Problem Of The Month

April 2021

Problem:

There are two boxes one coloured red and other coloured white. At the beginning each box contains 500 beads and there are infinitely many spare beads. Alice and Bob play a game alternatively making moves, Alice making first move. A player making a move performs one of the four actions: adds a bead to the red box \((R^+\)) , removes a bead from the red box \((R^-)\), adds a bead to the white box \((W^+)\), removes a bead from the white box \((W^-)\) depending on the last move of her/his opponent and according to the following table: (at the first move Alice can choose any one of the four actions):

<table>
<thead>
<tr>
<th>Last move</th>
<th>(R^+)</th>
<th>(R^-)</th>
<th>(W^+)</th>
<th>(W^-)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alice’s move</td>
<td>(R^+) or (W^+)</td>
<td>(R^-) or (W^-)</td>
<td>(W^+) or (R^-)</td>
<td>(W^-) or (R^+)</td>
</tr>
<tr>
<td>Bob’s move</td>
<td>(R^+) or (W^-)</td>
<td>(R^-) or (W^+)</td>
<td>(W^+) or (R^+)</td>
<td>(W^-) or (R^-)</td>
</tr>
</tbody>
</table>

A player making 1001 or 0 beads in any box loses the game. Who wins?

Note: Due to the table after Bob’s move \(R^+\) Alice can move either \(R^+\) or \(W^+\), after Alice’s move \(R^+\) Bob can move either \(R^+\) or \(W^-\), ... 

Solution: Answer: Alice wins.

Let us consider the \(n\)-game when at the beginning each box contains \(n\) beads and a player making \(2n + 1\) or 0 beads in any box loses the game. We will show that Alice wins the \(n\)-game for any \(n\).

The position when the white and red boxes contain \(a\) and \(b\) beads respectively will be denoted by \((a, b)\). The positions \((1, p), (q, 1), (2n, r)\) and \((s, 2n)\) will be called boundary positions. Note that Alice always makes moves from positions \((a, b)\) with \(a + b = 0 \pmod{2}\) and Bob always makes moves from positions \((a, b)\) with \(a + b = 1 \pmod{2}\).
Lemma. The player firstly making a move to any boundary position loses the game.

Proof: Suppose that Bob makes firstly a move to some boundary position. After that Alice will always choose a move to a neighbouring boundary position and consequently Bob will have only one legal move to some neighbouring boundary position. The process will terminate at either \((1, 2n)\) or \((2n, 1)\). At this position the only legal move of Bob is a losing move. If Alice makes firstly a move to some boundary situation then Bob wins by the similar strategy terminating at either \((1, 1)\) or \((2n, 2n)\). Done.

For example, suppose that Bob makes firstly a move to some boundary position \((q, 1)\). Then Alice makes a move to \((q - 1, 1)\), Bob is forced to move to \((q - 2, 1)\), Alice makes a move to \((q - 3, 1)\), and so on until they reach the position \((1, 1)\). Since \(1 + 1 = 0 \pmod{2}\) Alice makes a move to \((1, 2)\), Bob is forced to move to \((1, 3)\), and so on until they reach the position \((1, 2n)\). Since \(1 + 2n + 2 = 1 \pmod{2}\) Bob should make a losing move. The remaining cases starting at other boundary positions are similar.

By induction over \(n\), we will show that Alice wins the \(n\)-game and her last move is either \((1, 2n - 1) \rightarrow (1, 2n)\) or \((2n, 2) \rightarrow (2n, 1)\).

If \(n = 1\) then Alice wins by making a move \((1, 1) \rightarrow (1, 2)\).

Suppose that Alice wins the \(n\)-game. Consider the \((n + 1)\)-game. The winning strategy of Alice is the following. She is trying to apply the strategy of \(n\)-game and make her last move as either \((1, 2n - 1) \rightarrow (1, 2n)\) or \((2n, 2) \rightarrow (2n, 1)\). If Bob makes a move to one of the boundary positions \((1, p), (q, 1), (2n + 2, r), (s, 2n + 2)\) before Alice completes the \(n\)-game strategy, then wins by the lemma. Otherwise they reach either \((1, 2n)\) or \((2n, 1)\) and Bob is forced to make a move to some boundary position and again loses by the lemma.