, b) with $a + b = 1 \pmod 2$.

Lemma. The player firstly making a move to any boundary position loses the game.

Proof: Suppose that Bob makes firstly a move to some boundary position. After that Alice will always choose a move to a neighbouring boundary position and consequently Bob will have only one legal move to some neighbouring boundary position. The process will terminate at either $(1, 2n)$ or $(2n, 1)$. At this position the only legal move of Bob is a losing move. If Alice makes firstly a move to some boundary situation then Bob wins by the similar strategy terminating at either $(1, 1)$ or $(2n, 2n)$. Done.

For example, suppose that Bob makes firstly a move to some boundary position $(q, 1)$. Then Alice makes a move to $(q-1, 1)$, Bob is forced to move to $(q-2, 1)$, Alice makes a move to $(q-3, 1)$, and so on until they reach the position $(1, 1)$. Since $1+1 = 0 \pmod 2$ Alice makes a move to $(1, 2)$, Bob is forced to move to $(1, 3)$, and so on until they reach the position $(1, 2n)$. Since $1 + 2n + 2 = 1 \pmod 2$ Bob should make a losing move. The remaining cases starting at other boundary positions are similar.

By induction over n, we will show that Alice wins the n-game and her last move is either $(1, 2n-1) \to (1, 2n)$ or $(2n, 2) \to (2n, 1)$.

If $n = 1$ than Alice wins by making a move $(1, 1) \to (1, 2)$.

Suppose that Alice wins the n-game. Consider the $(n + 1)$-game. The winning strategy of Alice is the following. She is trying to apply the strategy of n-game and make her last move as either $((1, 2n-1) \to (1, 2n)$ or $(2n, 2) \to (2n, 1)$. If Bob makes a move to one of the boundary positions $(1, p), (q, 1), (2n + 2, r)$, $(s, 2n + 2)$ before Alice completes the n-game strategy, then wins by the lemma. Otherwise they reach either $(1, 2n)$ or $(2n, 1)$ and Bob is forced to make a move to some boundary position and again loses by the lemma.