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Problem Of The Month

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Problem:

In a country consisting of 100 cities there are direct round flights between some pairs of cities. The total number of flights is 2021 and each flight is operated by one of n air companies. There are at least two cities such that one is not reachable from the other one by less than three flights. Given that for any two cities there is an air company connecting these cities (by direct or not direct flights) find the maximal possible value of n.

Solution: Answer: The maximal possible value of n is 1923.

Let us reformulate the problem in terms of the graph theory: Let G be a connected graph with 100 vertices and 2021 edges such that the distance between two vertices of G is at least 3. The edges of G are coloured into n colours such that any two vertices are connected by some monochromatic path. What is the maximal possible value of n?

Let us construct an example for $n = 1923$. Suppose that all edges of some tree connecting all 100 vertices are identically coloured and all remaining edges are differently coloured. Then all conditions are readily held and $n = 2021 - 99 + 1 = 1923$.

Now we will prove that there are at most 1923 colors. Consider a colouring of G with maximal number of colors. Then the following statements are held:

S1: Identically coloured edges form a connected tree. Indeed, if there is a monochromatic cycle we can recolour one of its edges in a new color and increase the number of colors, if two identically coloured trees are not connected then we can recolour one of these trees in a new color and thereby increase the number of colors.

S2: Two trees can not have more than 2 intersections. Indeed, if there are more than two intersections the union of two trees includes at least two cycles and we can recolor these two trees identically and after that recolour one edge in each cycle in new colors and thereby increase the number of colors.

S3: W.l.o.g. we can suppose that if two trees intersect at two vertices then both vertices are endpoints (vertices of degree one) of these trees. Indeed if one of intersection vertices is not an endpoint there is a cycle; we can recolour these two trees identically and after that recolour one edge in a cycle by new color thereby the number of colors will not change.

Consider vertices u and v located at distance at least 3. Let X be the set of vertices of a tree connecting u and v and $Y = X - \{u, v\}$. Let A be the set of all neighbour vertices of u and B be the set of all vertices of G except u, v and A. The vertex u should be connected to each vertex from $B - Y$ be some monochromatic tree. Let us denote these trees by $T_1, T_2, \ldots, T_p$. Any two of these trees intersect at u. By S3 any of these two trees intersect just at u. Therefore, to each tree T_k, $k = 1, \ldots, p$ we can assign a different vertex of A. Thus, the union of $T_1, \ldots, T_p$ contain at least $|B| - |Y \cap B| + p$ vertices except u. The vertex v should be connected to each vertex from $A - Y$ be some monochromatic tree. Let us denote these trees by $S_1, S_2, \ldots, S_q$. Similarly, the union of the trees $S_1, S_2, \ldots, S_q$ contain at least $|A| - |Y \cap A| + q$ vertices except v. A tree on $1 + d$ vertices with identically coloured edges causes a $d - 1$ colour loss. Therefore, the total colour loss will be at least

$$|B| - |Y \cap B| + p - p + |A| - |Y \cap A| + q - q + |Y| - 2.$$

Since $|Y \cap B| + |Y \cap A| = |Y| - 2$ the total colour loss is at least $|A| + |B| = 98$. Thus, the total number of colours is at most $2021 - 98 = 1923$. Done.